Journal of Mathematical Research with Applications
Sept., 2014, Vol. 34, No. 5, pp. 575–586
DOI:10.3770/j.issn:2095-2651.2014.05.008
Http://jmre.dlut.edu.cn

Some Properties of Cauchy-type Singular Integral Operator on Unbounded Domains

Liping WANG, Yuying QIAO*, Weiping ZHANG
College of Mathematics and Information Science, Hebei Normal University, Hebei 050024, P. R. China

Abstract On the basis of introducing the modified Cauchy kernel, we discuss the Hölder continuity of the Cauchy-type singular integral operator on unbounded domains for regular functions by dividing into the following three cases: two points are on the boundary of region; one point is on the boundary and another point is in the interior(or exterior) of the region; two points are in the interior (or exterior) of the region.

Keywords regular function; Cauchy-type singular integral operator; unbounded domains; Hölder continuity.

MR(2010) Subject Classification 47H05; 30G30

1. Introduction

Clifford algebra $\mathcal{A}_n(R)$ was established by Clifford [1] in 1878, which is the extension of the plural, quaternion and exterior algebras. It possesses both theoretical and applicable values in many fields, such as quantum field theory [2], computer graphics [3], neural network [4] and so on. Clifford analysis is an important branch of modern analysis that studies functions defined on R^{n+1} and valued in Clifford algebra space $\mathcal{A}_n(R)$. Since 1970, on the basis of Dirac operator, Brackx, Delanghe, Sommen [5] etc. put forward the regular function, which is an extension of the holomorphic function in higher dimensional space and has been investigated by many scholars such as Xu [6], Wen [7], Huang, Qiao [8, 9] etc. In addition, because many problems in the actual application are proposed in the case of unbounded domains, Klaus Gürlebeck, Uwe Kähler, John Ryan [10] introduced the modified Cauchy kernel in 1997, which makes it possible to study the Cauchy integral on unbounded domains and have obtained a series of results such as the integral representation of the regular function and Plemelj formulae etc.

Cauchy-type integral operator is a singular integral operator and it is the core components of the solution of the boundary value problem. Its localized properties are the theoretical basis of the boundary value problem of the analytic function and generalized analytic function. Furthermore, the latter is closely related with the elasticity mechanics, shell theory and the air

Received July 8, 2013; Accepted February 24, 2014
Supported by the National Natural Science Foundation of China (Grant Nos. 11401162;11301136) and the Science Foundation of Hebei Province (Grant No. A2014208158).
*Corresponding author
E-mail address: wlpxjj@163.com (Liping WANG); yuyingqiao@163.com (Yuying QIAO)

dynamics etc. Thus there are many scholars who have carried a lot of research for it. For example, Lu [11] researched the properties and the corresponding boundary value problem of the Cauchy-type integral operator and Cauchy-type singular integral for density function containing parameter in complex plane. Gilbert [12] studied some properties and applications of Cauchy type integral operators and higher dimensional singular integral operators. Gong [13] discussed the stabilities of Cauchy-type singular integral operator when the boundary curve of integral domain is perturbed. In addition, because Clifford analysis is the extension of complex analysis, studying the properties of all kinds of singular integral operators has been a hot and significant topics in Clifford analysis. Yang [14] and Qiao [15], for example, have studied some properties of some singular integral operators and the corresponding boundary value problem in Quaternion analysis and Clifford analysis, respectively.

On the basis of the above reference, this article studies some properties of the Cauchy-type singular integral operator on unbounded domains. For example, we discuss the Hölder continuity of the Cauchy-type singular integral operator on unbounded domains for regular functions by dividing into the following three cases: two points are on the boundary of region; one point is on the boundary and another point is in the interior (or exterior) of the region; two points are in the interior (or exterior) of the region.

2. Preliminaries

Let $e_0, e_1, \ldots, e_n$ be an orthogonal basis of the Euclidean space R^{n+1} and $\mathcal{A}_n(R)$ be the 2^n-dimensional Clifford algebra with basis $\{e_A : e_A = e_{\alpha_1} \cdots e_{\alpha_h}\}$, where $A = \{\alpha_1, \ldots, \alpha_h\} \subseteq \{1, \ldots, n\}, 1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_h \leq n$ and $e_\emptyset = e_0 = 1$. The associative and noncommutative multiplication of the basis in $\mathcal{A}_n(R)$ is governed by the rules $e_i^2 = -1\,(i = 1, 2, \ldots, n)$, $e_ie_j = -e_je_i\,(1 \leq i, j \leq n, i \neq j)$, $e_0e_i = e_ie_0 = e_i\,(i = 0, 1, \ldots, n)$. Hence the real Clifford algebra is composed of elements having the type $a = \sum_A x_Ae_A$, $x_A \in R$. The norm for an element $a \in \mathcal{A}_n(R)$ is taken to be $|a| = (\sum_A |x_A|^2)^{\frac{1}{2}}$, and satisfies $|\overline{a}| = |a|, |a + b| \leq |a| + |b|, |ab| \leq 2^n|a||b|$.

Let $U \subset R^{n+1}$ be a domain. The function f which is defined in U with values in $\mathcal{A}_n(R)$ can be expressed as $f(x) = \sum_A f_A(x)e_A$, where all $f_A(x)$ are real-valued functions. Let $f(x) \in C^{(r)}(U, \mathcal{A}_n(R)) = \{f | f : U \to \mathcal{A}_n(R),\ f(x) = \sum_A f_A(x)e_A,\ f_A(x) \in C^r(U)\}$. And the Dirac operator is defined as

$$D_l(f) = \sum_{i=0}^{n} e_i \frac{\partial f}{\partial x_i} = \sum_{i=0}^{n} \sum_{A} e_ie_A \frac{\partial f_A}{\partial x_i}, \quad D_r(f) = \sum_{i=0}^{n} \frac{\partial f}{\partial x_i} e_i = \sum_{i=0}^{n} \sum_{A} \frac{\partial f_A}{\partial x_i} e_Ae_i.$$

If $D_l(f) = 0(D_r(f) = 0)$, then f is called a left (right) regular function.

In addition, denoting

$$\mathrm{d}\widehat{x_i} = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{i-1} \wedge \mathrm{d}x_{i+1} \cdots \wedge \mathrm{d}x_n, \quad i = 1, 2, \ldots, n,$$

$$\mathrm{d}\sigma = \sum_{i=1}^{n} (-1)^{i-1} e_i \mathrm{d}\widehat{x_i}, \quad \overrightarrow{n} = \sum_{i=1}^{n} e_in_i,$$

where n_i is i-th component of the unit outward normal vector $\overrightarrow{n}$ and $\mathrm{d}\sigma = \overrightarrow{n}\mathrm{d}s$. $\mathrm{d}s$ is the surface element and $\mathrm{d}x^n = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ is the volume element.

Next let $U \subset R^{n+1}$ be unbounded domains whose complement contains an internal point, and the boundary ∂U be a differentiable, oriented and compact Liapunov surface.

Since ∂U is a Liapunov surface, from the corresponding proof in [8], we have

$$|\mathrm{d}\sigma| = |\mathrm{d}s| = \Big|\frac{D(\xi_0, \xi_1, \dots, \xi_{n-1})}{D(\rho_0, \varphi_1, \dots, \varphi_{n-1})}\Big||\mathrm{d}\rho_0 \mathrm{d}\varphi_1 \mathrm{d}\varphi_2 \cdots \mathrm{d}\varphi_{n-1}| \leq M_0 \rho_0^{n-1} \mathrm{d}\rho_0,$$

where $M_0 > 0$ is a real constant number.

Definition 2.1 *Let $U, \partial U$ be as stated above. The integral*

$$(T_{\partial U}[f])(x) = \frac{1}{\omega_{n+1}} \int_{\partial U} l(y, x) \mathrm{d}\sigma(y) f(y) \tag{2.1}$$

is called Cauchy-type singular integral operator on unbounded domains, where $n(y)$ is the normal vector through y, $l(y, x) = \frac{\overline{y}-\overline{x}}{|y-x|^{n+1}} - \frac{\overline{y}-\overline{z}}{|y-z|^{n+1}}$ is a modified Cauchy kernel, z is a given point in the complement of $\overline{U}$ and ω_{n+1} is the area of the unit sphere in R^{n+1}. When $x \notin \partial U$, it is clear that the integral is well defined. When $x \in \partial U$, it is a singular integral on unbounded domains.

Definition 2.2 *Let $U, \partial U$ be as stated above and $x_0 \in \partial U$. We construct a sphere E with the center at x_0 and radiu $\delta > 0$. Then ∂U is divided into two parts by E and the part of ∂U lying in the interior of E is denoted by λ_δ. Suppose*

$$(T_{\partial U}[f])_\delta(x_0) = \frac{1}{\omega_{n+1}} \int_{\partial U - \lambda_\delta} l(y, x_0) \mathrm{d}\sigma(y) f(y). \tag{2.2}$$

If $\lim_{\delta \to 0}(T_{\partial U}[f])_\delta(x_0) = I(x_0)$, then $I(x_0)$ is called the Cauchy principal value of singular integral $(T_{\partial U}[f])(x_0)$ and we denote that $I(x_0) = (T_{\partial U}[f])(x_0)$.

Remark When $f : \partial U \to \mathcal{A}_n(R)$ is both bounded and Hölder continuous with exponent $\alpha \in (0, 1)$, the above Cauchy principal value of singular integral is well defined for each $x_0 \in \partial U$.

Lemma 2.3 ([10]) *There exists the positive constant $C(n) > 0$, such that*

$$|l(y, x)| \leq C(n)|x - z| \sum_{j=1}^{n} |y - x|^{-j} |y - z|^{j-(n+1)}.$$

Lemma 2.4 ([10]) *Let $U, \partial U$ be as stated above. If $f(x)$ is a bounded, left regular function on U and extends continuously in the L^∞ sense to the boundary of U. Then, we have*

$$\frac{1}{\omega_{n+1}} \int_{\partial U} l(y, x) \mathrm{d}\sigma(y) f(y) = \begin{cases} f(x), & x \in U, \\ 0, & x \in R^{n+1} - \bar{U}. \end{cases}$$

Lemma 2.5 ([10]) *Let $U, \partial U$ be as stated above. Then, in the sense of cauchy principal value, we have*

$$\frac{1}{\omega_{n+1}} \int_{\partial U} l(y, x_0) \mathrm{d}\sigma(y) = \frac{1}{2}, \quad x_0 \in \partial U.$$

Lemma 2.6 ([10]) *Let $U, \partial U$ be as stated above, $f \in H^\alpha_{\partial U}$ $(0 < \alpha < 1)$, f be bounded and $x_0 \in \partial U$. Suppose $(T_{\partial U}[f])^+(x_0) = \lim_{x \to x_0, x \in U^+}(T_{\partial U}[f])(x)$, $(T_{\partial U}[f])^-(x_0) = \lim_{x \to x_0 x \in U^-}(T_{\partial U}[f])(x)$, where U^+, U^- are the interior and exterior of U, respectively. Then, in the sense of cauchy prin-*

cipal value, we have

$$\begin{cases} (T_{\partial U}[f])^+(x_0) = \frac{1}{2}f(x_0) + (T_{\partial U}[f])(x_0), \\ (T_{\partial U}[f])^-(x_0) = -\frac{1}{2}f(x_0) + (T_{\partial U}[f])(x_0). \end{cases}$$

Lemma 2.7 ([16]) *(1) If $\varphi \in H^{\alpha}_{\partial U}, 0 < \beta \le \alpha < 1$, then $\varphi \in H^{\beta}_{\partial U}$; (2) If $f_1(x), f_2(x) \in H^{\alpha}_{\partial U}$, then $f_1(x) \pm f_2(x) \in H^{\alpha}_{\partial U}$; (3) Let $f(x) = \sum_A f_A(x)e_A$. If $f_A(x) \in H^{\alpha}_{\partial U}$, then $f \in H^{\alpha}_{\partial U}, 0 < \alpha < 1$.*

3. The Hölder continuity of the singular integral operator

Theorem 3.1 *Let $U, \partial U, U^-$ and $(T_{\partial U}[f])^-$ be stated as before, $\Omega \subseteq R^{n+1}$ be a bounded domain and $\Omega \cap \partial U \ne \emptyset$. If $f \in H^{\alpha}_{\partial U}(0 < \alpha < 1)$ and f is bounded. Then, in the sense of cauchy principal value, we have*

(1) $(T_{\partial U}[f])^- \in H^{\alpha}_{\partial U\cap\Omega}$, namely, there exists $M_1 > 0$ such that for any $x_1, x_2 \in \partial U \cap \Omega$, $(T_{\partial U}[f])^-$ satisfies

$$|(T_{\partial U}[f])^-(x_1) - (T_{\partial U}[f])^-(x_2)| \le M_1|x_1 - x_2|^{\alpha}.$$

(2) $\|(T_{\partial U}[f])^-\|_{\alpha} \le J\|f\|_{\alpha}$, where J is a constant independent of f.

Proof (1) For any $x_1, x_2 \in \partial U \cap \Omega$, let $|x_1 - x_2| = \delta$. Firstly, we suppose $6\delta < d$, where d is the same as the one in the definition of Liapunov surface. Because the following formula is right

$$\lim_{|y|\to\infty} \frac{|y - x_2|}{|y - z|} = 1,$$

we may construct a sphere E_1 with the center at x_1 and radius 3δ, and construct a sphere E_2 with the center at x_1 and radius R, where R is big enough. The part of ∂U lying in the interior of E_1 is denoted by ∂U_1, the part of ∂U lying between E_1 and E_2 is denoted by ∂U_2 and the part of ∂U lying in the exterior E_2 is denoted by ∂U_3. We take R to be big enough in order to guarantee the following formula is right on ∂U_3.

$$|y - x_2| \le L_1|y - z|, \tag{3.1}$$

where $L_1 > 0$ is an constant number.

In addition, by $|y - x_1| \le |y - x_2| + |x_2 - x_1|, |y - x_2| \le |y - x_1| + |x_1 - x_2|$ and $|y - x_1| > 3\delta (y \in \partial U - \partial U_1)$, we know the following formula is right on both ∂U_2 and ∂U_3.

$$\frac{1}{2} \le \frac{|y - x_2|}{|y - x_1|} \le 2. \tag{3.2}$$

By Lemmas 2.5 and 2.6, we have

$$\begin{aligned}
&|(T_{\partial U}[f])^-(x_1) - (T_{\partial U}[f])^-(x_2)| \\
&= |[(T_{\partial U}[f])(x_1) - \frac{1}{2}f(x_1)] - [(T_{\partial U}[f])(x_2) - \frac{1}{2}f(x_2)]| \\
&\le \frac{1}{\omega_{n+1}}\Big|\int_{\partial U} l(y, x_1)\mathrm{d}\sigma(y)[f(y) - f(x_1)] - \int_{\partial U} l(y, x_2)\mathrm{d}\sigma(y)[f(y) - f(x_2)]\Big| \\
&\le J_1\Big\{\Big|\int_{\partial U_1} l(y, x_1)\mathrm{d}\sigma(y)[f(y) - f(x_1)]\Big| + \Big|\int_{\partial U_1} l(y, x_2)\mathrm{d}\sigma(y)[f(y) - f(x_2)]\Big| +
\end{aligned}$$

$$\begin{aligned}
&\Big|\int_{\partial U_2} l(y,x_1)\mathrm{d}\sigma(y)[f(y)-f(x_1)]-\int_{\partial U_2} l(y,x_2)\mathrm{d}\sigma(y)[f(y)-f(x_2)]\Big|+\\
&\Big|\int_{\partial U_3} l(y,x_1)\mathrm{d}\sigma(y)[f(y)-f(x_1)]-\int_{\partial U_3} l(y,x_2)\mathrm{d}\sigma(y)[f(y)-f(x_2)]\Big|\Big\}\\
\le{}& J_1(I_1+I_2+I_3+I_4). \qquad (3.3)
\end{aligned}$$

Firstly, in the case of $3\delta < 6\delta < d$. Then, because $\frac{y-x_1}{y-z}$ and $\frac{x_1-z}{y-z}$ are continuous on ∂U_1, there exist constants $L_2, L_3 > 0$, such that the following formula is right for any $y \in \partial U_1$.

$$|y-x_1| < L_2|y-z|, \quad |x_1-z| < L_3|y-z|.$$

Again by lemma 2.3, we have

$$\begin{aligned}
|l(y,x_1)| &\le C(n)|x_1-z|\sum_{j=1}^{n}|y-x_1|^{-j}|y-z|^{j-(n+1)}\\
&\le C(n)J_2|y-z|\sum_{j=1}^{n}|y-x_1|^{-j}|y-z|^{j-(n+1)}\\
&= C(n)J_3\sum_{j=1}^{n}|y-x_1|^{-j}|y-z|^{j-n}\\
&\le C(n)J_3\sum_{j=1}^{n}|y-x_1|^{-j}(\frac{1}{L_2})^{j-n}|y-x_1|^{j-n} = J_4|y-x_1|^{-n}.
\end{aligned}$$

Hence, by $f \in H^{\alpha}_{\partial U}$ and local generalized spherical coordinate, we obtain

$$\begin{aligned}
I_1 &= \Big|\int_{\partial U_1} l(y,x_1)\mathrm{d}\sigma(y)[f(y)-f(x_1)]\Big|\\
&\le J_5H(f,\partial U,\alpha)\int_{\partial U_1}|y-x_1|^{-n+\alpha}|\mathrm{d}\sigma(y)|\\
&\le J_6H(f,\partial U,\alpha)\int_0^{3\delta}\rho_0^{\alpha-1}\mathrm{d}\rho_0 = J_7H(f,\partial U,\alpha)|x_1-x_2|^{\alpha}. \qquad (3.4)
\end{aligned}$$

Similarly, we have

$$I_2 \le J_8H(f,\partial U,\alpha)|x_1-x_2|^{\alpha}. \qquad (3.5)$$

Next, we calculate I_3,

$$\begin{aligned}
I_3 =&\Big|\int_{\partial U_2} l(y,x_1)\mathrm{d}\sigma(y)[f(y)-f(x_1)]-\int_{\partial U_2} l(y,x_2)\mathrm{d}\sigma(y)[f(y)-f(x_2)]\Big|\\
=&\Big|\int_{\partial U_2}[l(y,x_1)-l(y,x_2)]\mathrm{d}\sigma(y)[f(y)-f(x_1)]+\int_{\partial U_2} l(y,x_2)\mathrm{d}\sigma(y)[f(x_2)-f(x_1)]\Big|\\
\le&\Big[\Big|\int_{\partial U_2}[\frac{\overline{y}-\overline{x_1}}{|y-x_1|^{n+1}}-\frac{\overline{y}-\overline{x_2}}{|y-x_2|^{n+1}}]\mathrm{d}\sigma(y)[f(y)-f(x_1)]\Big|+\\
&\Big|\int_{\partial U_2}\frac{\overline{y}-\overline{x_2}}{|y-x_2|^{n+1}}\mathrm{d}\sigma(y)[f(x_2)-f(x_1)]\Big|\Big]+\\
&\Big|\int_{\partial U_2}\frac{\overline{y}-\overline{z}}{|y-z|^{n+1}}\mathrm{d}\sigma(y)[f(x_2)-f(x_1)]\Big|\\
=&I_{31}+I_{32}.
\end{aligned}$$

By [17], we have $I_{31} \leq J_9 H(f, \partial U, \alpha)|x_1 - x_2|^\alpha$. In addition, for I_{32}, because the integral has no singularity on bounded domain ∂U_2, it is well defined. There is no harm to suppose $|\int_{\partial U_2} \frac{\overline{y}-\overline{z}}{|y-z|^{n+1}} \mathrm{d}\sigma(y)| = J_{10}$. Thus we have

$$I_{32} \leq J_{10} H(f, \partial U, \alpha)|x_1 - x_2|^\alpha.$$

So

$$I_3 \leq J_{11} H(f, \partial U, \alpha)|x_1 - x_2|^\alpha. \tag{3.6}$$

Finally, we calculate I_4,

$$\begin{aligned} I_4 &= \Big| \int_{\partial U_3} l(y, x_1) \mathrm{d}\sigma(y)[f(y) - f(x_1)] - \int_{\partial U_3} l(y, x_2) \mathrm{d}\sigma(y)[f(y) - f(x_2)] \Big| \\ &\leq \Big| \int_{\partial U_3} [l(y, x_1) - l(y, x_2)] \mathrm{d}\sigma(y)[f(y) - f(x_1)] \Big| + \Big| \int_{\partial U_3} l(y, x_2) \mathrm{d}\sigma(y)[f(x_2) - f(x_1)] \Big| \\ &= I_{41} + I_{42}. \end{aligned}$$

From Hile's lemma [2], we get

$$\begin{aligned} |l(y, x_1) - l(y, x_2)| &= \Big| \frac{\overline{y} - \overline{x_1}}{|y - x_1|^{n+1}} - \frac{\overline{y} - \overline{x_2}}{|y - x_2|^{n+1}} \Big| \\ &\leq \sum_{k=0}^{n-1} \Big| \frac{y - x_1}{y - x_2} \Big|^{-(k+1)} |y - x_2|^{-(n+1)} |x_1 - x_2|. \end{aligned} \tag{3.7}$$

Again by (3.2), we have

$$\begin{aligned} I_{41} &\leq \int_{\partial U_3} \sum_{k=0}^{n-1} 2^{k+1} \Big| \frac{y - x_1}{y - x_2} \Big|^\alpha \frac{1}{|y - x_2|^{n+1-\alpha}} H(f, \partial U, \alpha)|x_1 - x_2| |\mathrm{d}\sigma(y)| \\ &\leq J_{12} H(f, \partial U, \alpha)|x_1 - x_2| \int_R^{+\infty} \rho_0^{\alpha-2} \mathrm{d}\rho_0 \leq J_{13} H(f, \partial U, \alpha)|x_1 - x_2|^\alpha. \end{aligned}$$

For I_{42}, from Lemma 2.3, (3.1), (3.2) and the boundedness of the region Ω, we obtain

$$\begin{aligned} |l(y, x_2)| &\leq C(n)|x_2 - z| \sum_{j=1}^{n} |y - x_2|^{-j} |y - z|^{j-(n+1)} \\ &\leq C(n)|x_2 - z| \sum_{j=1}^{n} |y - x_2|^{-j} (\frac{1}{M_2})^{j-(n+1)} |y - x_2|^{j-(n+1)} \\ &\leq J_{14} |y - x_2|^{-(n+1)} \leq J_{15} |y - x_1|^{-(n+1)}. \end{aligned}$$

Thus

$$\begin{aligned} I_{42} &= \Big| \int_{\partial U_3} l(y, x_2) \mathrm{d}\sigma(y)[f(x_2) - f(x_1)] \Big| \\ &\leq J_{16} H(f, \partial U, \alpha)|x_1 - x_2|^\alpha \int_{\partial U_3} |y - x_1|^{-(n+1)} |\mathrm{d}\sigma(y)| \\ &\leq J_{17} H(f, \partial U, \alpha)|x_1 - x_2|^\alpha \int_R^{+\infty} \rho_0^{-2} \mathrm{d}_{\rho_0} = J_{18} H(f, \partial U, \alpha)|x_1 - x_2|^\alpha. \end{aligned}$$

So

$$I_4 \leq J_{19} H(f, \partial U, \alpha)|x_1 - x_2|^\alpha. \tag{3.8}$$

Hence, by (3.3)–(3.8), we get

$$|(T_{\partial U}[f])^{-}(x_1) - (T_{\partial U}[f])^{-}(x_2)| \le J_{20} H(f, \partial U, \alpha)|x_1 - x_2|^{\alpha} \le J_{21}\|f\|_{\alpha}|x_1 - x_2|^{\alpha}. \tag{3.9}$$

Namely, when $6|x_1 - x_2| < d$, $(T_{\partial U}[f])^{-} \in H^{\alpha}_{\partial U \cap \Omega}$.

Secondly, when $6|x_1 - x_2| \ge d$. By Lemmas 2.5 and 2.6, we know for arbitrary given $x \in \partial U \cap \Omega$,

$$\begin{aligned} |(T_{\partial U}[f])^{-}(x)| &= |(T_{\partial U}[f])(x) - \frac{1}{2}f(x)| \\ &= \frac{1}{\omega_{n+1}}\Big|\int_{\partial U} l(y,x)\mathrm{d}\sigma(y)[f(y) - f(x)]\Big| \\ &\le \frac{1}{\omega_{n+1}} H(f, \partial U, \alpha)\int_{\partial U} |l(y,x)||y - x|^{\alpha}|\mathrm{d}\sigma(y)|. \end{aligned} \tag{3.10}$$

Again

$$\lim_{|y|\to\infty} \frac{|y - x|}{|y - z|} = 1.$$

Hence, there exists a constant $r > 0$ such that the following formula is right for any $y \in \partial U - \lambda_r$.

$$|y - x| \le L_4|y - z|, \tag{3.11}$$

where $\lambda_r = B(x, r) \cap \partial U$ and $L_4 > 0$ is a constant number. Thus

$$\begin{aligned} \int_{\partial U} |l(y,x)||y - x|^{\alpha}|\mathrm{d}\sigma(y)| &= \int_{\lambda_r} |l(y,x)||y - x|^{\alpha}|\mathrm{d}\sigma(y)| + \int_{\partial U - \lambda_r} |l(y,x)||y - x|^{\alpha}|\mathrm{d}\sigma(y)| \\ &= O_1 + O_2. \end{aligned}$$

Again

$$\begin{aligned} O_1 &= \int_{\lambda_r} |l(y,x)||y - x|^{\alpha}|\mathrm{d}\sigma(y)| \\ &\le \int_{\lambda_r} \frac{|\overline{y} - \overline{x}|}{|y - x|^{n+1}}|y - x|^{\alpha}|\mathrm{d}\sigma(y)| + \int_{\lambda_r} \frac{|\overline{y} - \overline{z}|}{|y - z|^{n+1}}|y - x|^{\alpha}|\mathrm{d}\sigma(y)| \\ &= O_{11} + O_{12}. \end{aligned}$$

For O_{11}, by the local generalized spherical coordinate, we know it is convergent. For O_{12}, because it is normal integral on bounded domain, it is convergent. So there is no harm to suppose

$$O_1 \le J_{22}. \tag{3.12}$$

For O_2, by Lemma 2.3, (3.11), the boundedness of the region Ω and the local generalized spherical coordinate, we have

$$\begin{aligned} O_2 &= \int_{\partial U - \lambda_r} |l(y,x)||y - x|^{\alpha}|\mathrm{d}\sigma(y)| \\ &\le \int_{\partial U - \lambda_r} C(n)|x - z|\sum_{j=1}^{n} |y - x|^{-j}|y - z|^{j-(n+1)}|y - x|^{\alpha}|\mathrm{d}\sigma(y)| \\ &\le J_{23}\int_{\partial U - \lambda_r} |y - x|^{\alpha-(n+1)}|\mathrm{d}\sigma(y)| \le J_{24}\int_{\partial U - \lambda_r} \rho_0^{\alpha-2}\mathrm{d}\rho_0 = J_{25}. \end{aligned} \tag{3.13}$$

Thus by (3.10), (3.12) and (3.13), we get

$$|(T_{\partial U}[f])^{-}(x)| \le J_{26} H(f, \partial U, \alpha) \le J_{26}\|f\|_{\alpha}. \tag{3.14}$$

So, when $6|x_1-x_2|\ge d$, we have

$$\begin{aligned}|(T_{\partial U}[f])^-(x_1)-(T_{\partial U}[f])^-(x_2)| &\le |(T_{\partial U}[f])^-(x_1)|+|(T_{\partial U}[f])^-(x_2)|\\ &\le 2J_{26}\|f\|_\alpha 6^\alpha\frac{|x_1-x_2|^\alpha}{\mathrm{d}^\alpha}\le J_{27}\|f\|_\alpha|x_1-x_2|^\alpha.\end{aligned}\tag{3.15}$$

Namely, when $6|x_1-x_2|\ge d$, $(T_{\partial U}[f])^-\in H^\alpha_{\partial U\cap\Omega}$. Hence, we have

$$(T_{\partial U}[f])^-\in H^\alpha_{\partial U\cap\Omega},\quad 0<\alpha<1.$$

(2) From (3.14), we know

$$\max_{x\in\partial U\cap\Omega}|(T_{\partial U}[f])^-(x)|\le J_{26}\|f\|_\alpha.$$

In addition, from (3.9) and (3.15), we get

$$\sup_{x_1,x_2\in\partial U\cap\Omega,x_1\ne x_2}\frac{|(T_{\partial U}[f])^-(x_1)-(T_{\partial U}[f])^-(x_2)|}{|x_1-x_2|^\alpha}\le J_{28}\|f\|_\alpha.$$

So, $\|(T_{\partial U}[f])^-(x)\|_\alpha\le J_{29}||f||_\alpha$. Taking $J=J_{29}$, we get $\|(T_{\partial U}[f])^-(x)\|_\alpha\le J||f||_\alpha$, where J is a constant independent of f. □

Theorem 3.2 *Let $U,\partial U,U^-$ and Ω be stated as above. If $f\in H^\alpha_{\partial U}(0<\alpha<1)$ and f is bounded. Then we can obtain*

(1) $T_{\partial U}[f]\in H^\alpha_{\partial U\cap\Omega}$.

(2) $\|T_{\partial U}[f]\|_\alpha\le J'\|f\|_\alpha$, where J' is a constant independent of f.

Proof From Lemma 2.6, we know

$$(T_{\partial U}[f])(x)=(T_{\partial U}[f])^-(x)+\frac12 f(x),\quad x\in\partial U\cap\Omega.$$

Again by Theorem 3.1, we have $(T_{\partial U}[f])^-\in H^\alpha_{\partial U\cap\Omega}$. Hence, by Lemma 2.7, we get $T_{\partial U}[f]\in H^\alpha_{\partial U\cap\Omega}$.

(2) From Lemma 2.6, we know $|(T_{\partial U}[f])(x)|=|(T_{\partial U}[f])^-(x)+\frac12 f(x)|$. Thus

$$\max_{x\in\partial U\cap\Omega}|(T_{\partial U}[f])(x)|\le\max_{x\in\partial U\cap\Omega}|(T_{\partial U}[f])^-(x)|+\frac12\max_{x\in\partial U}|f(x)|.$$

Again by Lemma 2.6, we have

$$\begin{aligned}&|(T_{\partial U}[f])(x_1)-(T_{\partial U}[f])(x_2)|\\ &\quad=|(T_{\partial U}[f])^-(x_1)+\frac12 f(x_1)-(T_{\partial U}[f])^-(x_2)-\frac12 f(x_2)|\\ &\quad\le|(T_{\partial U}[f])^-(x_1)-(T_{\partial U}[f])^-(x_2)|+\frac12|f(x_1)-f(x_2)|.\end{aligned}$$

So

$$\begin{aligned}&\sup_{x_1,x_2\in\partial U\cap\Omega,x_1\ne x_2}\frac{|(T_{\partial U}[f])(x_1)-(T_{\partial U}[f])(x_2)|}{|x_1-x_2|^\alpha}\\ &\quad\le\sup_{x_1,x_2\in\partial U\cap\Omega,x_1\ne x_2}\frac{|(T_{\partial U}[f])^-(x_1)-(T_{\partial U}[f])^-(x_2)|}{|x_1-x_2|^\alpha}+\frac12\sup_{x_1,x_2\in\partial U,x_1\ne x_2}\frac{|f(x_1)-f(x_2)|}{|x_1-x_2|^\alpha}.\end{aligned}$$

Namely,

$$H(T_{\partial U}[f],\partial U\cap\Omega,\alpha)\le H((T_{\partial U}[f])^-,\partial U\cap\Omega,\alpha)+\frac12 H(f,\partial U,\alpha).$$

So

$$\begin{aligned}\|T_{\partial U}[f]\|_\alpha &\le \|(T_{\partial U}[f])^-\|_\alpha + \frac{1}{2}H(f,\partial U,\alpha)\\ &\le J_{29}\|f\|_\alpha + \frac{1}{2}\|f\|_\alpha = J_{30}\|f\|_\alpha.\end{aligned}$$

Taking $J' = J_{30}$, we can obtain $\|T_{\partial U}[f]\|_\alpha \le J'\|f\|_\alpha$, where J' is a constant independent of f. □

Theorem 3.3 *Let $U, \partial U, U^+, \Omega$ be stated as above. If $f \in H^\alpha_{\partial U}(0<\alpha<1)$ and f is bounded, then for any points $x_0 \in \partial U \cap \Omega$, $x \in U^+ \cap \Omega$, we have*

$$|T_{\partial U}[f](x) - (T_{\partial U}[f])^+(x_0)| \le M_2|x-x_0|^\alpha,$$

where $M_2 > 0$ is a constant independent of x, x_0.

Proof Let $|x-x_0| = \delta$. We suppose $6\delta < d$. Similarly to the theorem 3.1, we construct a sphere E_1 with the center at x_0 and radius 3δ. Then we construct a sphere E_2 with the center at x_0 and radius R, where R is big enough. The part of ∂U lying in the interior of E_1 is denoted by ∂U_1, the part of ∂U lying between E_1 and E_2 is denoted by ∂U_2, and the part of ∂U lying in the exterior E_2 is denoted by ∂U_3. Thus by Lemmas 2.4, 2.5 and 2.6, we have

$$\begin{aligned}&|T_{\partial U}[f](x) - (T_{\partial U}[f])^+(x_0)|\\
&= |T_{\partial U}[f](x) - T_{\partial U}[f](x_0) - \frac{1}{2}f(x_0)|\\
&= \Big|\frac{1}{\omega_{n+1}}\int_{\partial U} l(y,x)\mathrm{d}\sigma(y)f(y) - \frac{1}{\omega_{n+1}}\int_{\partial U} l(y,x_0)\mathrm{d}\sigma(y)f(y) - \frac{1}{2}f(x_0)\Big|\\
&= \Big|\frac{1}{\omega_{n+1}}\int_{\partial U} l(y,x)\mathrm{d}\sigma(y)f(y) - f(x_0) + f(x_0) - \frac{1}{\omega_{n+1}}\int_{\partial U} l(y,x_0)\mathrm{d}\sigma(y)f(y) - \frac{1}{2}f(x_0)\Big|\\
&= \Big|\frac{1}{\omega_{n+1}}\int_{\partial U} l(y,x)\mathrm{d}\sigma(y)(f(y)-f(x_0)) + \frac{1}{2}f(x_0) - \frac{1}{\omega_{n+1}}\int_{\partial U} l(y,x_0)\mathrm{d}\sigma(y)f(y)\Big|\\
&= \Big|\frac{1}{\omega_{n+1}}\int_{\partial U} l(y,x)\mathrm{d}\sigma(y)(f(y)-f(x_0)) + \frac{1}{\omega_{n+1}}\int_{\partial U} l(y,x_0)\mathrm{d}\sigma(y)(f(x_0)-f(y))\Big|\\
&= \Big|\frac{1}{\omega_{n+1}}\int_{\partial U} (l(y,x)-l(y,x_0))\mathrm{d}\sigma(y)(f(y)-f(x_0))\Big|\\
&\le \Big|\frac{1}{\omega_{n+1}}\int_{\partial U_1} (l(y,x)-l(y,x_0))\mathrm{d}\sigma(y)(f(y)-f(x_0))\Big| +\\
&\quad \Big|\frac{1}{\omega_{n+1}}\int_{\partial U_2} (l(y,x)-l(y,x_0))\mathrm{d}\sigma(y)(f(y)-f(x_0))\Big| +\\
&\quad \Big|\frac{1}{\omega_{n+1}}\int_{\partial U_3} (l(y,x)-l(y,x_0))\mathrm{d}\sigma(y)(f(y)-f(x_0))\Big|\\
&= I_5 + I_6 + I_7.\end{aligned}$$

Again, when $y \in \partial U_1$, the following inequalities hold.

$$|y-x| \le L_5|y-z|, |x-z| \le L_5|y-z|, |y-x_0| \le L_6|y-z|, |x_0-z| \le L_6|y-z|,$$

where $L_5 > 0, L_6 > 0$ is a constant number.

And

$$\begin{aligned}
I_5 &= \Big|\frac{1}{\omega_{n+1}}\int_{\partial U_1}(l(y,x)-l(y,x_0))\mathrm{d}\sigma(y)(f(y)-f(x_0))\Big| \\
&\le \Big|\frac{1}{\omega_{n+1}}\int_{\partial U_1}|l(y,x)|\mathrm{d}\sigma(y)|f(y)-f(x_0)\Big| + \Big|\frac{1}{\omega_{n+1}}\int_{\partial U_1}|l(y,x_0)|\mathrm{d}\sigma(y)|f(y)-f(x_0))\Big| \\
&= I_{51}+I_{52}.
\end{aligned}$$

So, similarly to the estimation method of I_1, I_2 in Theorem 3.1, we have

$$I_{51}\le J_{31}|x-x_0|^\alpha,\quad I_{52}\le J_{32}|x-x_0|^\alpha.$$

And similarly to the estimation method of I_{31}, I_{41} in Theorem 3.1, we have

$$I_6\le J_{33}|x-x_0|^\alpha,\quad I_7\le J_{34}|x-x_0|^\alpha.$$

Hence, we obtain

$$|T_{\partial U}[f](x)-(T_{\partial U}[f])^+(x_0)|\le M_2|x-x_0|^\alpha,\quad x\in U^+\cap\Omega, x_0\in\partial U\cap\Omega,$$

where $M_2=J_{31}+J_{32}+J_{33}+J_{34}$.

In addition, the case when $6|x_1-x_2|\ge d$ is similar to the proof of Theorem 3.1. □

Theorem 3.4 *Let $U, \partial U, U^+, \Omega$ be stated as above. If $f\in H^\alpha_{\partial U}(0<\alpha<1)$ and f is bounded, then for any points $x_1, x_2\in U^+\cap\Omega$, we have*

$$|T_{\partial U}[f](x_1)-(T_{\partial U}[f])^+(x_2)|\le M_3|x_1-x_2|^\alpha,$$

where $M_3>0$ is a constant independent of x_1, x_2.

Proof Let $|x_1-x_2|=\delta$. Because $\overline{x_1x_2}$ and $\partial U\cap\Omega$ are compact, there exist a point $\widetilde{x}\in\overline{x_1x_2}$ and a point $\widetilde{y_0}\in\partial U\cap\Omega$ such that

$$\widetilde{x}-\widetilde{y_0}=\inf_{x\in\overline{x_1x_2}, y\in\partial U\cap\Omega}|x-y|.$$

Let $|\widetilde{x}-\widetilde{y_0}|=\delta_0$. Next we discuss $|T_{\partial U}[f](x_1)-T_{\partial U}[f](x_2)|$ in three cases.

(1) If $\delta_0=0$, then $\widetilde{x}=\widetilde{y_0}\in\partial U\cap\Omega$. Thus from Theorem 3.3, we have

$$\begin{aligned}
&|T_{\partial U}[f](x_1)-T_{\partial U}[f](x_2)| \\
&\quad\le|T_{\partial U}[f](x_1)-T_{\partial U}[f](\widetilde{x})|+|T_{\partial U}[f](\widetilde{x})-T_{\partial U}[f](x_2)|\le 2M_2|x_1-x_2|^\alpha.
\end{aligned}$$

(2) If $\delta_0>0$ and $\delta\ge\delta_0$, then we have

$$|x_1-\widetilde{y_0}|\le|x_1-\widetilde{x}|+|\widetilde{x}-\widetilde{y_0}|\le|x_1-x_2|+|\widetilde{x}-\widetilde{y_0}|=\delta+\delta_0\le 2\delta.$$

Similarly, we have $|x_2-\widetilde{y_0}|\le 2\delta$. Thus, by Theorem 3.3, we get

$$\begin{aligned}
|T_{\partial U}[f](x_1)-T_{\partial U}[f](x_2)| &\le|T_{\partial U}[f](x_1)-T_{\partial U}[f](\widetilde{y_0})|+|T_{\partial U}[f](\widetilde{y_0})-T_{\partial U}[f](x_2)| \\
&\le M_2|x_1-\widetilde{y_0}|^\alpha+M_2|\widetilde{y_0}-x_2|^\alpha \\
&\le 2^{\alpha+1}M_2|x_1-x_2|^\alpha.
\end{aligned}$$

(3) If $\delta_0>0$ and $\delta<\delta_0$, then by Lemma 2.4, we have

$$|T_{\partial U}[f](x_1)-T_{\partial U}[f](x_2)|$$

$$
\begin{aligned}
&= |T_{\partial U}[f](x_1) - f(\widetilde{y_0}) + f(\widetilde{y_0}) - T_{\partial U}[f](x_2)| \\
&= \Big|\frac{1}{\omega_{n+1}}\int_{\partial U} l(y,x_1)\mathrm{d}\sigma(y)f(y) - \frac{1}{\omega_{n+1}}\int_{\partial U} l(y,x_1)\mathrm{d}\sigma(y)f(\widetilde{y_0}) + \\
&\quad \frac{1}{\omega_{n+1}}\int_{\partial U} l(y,x_2)\mathrm{d}\sigma(y)f(\widetilde{y_0}) - \frac{1}{\omega_{n+1}}\int_{\partial U} l(y,x_2)\mathrm{d}\sigma(y)f(y)\Big| \\
&= \Big|\frac{1}{\omega_{n+1}}\int_{\partial U} [l(y,x_1) - l(y,x_2)]\mathrm{d}\sigma(y)(f(y) - f(\widetilde{y_0}))\Big|.
\end{aligned}
$$

Similarly to the theorem 3.1, we construct a sphere E_1 with the center at $\widetilde{y_0}$ and radius 3δ. Then we construct a sphere E_2 with the center at $\widetilde{y_0}$ and radius R, where R is big enough. The part of ∂U lying in the interior of E_1 is denoted by ∂U_1, the part of ∂U lying between E_1 and E_2 is denoted by ∂U_2, and the part of ∂U lying in the exterior E_2 is denoted by ∂U_3.

In this case, for any $y \in \partial U$, we have

$$
|y - x_1| \le |y - x_2| + |x_2 - x_1| = |y - x_2| + \delta \le |y - x_2| + \delta_0 \le 2|y - x_2|.
$$

Similarly, we have $|y - x_2| \le 2|y - x_1|$.

Hence, for any $y \in \partial U$, we have

$$
\frac{1}{2} \le \frac{|y - x_1|}{|y - x_2|} \le 2.
$$

In addition, for any $y \in \partial U$, we have

$$
|y - \widetilde{y_0}| \le |y - x_2| + |x_2 - \tilde{x}| + |\tilde{x} - \widetilde{y_0}| = |y - x_2| + \delta + \delta_0 \le 3|y - x_2|.
$$

Thus, by (3.7), $f \in H^{\alpha}_{\partial U}$, $|x_1 - x_2| = \delta < \delta_0 \le |y - x_2|$ and the above inequalities, we have

$$
\begin{aligned}
&\Big|\frac{1}{\omega_{n+1}}\int_{\partial U_1} [l(y,x_1) - l(y,x_2)]\mathrm{d}\sigma(y)(f(y) - f(\widetilde{y_0}))\Big| \\
&\le H(f,\partial U,\alpha)\Big|\frac{1}{\omega_{n+1}}\int_{\partial U_1}\sum_{k=0}^{n-1}\Big|\frac{y - x_1}{y - x_2}\Big|^{-(k+1)}|y - x_2|^{-(n+1)}|x_1 - x_2||y - \widetilde{y_0}|^{\alpha}\mathrm{d}\sigma(y) \\
&\le J_{35}\Big|\frac{1}{\omega_{n+1}}\int_{\partial U_1} |y - x_2|^{-n}|y - \widetilde{y_0}|^{\alpha}\mathrm{d}\sigma(y) \\
&\le J_{36}\Big|\frac{1}{\omega_{n+1}}\int_{\partial U_1} \frac{1}{|y - \widetilde{y_0}|^{n-\alpha}}\mathrm{d}\sigma(y) \\
&\le J_{37}\Big|\frac{1}{\omega_{n+1}}\int_0^{3\delta} \frac{1}{\rho_0^{1-\alpha}}\mathrm{d}\rho_0 \\
&= J_{37}\frac{1}{\alpha}3^{\alpha}\delta^{\alpha} = J_{38}|x_1 - x_2|^{\alpha}.
\end{aligned}
$$

And

$$
\begin{aligned}
&\Big|\frac{1}{\omega_{n+1}}\int_{\partial U_2} [l(y,x_1) - l(y,x_2)]\mathrm{d}\sigma(y)(f(y) - f(\widetilde{y_0}))\Big| \\
&\le H(f,\partial U,\alpha)|\frac{1}{\omega_{n+1}}\int_{\partial U_2}\sum_{k=0}^{n-1}\Big|\frac{y - x_1}{y - x_2}\Big|^{-(k+1)}|y - x_2|^{-(n+1)}|x_1 - x_2||y - \widetilde{y_0}|^{\alpha}\mathrm{d}\sigma(y) \\
&\le J_{38}\Big|\frac{1}{\omega_{n+1}}\int_{\partial U_2} |y - x_2|^{-(n+1)}|y - \widetilde{y_0}|^{\alpha}\mathrm{d}\sigma(y)|x_1 - x_2|
\end{aligned}
$$

$$
\begin{aligned}
&\le J_{39}\Big|\frac{1}{\omega_{n+1}}\int_{\partial U_2}\frac{1}{|y-\widetilde{y_0}|^{n+1-\alpha}}\mathrm{d}\sigma(y)|x_1-x_2| \\
&\le J_{40}\Big|\frac{1}{\omega_{n+1}}\int_{3\delta}^{R}\frac{1}{\rho_0^{2-\alpha}}\mathrm{d}\rho_0|x_1-x_2| \\
&\le J_{41}|x_1-x_2|^{\alpha}.
\end{aligned}
$$

In addition, by the boundedness of f, we have

$$
\begin{aligned}
&\Big|\frac{1}{\omega_{n+1}}\int_{\partial U_3}[l(y,x_1)-l(y,x_2)]\mathrm{d}\sigma(y)(f(y)-f(\widetilde{y_0}))\Big| \\
&\quad\le J_{42}\Big|\frac{1}{\omega_{n+1}}\int_{R}^{+\infty}\frac{1}{\rho_0^{2}}\mathrm{d}\rho_0|x_1-x_2| \\
&\quad\le J_{43}|x_1-x_2|^{\alpha}.
\end{aligned}
$$

Therefore, from the above inequality, we obtain

$$
|T_{\partial U}[f](x_1)-T_{\partial U}[f](x_2)|\le M_3|x_1-x_2|^{\alpha},
$$

where $M_3=J_{38}+J_{41}+J_{43}$. $\square$

References

[1] W. K. CLIFFORD. *Applications of Grassmann's extensive algebra.* Amer. J. Math., 1878, **1**(4): 350–358.

[2] R. P. GILBERT. *First Order Elliptic Systems, A Function Theoretic Approach.* Academic Press, Inc., Orlando, FL, 1983.

[3] Chen XU, Hui LIU, Wenming CAO, et al. *Multispectral image edge detection via Clifford gradient.* Sci. China Inf. Sci., 2012, **55**(2): 260–269.

[4] J. RIVERA-ROVELO, E. BAYRO-CORROCHANO. *Medical Image Segmentation using a self-organizing Neural Network.* 2006, **6**: 3538–3545.

[5] F. BRACKX, R. DELANGHE, F. SOMMEN. *Clifford Analysis.* Pitman, Boston, MA, 1982.

[6] Zhenyuan XU. *On Riemann problem for regular functions with values in Clifford Algebra.* Science Bulletin, 1987, **32**(23): 476–477. (in Chinese)

[7] Guochun WEN. *Clifford Analysis and Elliptic Systems, Hyperbolic Systems of First Order Equations.* World Sci. Publ., Teaneck, NJ, 1991.

[8] Sha HUANG. *Nonlinear boundary value problem for biregular functions in Clifford analysis.* Sci. China Ser. A, 1996, **39**(11): 1152–1163.

[9] Sha HUANG, Yuying QIAO, Guochun WEN. *Real and Complex Clifford Analysis.* Springer, 2005.

[10] K. GÜRLEBECK, U. KÄHLER, J. RYAN. *Clifford analysis over unbounded domains.* Adv. in Appl. Math., 1997, **19**(2): 216–239.

[11] Jianke LU. *Boundary Value Problem of Analytical Function (the Second Edition).* Wuhan University Press, Hubei, 2004. (in Chinese)

[12] R. P. GILBERT, Zongyi HOU, Xiangwu MENG. *Vekua theory in higher-dimensional complex spaces: the Π-operator in C^n.* Complex Variables Theory Appl., 1993, **21**(1-2): 99–105.

[13] Xiaolin WANG, Yafang GONG. *The stabilities of Cauchy kernel integral about the integral surface.* Chin. Ann. Math. Ser. A, 2001, **22**(4): 447–452.

[14] Piwen YANG. *Hölder continuity of $T_G f$ and Riemann-Hilbert boundary value problems in quaternionic analysis.* Acta Math. Sinica (Chin. Ser.), 2003, **46**(5): 993–998. (in Chinese)

[15] Xiaoling LI, Yuying QIAO, Weiwei HU, et al. *Some Properties of the k-regular Functions and Boundary Behavior of the higher order Cauchy-type integral in Clifford analysis.* Adv. Math. (China), 2010, **39**(1): 31–48. (in Chinese)

[16] V. IFTIMIE. *Functions Hypercomplexes.* Bull. Math. de la Soc. Sci. Math. de la R.S., 1965, **9**(57): 279–332.

[17] Liping WANG, Weiling PENG, Zhuofeng XIAO. *The fixed point theorem and iterative approximation of modified Cauchy integral operator of regular function.* J. Math. Res. Exposition, 2008, **28**(3): 593–604.