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Variation of Constant Formulae for Time Invariant and Time Varying Caputo Fractional Delay Differential Systems

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Abstract This paper studies the variation of constant formulae for linear Caputo fractional delay differential systems. We discuss the exponential estimates of the solutions for linear time invariant fractional delay differential systems by using the Gronwall's integral inequality. The variation of constant formula for linear time invariant fractional delay differential systems is obtained by using the Laplace transform method. In terms of the superposition principle of linear systems and fundamental solution matrix, we also establish the variation of constant formula for linear time varying fractional delay differential systems. The obtained results generalize the corresponding ones of integer-order delayed differential equations.

Keywords fractional delay differential systems; variation formula; exponential estimates; Laplace transform; Gronwall's integral inequality.

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1. Introduction

In this paper, we consider the following linear time invariant Caputo fractional delay differential system

$$\begin{cases} {}^CD^\alpha x(t) = Ax(t) + Bx(t-\tau) + f(t), \quad t \geq 0, \\ x(t) = \varphi(t), \quad t \in [-\tau, 0] \end{cases} \tag{1.1}$$

and linear time varying Caputo fractional delay differential system

$$\begin{cases} {}^CD^\alpha x(t) = A(t)x(t) + B(t)x(t-\tau) + f(t), \quad t \geq 0, \\ x(t) = \varphi(t), \quad t \in [-\tau, 0], \end{cases} \tag{1.2}$$

where ${}^CD^\alpha x(t)$ denotes an α order Caputo fractional derivative of $x(t)$, $0 < \alpha < 1$, A, B are $n \times n$ constant matrices, $A(t), B(t)$ are $n \times n$ continuous function matrices, τ is a constant with $\tau > 0$, and $f(t)$ is an n-dimensional continuous vector-valued function.

In the past few decades, the widely investigated subjects of fractional calculus and fractional-order differential systems have remarkably gained importance and popularity due to its demonstrated applications in numerous diverse fields of science and engineering. For more details on

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fractional calculus theory, one can see the monographs of Miller and Ross [1], Podlubny [2], Kilbas et al. [3] and Diethelm [4]. Fractional differential equations involving the Riemann-Liouville fractional derivative or the Caputo fractional derivative have been paid more and more attention in [1–22] and references therein. For example, Kilbas et al. [3] investigated the explicit solutions of linear fractional ordinary differential equations based on the method of successive approximations. Odibat and Momani [5] discussed the approximate solutions of nonlinear fractional order differential equations by using variational iteration method. Bonilla et al. [6] discussed the explicit solutions for linear fractional ordinary differential equations employing the exponential matrix function and the fractional Green function. Lakhmikantham [7] established the theory of inequalities, local existence, extremal solutions, comparison results and global existence of the solutions of fractional differential equations. In [10–12], the authors investigated the initial value problem of impulsive fractional differential equations. Ahmad et al. [13, 14] obtained the existence results for boundary value problems of fractional differential equations by fixed point theorems. Odibat [15] derived the exact solutions of linear fractional ordinary differential systems by the analytical approaches.

On the other hand, time delay has an important effect on the stability and performance of system. Recently, there has been increasing interest in the investigation of fractional differential equations with delay [23–28], due to their significance in both theory and applications. In [23–26], some existence results of fractional differential equations with delay have been obtained by using the different methods, such as fractional functional differential inequalities [23], method of steps [24], fixed point theorem [25, 26].

As we all know, the Laplace transform method is an effective and convenient method for solving linear integer order differential equations with constant coefficient. In [17, 19], the Laplace transform method was extended to solve fractional differential equations, in which the authors focused especially on the non-delayed case for $\tau \equiv 0$. In this paper, motivated by the above mentioned works, we investigate the variation of constant formulae for system (1.1) and system (1.2). Firstly, by using the Gronwall's integral inequality [3], we derive the exponential estimates of the solutions for system (1.1), which are basic to the applications of the Laplace transform [17]. Next, by using the Laplace transform method, we establish the variation of constant formula for such system. In terms of the superposition principle of linear systems and fundamental solution matrix, the variation of constant formula for system (1.2) is also obtained.

This paper is organized as follows. In the next section, we recall some definitions and preliminary lemmas used in the paper. In Section 3, we derive the exponential estimates of the solutions for system (1.1) by using the Gronwall's integral inequality. In Section 4, the variation of constant formula for system (1.1) is obtained by using the Laplace transform method. In Section 5, we establish the variation of constant formula for system (1.2) based on the superposition principle of linear systems and fundamental solution matrix.

2. Preliminaries

In this section, some definitions about fractional calculus and preliminary lemmas are recalled. For more details, one can see [1–4].

Throughout this paper, denote by $C([-\tau,0],\mathbb{R}^n)$ the Banach space of all continuous functions mapping the interval $[-\tau,0]$ into $\mathbb{R}^n$. For any element $\varphi\in C([-\tau,0],\mathbb{R}^n)$, the norm $\|\cdot\|$ is defined as $\|\varphi\|=\sup_{\theta\in[-\tau,0]}|\varphi(\theta)|$.

Definition 2.1 ([2]) *The Riemann-Liouville's fractional integral of order $p>0$ for a function $f:R^+\to R^n$ is defined as*

$$D^{-p}f(t)=\frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1}f(s)\mathrm{d}s,$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2 ([2]) *The Caputo's fractional derivative of order p $(0\le m-1<p<m)$ for a function $f:R^+\to R^n$ is defined as*

$$^{C}D^{p}f(t)=\frac{1}{\Gamma(m-p)}\int_0^t (t-s)^{m-p-1}f^{(m)}(s)\mathrm{d}s.$$

Remark 2.3 From Definition 2.2, the Caputo fractional partial derivative of α order $0<\alpha<1$ for a function $f(t,s)$ can be defined as

$$^{C}_{a}D^{\alpha}_{t}\left[f(t,s)\right]=\frac{\partial^\alpha f(t,s)}{\partial^\alpha t}=\frac{1}{\Gamma(1-\alpha)}\int_a^t \frac{\partial f(\theta,s)}{\partial\theta}\frac{1}{(t-\theta)^\alpha}\mathrm{d}\theta. \tag{2.1}$$

Lemma 2.4 (Gronwall's integral inequality [3]) *If $x(t)$ and $u(t)$ are real valued continuous functions on $[a,b]$, and $u(t)$ is a nondecreasing function on $[a,b]$. In addition, $g(t)\ge 0$ is integrable on $[a,b]$ with*

$$x(t)\le u(t)+\int_a^t g(s)x(s)\mathrm{d}s,\quad t\in[a,b],$$

then the following inequality holds

$$x(t)\le u(t)e^{\int_a^t g(s)\mathrm{d}s},\quad t\in[a,b]. \tag{2.2}$$

Lemma 2.5 (Existence and convolution of Laplace transform [3]) *If $f:[0,+\infty)\to\mathbb{R}^n$ is measurable and satisfies*

$$\|f(t)\|\le Me^{\sigma t},\quad t\in[0,+\infty),$$

for real constants M and σ, then the Laplace transform $\pounds[f(t)]$ defined by

$$\pounds[f(t)]=\int_0^{+\infty} e^{-st}f(t)\mathrm{d}t,\quad s\in\mathbb{C}$$

exists and is an analytic function of s for $\mathfrak{R}(s)>\sigma$, where $\mathfrak{R}(s)$ represents the real part of the complex number s, and $\mathbb{C}$ denotes the complex plane. If the function $f*g$ is defined by

$$f*g=\int_0^t f(t-u)g(u)\mathrm{d}u,$$

then $\pounds[f*g]$ has the following property

$$\pounds[f*g]=\pounds[f(t)]\pounds[g(t)]. \tag{2.3}$$

Lemma 2.6 (Inversion theorem of Laplace transform [3]) *Suppose $f : [0, +\infty) \to \mathbb{R}^n$ is a given function, $\sigma > 0$ is a given constant such that f is of bounded variation on any compact set and $f(t)e^{-\sigma t}$ is Lebesque integral on $[0, +\infty)$. Then, for any $c > \sigma$,*

$$\int_{(c)} \pounds[f(t)]e^{st}\mathrm{d}s =: \lim_{T\to+\infty} \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \pounds[f(t)]e^{st}\mathrm{d}s = \begin{cases} \frac{1}{2}[f(t^+) + f(t^-)], & t > 0, \\ \frac{1}{2}f(0^+), & t = 0. \end{cases} \tag{2.4}$$

3. Exponential estimates and fundamental solutions of system (1.1)

In this section, we derive the exponential estimation of solution $x(\varphi, f)$ for system (1.1) depends on φ and f. These estimates are basic to the applications of the Laplace transform [17].

Theorem 3.1 *If $f(t)$ is continuous on $[0, +\infty)$, then there exists a positive constant $\sigma > 0$, such that the solution $x(t)$ satisfies the exponential estimation*

$$\|x(t)\| \le \Big[\|\varphi\| + \frac{\sigma\tau^\alpha t^\alpha}{\Gamma(\alpha+1)M^\alpha}\|F(t)\|\Big]e^{\sigma t}, \quad t \ge 0, \tag{3.1}$$

where $\|F(t)\| = \sup_{s\in[0,t]} \|f(s)\|$, $M = \frac{\|A\|+\|B\|}{\Gamma(\alpha+1)}$.

Proof By the method of steps which has been presented in [24], there exists a unique solution to system (1.1).

Since $f(t)$ is continuous on $[0, +\infty)$, it follows from [23] that system (1.1) is equivalent to the following Volterra fractional integral with memory

$$\begin{cases} x(t) = \varphi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[Ax(s) + Bx(s-\tau) + f(s)]\mathrm{d}s, \quad t \ge 0, \\ x(t) = \varphi(t), \quad t \in [-\tau, 0], \end{cases} \tag{3.2}$$

that is, every solution of (3.2) is also a solution of (1.1) and vice versa. Applying the appropriate properties of the norm $\|\cdot\|$ and integral for $t \ge 0$ gives

$$\|x(t)\| \le \|\varphi\| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[\|A\|\|x(s)\| + \|B\|\|x(s-\tau)\| + \|f(s)\|]\mathrm{d}s.$$

Let $\|y(t)\| = \sup_{\theta\in[-\tau,0]} \|x(t+\theta)\|$. For $0 \le t \le \tau$, one can obtain

$$\begin{aligned} \|y(t)\| &\le \|\varphi\| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[\|A\|\|y(s)\| + \|B\|\|y(s)\| + \|f(s)\|]\mathrm{d}s \\ &= \|\varphi\| + \frac{t^\alpha}{\Gamma(\alpha+1)}\|F(t)\| + \frac{\|A\|+\|B\|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|y(s)\|\mathrm{d}s, \end{aligned}$$

where $\|F(t)\| = \sup_{s\in[0,t]} \|f(s)\|$. Obviously, one can introduce a nondecreasing function $u(t)$ as

$$u(t) = \|\varphi\| + \frac{t^\alpha}{\Gamma(\alpha+1)}\|F(t)\|, \quad t \in [0, \tau].$$

An application of Lemma 2.4 yields that

$$\|y(t)\| \le \Big[\|\varphi\| + \frac{t^\alpha}{\Gamma(\alpha+1)}\|F(t)\|\Big]e^{\frac{\|A\|+\|B\|}{\Gamma(\alpha+1)}t^\alpha}, \quad t \in [0, \tau].$$

Taking $M=\frac{\|A\|+\|B\|}{\Gamma(\alpha+1)}$, we have

$$\|y(t)\| \leq \Big[\|\varphi\| + \frac{t^\alpha}{\Gamma(\alpha+1)}\|F(t)\|\Big]e^{Mt^\alpha}, \quad t\in[0,\tau]. \tag{3.3}$$

Moreover, the same argument implies the following estimation

$$\|y(t)\| \leq \Big[\|y(t_0)\| + \frac{(t-t_0)^\alpha}{\Gamma(\alpha+1)}\|F(t)\|\Big]e^{M(t-t_0)^\alpha}, \quad t\in[t_0,t_0+\tau].$$

Next, we need to prove that

$$\|y(t)\| \leq \Big[\|\varphi\| + \frac{\sigma\tau^\alpha t^\alpha}{\Gamma(\alpha+1)M^\alpha}\|F(t)\|\Big]e^{\sigma t}, \quad t\in[0,T^*],\ \forall T^*>0, \tag{3.4}$$

where $\sigma > M+\tau^{-\alpha}$. According to (3.3), we know that (3.4) is true for $t\in[0,\tau]$. Assume that (3.4) is true for $t\in[0,k\tau], k\geq 1$. By this hypothesis, we need to prove that (3.4) is true for $t\in[0,(k+1)\tau]$. For $t\in[\tau,(k+1)\tau]$, we denote $t_1=t-\tau\in[0,k\tau]$, then it yields

$$\begin{aligned}
\|y(t)\| &\leq \Big[\|y(t_1)\| + \frac{(t-t_1)^\alpha}{\Gamma(\alpha+1)}\|F(t)\|\Big]e^{M(t-t_1)^\alpha}\\
&= \Big[\|y(t_1)\| + \frac{\tau^\alpha}{\Gamma(\alpha+1)}\|F(t)\|\Big]e^{M\tau^\alpha}\\
&\leq \Big\{\big[\|\varphi\| + \frac{\sigma\tau^\alpha(t-\tau)^\alpha\|F(t-\tau)\|}{\Gamma(\alpha+1)M^\alpha}\big]e^{\sigma(t-\tau)^\alpha} + \frac{\tau^\alpha\|F(t)\|}{\Gamma(\alpha+1)}\Big\}e^{M\tau^\alpha}\\
&\leq \Big\{\big[\|\varphi\| + \frac{\sigma\tau^\alpha(t-\tau)^\alpha\|F(t-\tau)\|}{\Gamma(\alpha+1)M^\alpha}\big]e^{\sigma(t-\tau)^\alpha+M\tau^\alpha} + \frac{\tau^\alpha\|F(t)\|}{\Gamma(\alpha+1)}\Big\}e^{M\tau^\alpha}\\
&\leq \Big\{\big[\|\varphi\| + \frac{\sigma\tau^\alpha(t-\tau)^\alpha\|F(t)\|}{\Gamma(\alpha+1)M^\alpha}\big]e^{\sigma(t-\tau)^\alpha+M\tau^\alpha} + \frac{\tau^\alpha\|F(t)\|}{\Gamma(\alpha+1)}\Big\}e^{M\tau^\alpha}\\
&\leq \Big[\|\varphi\| + \frac{\sigma\tau^\alpha(t-\tau)^\alpha+\tau^\alpha M^\alpha}{\Gamma(\alpha+1)M^\alpha}\|F(t)\|\Big]e^{\sigma t^\alpha}\\
&\leq \Big[\|\varphi\| + \frac{\sigma\tau^\alpha t^\alpha}{\Gamma(\alpha+1)M^\alpha}\|F(t)\|\Big]e^{\sigma t}.
\end{aligned} \tag{3.5}$$

From the above analysis, we have three cases as follows:

Case 1 When $(k+1)\tau=T^*$, we know that (3.4) is true.

Case 2 When $0<T^*-(k+1)\tau<\tau$, we will prove (3.4) is true for $t\in[\tau,T^*]$. For $t\in[\tau,T^*]$, we denote $t_1=t-\tau\in[0,T^*-\tau]\subset[0,(k+1)\tau]$, and we can use the same proof as in the proof of (3.5) to finish the theorem.

Case 3 When $T^*-(k+1)\tau>\tau$, we can repeat the above process until the condition of Case 2 is satisfied.

Since T^* is arbitrary, we know that (3.4) holds on $[0,+\infty)$. Therefore, the proof of the theorem is completed. □

Next, we consider the fundamental solution and characteristic equation for linear fractional homogeneous equation of the form

$${}^C D^\alpha x(t) = Ax(t) + Bx(t-\tau), \quad t\geq 0,\ 0<\alpha<1. \tag{3.6}$$

The fundamental solution of the homogeneous system (3.6) is defined as follows:

Definition 3.2 *Assume that $X(t) \in \mathbb{R}^{n\times n}$. If $X(t)$ satisfies*

$$\begin{cases} {}^C D^\alpha X(t) = AX(t) + BX(t-\tau), \\ X(t) = \begin{cases} I, & t = 0, \\ 0, & t \in [-\tau, 0), \end{cases} \end{cases} \tag{3.7}$$

then $X(t)$ is called the fundamental solution of system (3.6).

The characteristic equation for equation (3.6) is obtained from the equation by looking for nontrivial solution of the form $e^{st}u$, where u is an n-dimensional vector. Equation (3.6) has a nontrivial solution $e^{st}u$ if and only if

$$H(s) =: s^\alpha I - A - Be^{-s\tau} = 0. \tag{3.8}$$

Theorem 3.3 *If $X(t)$ is the fundamental solution of system (3.6), then $X(t)$ satisfies*

$$X(t) = \pounds^{-1}[s^{\alpha-1}H^{-1}(s)]. \tag{3.9}$$

Moreover, if there exists a positive number c satisfying $c > \sigma$, then $X(t)$ can be further represented as

$$X(t) = \int_{(c)} s^{\alpha-1}H^{-1}(s)e^{st}\mathrm{d}s, \quad t > 0, \tag{3.10}$$

where $\pounds^{-1}$ denotes the inverse Laplace transform, and σ is the exponent associated with the bound on $X(t)$ in Theorem 3.1.

Proof Since $X(t)$ satisfies the exponential bounds in Theorem 3.1, $\pounds[X(t)]$ exists and is analytic for $\Re(s) > \sigma$. The Laplace transform of an α order Caputo fractional derivative of function $x(t)$ is (see [3])

$$\pounds[{}^C D^\alpha x(t)] = s^\alpha \pounds[x(t)] - s^{\alpha-1}x(0), \quad 0 < \alpha < 1.$$

Taking the Laplace transform with respect to t in both sides of system (3.6), we have

$$s^\alpha \pounds[x(t)] - s^{\alpha-1}x(0) = A\pounds[x(t)] + B\pounds[x(t-\tau)].$$

It follows from the properties of integral that

$$\pounds[x(t-\tau)] = \int_0^{+\infty} e^{-st}x(t-\tau)\mathrm{d}t = e^{-s\tau}\pounds[x(t)] + e^{-s\tau}\int_{-\tau}^0 e^{-st}x(t)\mathrm{d}t.$$

Note that $X(0) = I$ and $X(t) = 0, t \in [-\tau, 0)$. Then we have

$$[s^\alpha I - A - Be^{-s\tau}]\pounds[X(t)] = s^{\alpha-1}I,$$

which implies that

$$X(t) = \pounds^{-1}[s^{\alpha-1}H^{-1}(s)].$$

Since $X(t)$ is of bounded variation on compact sets and continuous for $t \geq 0$, applying the inverse Laplace transform (Lemma 2.6), we obtain

$$X(t) = \int_{(c)} s^{\alpha-1}H^{-1}(s)e^{st}\mathrm{d}s, \quad t > 0.$$

Therefore, the proof is completed. □

4. Variation formula for system (1.1)

In this section, we derive the variation of constant formula for linear time invariant Caputo fractional delay differential equation.

In terms of the fundamental solution $X(t)$ of system (3.6), the general solution $x(t,\varphi,0)$ for system (3.6) can be represented in the following theorem.

Theorem 4.1 *If $X(t)$ is the fundamental solution of system (3.6), then the general solution $x(t,\varphi,0)$ of system (3.6) can be represented in the following form*

$$x(t,\varphi,0)=X(t)\varphi(0)+B\int_{-\tau}^{0}\Big[({}^{C}D^{1-\alpha}X)(t-\tau-\theta)+\frac{(t-\tau-\theta)^{\alpha-1}}{\Gamma(\alpha)}I\Big]\varphi(\theta)\mathrm{d}\theta. \tag{4.1}$$

Proof Applying the Laplace transform to system (3.6), we have

$$s^{\alpha}\pounds[x(t)]-s^{\alpha-1}\varphi(0)=A\pounds[x(t)]+B\pounds[x(t-\tau)]. \tag{4.2}$$

Applying the properties of integral, it is easy to obtain

$$\pounds[x(t-\tau)]=\int_{0}^{+\infty}e^{-st}x(t-\tau)\mathrm{d}t=e^{-s\tau}\pounds[x(t)]+e^{-s\tau}\int_{-\tau}^{0}e^{-st}\varphi(t)\mathrm{d}t. \tag{4.3}$$

Combining (4.2) with (4.3) yields that

$$[s^{\alpha}I-A-Be^{-s\tau}]\pounds[x(t)]=s^{\alpha-1}\varphi(0)+Be^{-s\tau}\int_{-\tau}^{0}e^{-st}\varphi(t)\mathrm{d}t.$$

Note that $H(s)=s^{\alpha}I-A-Be^{-s\tau}$ and $s^{\alpha-1}H^{-1}(s)=\pounds[X(t)]$, one can get

$$\pounds[x(t)]=\pounds[X(t)]\varphi(0)+\pounds[X(t)]Bs^{1-\alpha}e^{-s\tau}\int_{-\tau}^{0}e^{-st}\varphi(t)\mathrm{d}t. \tag{4.4}$$

In order to apply Lemma 2.5 to the second term of the right in (4.4), we define a function $\omega(\cdot)$ and extend the initial function $\varphi(\cdot)$ as follows:

$$\omega(t)=\begin{cases}0, & t\ge 0,\\ 1, & t\in[-\tau,0),\end{cases}\qquad \hat{\varphi}(t)=\begin{cases}\varphi(0), & t\ge 0,\\ \varphi(t), & t\in[-\tau,0).\end{cases}$$

Therefore, we have

$$\begin{aligned}\pounds[X(t)]Bs^{1-\alpha}e^{-s\tau}\int_{-\tau}^{0}e^{-st}\varphi(t)\mathrm{d}t&=Bs^{1-\alpha}\pounds[X(t)]e^{-s\tau}\int_{-\tau}^{+\infty}e^{-st}\hat{\varphi}(t)\omega(t)\mathrm{d}t\\&=Bs^{1-\alpha}\pounds[X(t)]\int_{0}^{+\infty}e^{-st}\hat{\varphi}(t-\tau)\omega(t-\tau)\mathrm{d}t\\&=Bs^{1-\alpha}\pounds[X(t)]\pounds[\hat{\varphi}(t-\tau)\omega(t-\tau)].\end{aligned}$$

According to [3], for $\alpha\in(0,1)$, one can get the following equalities

$$\pounds[({}^{C}D^{1-\alpha}X)(t)]=s^{1-\alpha}\pounds[X(t)]-s^{-\alpha}X(0),\quad s^{-\alpha}\pounds[f(t)]=\pounds[D^{-\alpha}f(t)].$$

Thus, (4.4) is equivalent to

$$\pounds[x(t)] = \pounds[X(t)]\varphi(0)+B\pounds[{}^C D^{1-\alpha}X(t)]\pounds[\hat{\varphi}(t-\tau)\omega(t-\tau)]+B\pounds\Big\{D^{-\alpha}\big[\hat{\varphi}(t-\tau)\omega(t-\tau)\big]\Big\}. \quad (4.5)$$

Lemmas 2.5 and 2.6 applied to (4.5) yields the form

$$\begin{aligned} x(t) &= X(t)\varphi(0) + B\big[({}^C D^{1-\alpha}X)(t)\big] * \big[\hat{\varphi}(t-\tau)\omega(t-\tau)\big] + BD^{-\alpha}\big[\hat{\varphi}(t-\tau)\omega(t-\tau)\big] \\ &= X(t)\varphi(0) + B\int_0^t \Big[({}^C D^{1-\alpha}X)(t-s) + \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}I\Big]\hat{\varphi}(s-\tau)\omega(s-\tau)\mathrm{d}s \\ &= X(t)\varphi(0) + B\int_0^\tau \Big[({}^C D^{1-\alpha}X)(t-s) + \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}I\Big]\varphi(s-\tau)\mathrm{d}s. \end{aligned} \quad (4.6)$$

Let $s=\tau+\theta$. Hence, it follows from (4.6) that

$$x(t) = X(t)\varphi(0) + B\int_{-\tau}^0 \Big[({}^C D^{1-\alpha}X)(t-\tau-\theta) + \frac{(t-\tau-\theta)^{\alpha-1}}{\Gamma(\alpha)}I\Big]\varphi(\theta)\mathrm{d}\theta.$$

The proof is completed. □

Based on the Laplace transform method, we derive the variation formula for linear nonhomogeneous time invariant Caputo fractional delay differential equation.

Theorem 4.2 *If $f(t)$ is continuous on $[0,+\infty)$ and exponentially bounded, $X(t)$ is the fundamental solution of system (3.6) and $x(t,\varphi,0)$ is a solution of system (3.6), then the solution $x^*(t,\varphi,f)$ of system (1.1) can be represented in the following form*

$$x^*(t,\varphi,f) = x(t,\varphi,0) + D^{-\alpha}f(t) + \int_0^t ({}^C D^{1-\alpha}X)(t-s)f(s)\mathrm{d}s. \quad (4.7)$$

Proof Taking the Laplace transform with respect to t in both sides of system (1.1), we have

$$s^\alpha\pounds[x^*(t)] - s^{\alpha-1}\varphi(0) = A\pounds[x^*(t)] + B\pounds[x^*(t-\tau)] + \pounds[f(t)].$$

Note that $H(s) = s^\alpha I - A - Be^{-s\tau}$ and $s^{\alpha-1}H^{-1}(s) = \pounds[X(t)]$, one can get

$$\pounds[x^*(t)] = \pounds[X(t)]\varphi(0) + \pounds[X(t)]Bs^{1-\alpha}e^{-s\tau}\int_{-\tau}^0 e^{-st}\varphi(t)\mathrm{d}t + s^{1-\alpha}\pounds[X(t)]\pounds[f(t)].$$

According to the proof of Theorem 4.2, we have

$$\pounds[x^*(t)] = \pounds[x(t)] + s^{1-\alpha}\pounds[X(t)]\pounds[f(t)]. \quad (4.8)$$

It follows from [3] that the following relations

$$\pounds[({}^C D^{1-\alpha}X)(t)] = s^{1-\alpha}\pounds[X(t)] - s^{-\alpha}X(0),\ X(0)=I,\ s^{-\alpha}\pounds[f(t)] = \pounds[D^{-\alpha}f(t)],$$

which imply that equality (4.8) can be presented in the following manner

$$\pounds[x^*(t)] = \pounds[x(t)] + \pounds[({}^C D^{1-\alpha}X)(t)]\pounds[f(t)] + \pounds[D^{-\alpha}f(t)]. \quad (4.9)$$

The convolution theorem and inverse theorem of the Laplace transform applied to (4.9) yields the form

$$x^*(t,\varphi,f) = x(t,\varphi,0) + D^{-\alpha}f(t) + \int_0^t ({}^C D^{1-\alpha}X)(t-s)f(s)\mathrm{d}s.$$

The proof is completed. □

According to Theorems 4.1 and 4.2, we can get the general solution expression of time-invariant Caputo fractional delay differential system (1.1).

Corollary 4.3 *If $f(t)$ is continuous on $[0,+\infty)$ and exponentially bounded, $X(t)$ is the fundamental solution of system (3.6), then the general solution $x^*(t,\varphi,f)$ of system (1.1) can be written as*

$$\begin{aligned} x^*(t,\varphi,f) =& X(t)\varphi(0) + B\int_{-\tau}^{0}\Big[({}^CD^{1-\alpha}X)(t-\tau-\theta)+\frac{(t-\tau-\theta)^{\alpha-1}}{\Gamma(\alpha)}I\Big]\varphi(\theta)\mathrm{d}\theta+ \\ & D^{-\alpha}f(t)+\int_0^t({}^CD^{1-\alpha}X)(t-s)f(s)\mathrm{d}s. \end{aligned}$$

5. Variation formula for system (1.2)

In this section, we establish the variation formula for system (1.2) in terms of the superposition principle of linear systems and fundamental solution matrix.

Consider the following two time varying Caputo fractional delay differential systems

$$\begin{cases} {}^CD^{\alpha}x(t)=A(t)x(t)+B(t)x(t-\tau)+f(t), & t\ge 0, \\ x(t)\equiv 0, \quad t\in[-\tau,0] \end{cases} \tag{5.1}$$

and

$$\begin{cases} {}^CD^{\alpha}x(t)=A(t)x(t)+B(t)x(t-\tau), & t\ge 0, \\ x(t)=\varphi(t), \quad t\in[-\tau,0]. \end{cases} \tag{5.2}$$

The following lemma can be derived from the superposition principle of linear systems.

Lemma 5.1 *If $x_1(t)$, $x_2(t)$ are the solutions of systems (5.1) and (5.2), respectively, then $x(t)=x_1(t)+x_2(t)$ is the solution of system (1.2).*

Definition 5.2 *Assume that $X(t,s)\in\mathbb{R}^{n\times n}$. If $X(t,s)$ satisfies*

$$\begin{cases} {}^CD^{\alpha}X(t,s)=A(t)X(t,s)+B(t)X(t-\tau,s), & t\ge s, \\ X(t,s)=\begin{cases} I, & t=s, \\ 0, & t<s, \end{cases} \end{cases} \tag{5.3}$$

then $X(t,s)$ is called the fundamental solution matrix of system (5.1).

Theorem 5.3 *If $X(t,s)$ is the fundamental solution matrix of system (5.1), then*

$$x(t)=\int_0^t X(t,s)D^{-\alpha}[f(s)]\mathrm{d}s \tag{5.4}$$

is the solution of system (5.1).

Proof Let $G(t,s)=X(t,s)D^{-\alpha}[f(s)]$. It follows from Definition 2.2 that

$$\begin{aligned} {}^CD^{\alpha}x(t) &= {}^CD^{\alpha}\Big\{\int_0^t X(t,s)D^{-\alpha}[f(s)]\mathrm{d}s\Big\} \\ &= {}^CD^{\alpha}\Big[\int_0^t G(t,s)\mathrm{d}s\Big] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-\theta)^\alpha} \frac{\mathrm{d}}{\mathrm{d}\theta} \Big[\int_0^\theta G(\theta,s)\mathrm{d}s \Big] \mathrm{d}\theta \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-\theta)^\alpha} \Big[\int_0^\theta \frac{\partial G(\theta,s)}{\partial \theta} \mathrm{d}s + \lim_{s\to\theta^-} G(\theta,s) \Big] \mathrm{d}\theta.
\end{aligned} \tag{5.5}
$$

Exchanging the integral order of the first term of the right in (5.5) yields

$$\int_0^t \frac{1}{(t-\theta)^\alpha} \Big[\int_0^\theta \frac{\partial G(\theta,s)}{\partial \theta} \mathrm{d}s \Big] \mathrm{d}\theta = \int_0^t \mathrm{d}s \int_s^t \frac{1}{(t-\theta)^\alpha} \frac{\partial G(\theta,s)}{\partial \theta} \mathrm{d}\theta,$$

which implies that (5.5) is equivalent to

$${}^C D^\alpha x(t) = \int_0^t \Big[{}_s^C D_t^\alpha G(t,s) \Big] \mathrm{d}s + \frac{1}{\Gamma(1-\alpha)} \int_0^t \lim_{s\to\theta^-} G(\theta,s) \frac{1}{(t-\theta)^\alpha} \mathrm{d}\theta. \tag{5.6}$$

On the one hand, from Definition 5.1, we can get

$$
\begin{aligned}
\int_0^t \Big[{}_s^C D_t^\alpha G(t,s) \Big] \mathrm{d}s &= \int_0^t \Big[{}_s^C D_t^\alpha X(t,s) D^{-\alpha}[f(s)] \Big] \mathrm{d}s \\
&= \int_0^t \Big[{}^C D^\alpha X(t,s) D^{-\alpha}[f(s)] \Big] \mathrm{d}s \\
&= \int_0^t \big[A(t)X(t,s) + B(t)X(t-\tau,s) \big] D^{-\alpha}[f(s)] \mathrm{d}s \\
&= \int_0^t A(t)X(t,s)D^{-\alpha}[f(s)]\mathrm{d}s + \int_0^{t-\tau} B(t)X(t-\tau,s)D^{-\alpha}[f(s)]\mathrm{d}s + \\
&\quad \int_{t-\tau}^t B(t)X(t-\tau,s)D^{-\alpha}[f(s)]\mathrm{d}s \\
&= A(t) \int_0^t X(t,s)D^{-\alpha}[f(s)]\mathrm{d}s + B(t) \int_0^{t-\tau} X(t-\tau,s)D^{-\alpha}[f(s)]\mathrm{d}s.
\end{aligned}
$$

From (5.4), we have

$$\int_0^t \Big[{}_s^C D_t^\alpha G(t,s) \Big] \mathrm{d}s = A(t)x(t) + B(t)x(t-\tau). \tag{5.7}$$

On the other hand, noting that ${}^C D^\alpha [D^{-\alpha} f(t)] = f(t)$, $0<\alpha<1$ (see [3]), we derive that

$$
\begin{aligned}
&\frac{1}{\Gamma(1-\alpha)} \int_0^t \lim_{s\to\theta^-} G(\theta,s) \frac{1}{(t-\theta)^\alpha} \mathrm{d}\theta \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^t \lim_{s\to\theta^-} X(\theta,s) D^{-\alpha}[f(s)] \frac{1}{(t-\theta)^\alpha} \mathrm{d}\theta \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^t X(\theta,\theta) D^{-\alpha}[f(\theta)] \frac{1}{(t-\theta)^\alpha} \mathrm{d}\theta \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^t D^{-\alpha}[f(\theta)] \frac{1}{(t-\theta)^\alpha} \mathrm{d}\theta.
\end{aligned}
$$

According to Definition 2.2, we get

$$\frac{1}{\Gamma(1-\alpha)} \int_0^t \lim_{s\to\theta^-} G(\theta,s) \frac{1}{(t-\theta)^\alpha} \mathrm{d}\theta = {}^C D^\alpha \big[D^{-\alpha} f(t) \big] = f(t). \tag{5.8}$$

Combining (5.6) with (5.7) and (5.8), we find that

$$x(t) = \int_0^t X(t,s)D^{-\alpha}[f(s)]\mathrm{d}s \tag{5.9}$$

is the solution of system (5.1). The proof is completed. □

From Lemma 5.1 and Theorem 5.3, we can get the variation formula for time varying Caputo fractional delay differential system (1.2).

Theorem 5.4 *If $X(t,s)$ is the fundamental solution of system (5.1) and $x(t,\varphi,0)$ is a solution of system (5.2), then the general solution $x^*(t,\varphi,f)$ of system (1.2) can be represented in the following form*

$$x^*(t,\varphi,f) = x(t,\varphi,0) + \int_0^t X(t,s)D^{-\alpha}[f(s)]\mathrm{d}s. \tag{5.10}$$

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