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Convergence Properties for Arrays of Rowwise φ-Mixing Random Variables

Xiaoqin LI
School of Mathematical Science, Anhui University, Auhui 230039*, P. R. China*

Abstract Convergence properties for arrays of rowwise φ-mixing random variables are studied. As an application, the Chung-type strong law of large numbers for arrays of rowwise φ-mixing random variables is obtained. Our results extend the corresponding ones for independent random variables to the case of φ-mixing random variables.

Keywords φ-mixing random variables; arrays of rowwise φ-mixing random variables; strong law of large numbers.

MR(2010) Subject Classification 60F15

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. We say that the sequence $\{X_n,\ n \geq 1\}$ satisfies the strong law of large numbers if there exist some increasing sequence $\{a_n, n \geq 1\}$ and some sequence $\{c_n, n \geq 1\}$ such that

$$\frac{1}{a_n}\sum_{i=1}^{n}(X_i - c_i) \to 0 \ \text{ a.s. as } n \to \infty.$$

Many authors have extended the strong law of large numbers for sequences of random variables to the case of triangular array of random variables and arrays of rowwise random variables. In the case of independent random variables, Hu and Taylor [1] proved the following strong law of large numbers.

Theorem 1.1 *Let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be a triangular array of rowwise independent random variables. Let $\{a_n, n \geq 1\}$ be a sequence of positive real numbers such that $0 < a_n \uparrow \infty$. Let $g(t)$ be a positive, even function such that $g(|t|)/|t|^p$ is an increasing function of $|t|$ and $g(|t|)/|t|^{p+1}$ is a decreasing function of $|t|$, respectively, that is,*

$$\frac{g(|t|)}{|t|^p}\uparrow, \quad \frac{g(|t|)}{|t|^{p+1}}\downarrow \quad \textit{as} \quad |t| \uparrow$$

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E-mail address: lixiaoqin1983@163.com

for some nonnegative integer p. If $p \geq 2$ and

$$EX_{ni} = 0,$$

$$\sum_{n=1}^{\infty} \sum_{i=1}^{n} E \frac{g(X_{ni})}{g(a_n)} < \infty,$$

$$\sum_{n=1}^{\infty} \Big(\sum_{i=1}^{n} E\big(\frac{X_{ni}}{a_n}\big)^2 \Big)^{2k} < \infty,$$

where k is a positive integer, then

$$\frac{1}{a_n} \sum_{i=1}^{n} X_{ni} \to 0 \quad a.s. \text{ as } n \to \infty. \tag{1.1}$$

Zhu [2] generalized and improved the result of Hu and Taylor [1] for triangular arrays of rowwise independent random variables to the case of arrays of rowwise $\tilde{\rho}$-mixing random variables. Wang et al. [3] generalized and improved the result of Hu and Taylor [1] to negatively associated and linearly negative quadrant dependent random variables. Shen [4] provided some sufficient conditions to prove the strong law of large numbers for arrays of negatively orthant dependent random variables. Shen and Hu [5] obtained some strong law of large numbers for arrays of rowwise $\tilde{\rho}$-mixing random variables under some simple and weak conditions. Inspired by Zhu [2], Shen [4], Shen and Hu [5] and other papers above, we investigate convergence properties for arrays of rowwise φ-mixing random variables. Firstly, let us recall the definitions of sequence of φ-mixing random variables and array of rowwise φ-mixing random variables.

Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. Let n and m be positive integers. Write $\mathcal{F}_n^m = \sigma(X_i, n \leq i \leq m)$. Given σ-algebras $\mathcal{B}, \mathcal{R}$ in $\mathcal{F}$, let

$$\varphi(\mathcal{B}, \mathcal{R}) = \sup_{A \in \mathcal{B}, B \in \mathcal{R}, P(A) > 0} |P(B|A) - P(B)|.$$

Define the φ-mixing coefficients by

$$\varphi(n) = \sup_{k \geq 1} \varphi(\mathcal{F}_1^k, \mathcal{F}_{k+n}^{\infty}), \quad n \geq 0.$$

Definition 1.2 *A sequence $\{X_n, n \geq 1\}$ of random variables is said to be a φ-mixing sequence if $\varphi(n) \downarrow 0$ as $n \to \infty$.*

An array $\{X_{ni}, i \geq 1, n \geq 1\}$ of random variables is called rowwise φ-mixing random variables if for every $n \geq 1$, $\{X_{ni}, i \geq 1\}$ is a sequence of φ-mixing random variables.

φ-mixing random variables were introduced by Dobrushin [6] and many applications have been found. See for example, Dobrushin [6], Utev [7] and Chen [8] for central limit theorem, Herrndorf [9] and Peligrad [10] for weak invariance principle, Utev [11] for weak convergence rate, Sen [12,13] for weak convergence of empirical processes, Peligrad [14] for Ibragimov-Iosifescu conjecture, Shao [15] for almost sure invariance principles, Hu and Wang [16] for large deviations, Wang et al. [17] for Hájek-Rényi-type inequality and the strong law of large numbers for φ-mixing sequence, and so forth.

Our goal in this paper is to study convergence properties for arrays of rowwise φ-mixing random variables. As an application, the Chung-type strong law of large numbers for arrays of rowwise φ-mixing random variables is obtained. We will give some sufficient conditions for the strong law of large numbers for an array of rowwise φ-mixing random variables without assumption of identical distribution. The results presented in this paper are obtained by using the truncated method and the classical maximal type inequality of φ-mixing random variables (Lemma 1.3 below).

Throughout the paper, let $I(A)$ be the indicator function of the set A. C denotes a positive constant which may be different in various places.

The following lemma is useful for the proofs of the main results.

Lemma 1.3 ([17, Lemma 1.7]) *Let $\{X_n, n \geq 1\}$ be a sequence of φ-mixing random variables satisfying $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$. Assume that $EX_n = 0$ and $E|X_n|^q < \infty$ for some $q \geq 2$ and each $n \geq 1$. Then there exists a constant C depending only on q such that*

$$E\Big(\max_{1\leq j\leq n}\Big|\sum_{i=1}^{j} X_i\Big|^q\Big) \leq C\Big\{\sum_{i=1}^{n} E|X_i|^q + \Big(\sum_{i=1}^{n} EX_i^2\Big)^{q/2}\Big\}$$

for every $n \geq 1$.

2. Main results and their proofs

In the section, we assume that $\{X_{ni}, i \geq 1, n \geq 1\}$ is an array of rowwise φ-mixing random variables with mixing coefficients $\{\varphi(n), n \geq 1\}$ in each row. We will provide convergence properties for arrays of rowwise φ-mixing random variables. As an application, the Chung-type strong law of large numbers for arrays of rowwise φ-mixing random variables is obtained.

Theorem 2.1 *Let $\{X_{ni},\ i \geq 1, n \geq 1\}$ be an array of rowwise φ-mixing random variables satisfying $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$ with $EX_{ni} = 0$, $i \geq 1$, $n \geq 1$ and $\{a_n, n \geq 1\}$ be a sequence of positive real numbers. Assume that $\{g_n(t), n \geq 1\}$ is a positive sequence of even functions such that $g_n(|t|)/|t|$ is an increasing function of $|t|$ and $g_n(|t|)/|t|^p$ is a decreasing function of $|t|$ for every $n \geq 1$, respectively, that is*

$$\frac{g_n(|t|)}{|t|}\uparrow,\ \frac{g_n(|t|)}{|t|^p}\downarrow \quad \text{as} \quad |t|\uparrow$$

for some positive constant $p > 1$. If $1 < p \leq 2$, we assume

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n} \frac{Eg_n(X_{ni})}{g_n(a_n)} < \infty, \tag{2.1}$$

then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\Big(\max_{1\leq j\leq n}\Big|\frac{1}{a_n}\sum_{i=1}^{j} X_{ni}\Big| > \varepsilon\Big) < \infty. \tag{2.2}$$

If $p > 2$, we also assume that (2.1) holds and

$$\sum_{n=1}^{\infty}\Big(\sum_{i=1}^{n}E\Big(\frac{X_{ni}}{a_n}\Big)^2\Big)^{v/2} < \infty, \tag{2.3}$$

where v is a positive constant and $v \geq p$. Then for any $\varepsilon > 0$, (2.2) holds. Moreover,

$$\frac{1}{a_n}\sum_{i=1}^{n}X_{ni} \to 0 \ \text{ a.s. as } \ n \to \infty.$$

Proof For fixed $n \geq 1$, define,

$$X_i^{(n)} = X_{ni}I(|X_{ni}| \leq a_n), \quad i \geq 1,$$

$$T_j^{(n)} = \frac{1}{a_n}\sum_{i=1}^{j}\Big(X_i^{(n)} - EX_i^{(n)}\Big), \quad j = 1, 2, \ldots, n.$$

It is easy to check that for any $\varepsilon > 0$,

$$\Big(\max_{1\leq j\leq n}\Big|\frac{1}{a_n}\sum_{i=1}^{j}X_{ni}\Big| > \varepsilon\Big) \subset \Big(\max_{1\leq i\leq n} \mid X_{ni} \mid > a_n\Big) \cup \Big(\max_{1\leq j\leq n}\Big|\frac{1}{a_n}\sum_{i=1}^{j}X_i^{(n)}\Big| > \varepsilon\Big),$$

which implies

$$\begin{aligned} P\Big(\max_{1\leq j\leq n}\Big|\frac{1}{a_n}\sum_{i=1}^{j}X_{ni}\Big| > \varepsilon\Big) &\leq P\Big(\max_{1\leq i\leq n} \mid X_{ni} \mid > a_n\Big) + P\Big(\max_{1\leq j\leq n}\Big|\frac{1}{a_n}\sum_{i=1}^{j}X_i^{(n)}\Big| > \varepsilon\Big) \\ &\leq \sum_{i=1}^{n}P(\mid X_{ni} \mid > a_n) + P\Big(\max_{1\leq j\leq n} \mid T_j^{(n)} \mid > \varepsilon - \max_{1\leq j\leq n}\Big|\frac{1}{a_n}\sum_{i=1}^{j}EX_i^{(n)}\Big|\Big). \end{aligned} \tag{2.4}$$

By conditions $EX_{ni} = 0$, $g_n(|t|)/|t| \uparrow$ as $|t| \uparrow$ and (2.1), we have that

$$\begin{aligned} \max_{1\leq j\leq n}\Big|\frac{1}{a_n}\sum_{i=1}^{j}EX_i^{(n)}\Big| &\leq \sum_{i=1}^{n}\frac{E|X_{ni}|I(|X_{ni}| > a_n)}{a_n} \\ &\leq \sum_{i=1}^{n}\frac{Eg_n(X_{ni})I(|X_{ni}| > a_n)}{g_n(a_n)} \\ &\leq \sum_{i=1}^{n}\frac{Eg_n(X_{ni})}{g_n(a_n)} \to 0 \ \text{ as } \ n \to \infty. \end{aligned}$$

So we have

$$\max_{1\leq j\leq n}\Big|\frac{1}{a_n}\sum_{i=1}^{j}EX_i^{(n)}\Big| \to 0 \ \text{ as } \ n \to \infty. \tag{2.5}$$

Obviously, combining (2.4) with (2.5), we obtain that for n large enough,

$$\sum_{n=1}^{\infty}P\Big(\max_{1\leq j\leq n}\Big|\frac{1}{a_n}\sum_{i=1}^{j}X_{ni}\Big| > \varepsilon\Big) \leq \sum_{n=1}^{\infty}\sum_{i=1}^{n}P(\mid X_{ni} \mid > a_n) + \sum_{n=1}^{\infty}P\Big(\max_{1\leq j\leq n}\Big|T_j^{(n)}\Big| > \frac{\varepsilon}{2}\Big). \tag{2.6}$$

Consequently, by (2.6), in order to prove (2.2), we need to show

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n}P(|X_{ni}| > a_n) < \infty \tag{2.7}$$

and

$$\sum_{n=1}^{\infty} P\Big(\max_{1\le j\le n} |T_j^{(n)}| > \frac{\varepsilon}{2}\Big) < \infty. \tag{2.8}$$

By $g_n(|t|)/|t| \uparrow$ as $|t| \uparrow$, Markov's inequality and (2.1), it follows that

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{i=1}^{n} P(|X_{ni}| > a_n) &\le \sum_{n=1}^{\infty}\sum_{i=1}^{n} P\Big(g_n(X_{ni}) > g_n(a_n)\Big)\\ &\le \sum_{n=1}^{\infty}\sum_{i=1}^{n} \frac{Eg_n(X_{ni})}{g_n(a_n)} < \infty,\end{aligned}$$

which implies (2.7).

Meanwhile, it is a fact that $\{\frac{1}{a_n}(X_i^{(n)} - EX_i^{(n)}), 1 \le i \le j\}$ is a sequence of φ-mixing random variables with same mixing coefficients. Next we prove that (2.8) holds for the case $p > 2$ and $1 < p \le 2$. For the case $p > 2$, by $v \ge p$ and $g_n(|t|)/|t|^p \downarrow$ as $|t| \uparrow$, it follows that $g_n(|t|)/|t|^v \downarrow$ as $|t| \uparrow$. So, we get by Markov's inequality, C_r's inequality, Lemma 1.3, (2.1) and (2.3) that

$$\begin{aligned}&\sum_{n=1}^{\infty} P\Big(\max_{1\le j\le n} \Big|T_j^{(n)}\Big| > \frac{\varepsilon}{2}\Big) \le C\sum_{n=1}^{\infty} E\Big(\max_{1\le j\le n} |T_j^{(n)}|^v\Big)\\ &\le C\sum_{n=1}^{\infty}\frac{1}{a_n^v}\Big\{\sum_{j=1}^{n} E\Big|X_j^{(n)} - EX_j^{(n)}\Big|^v + \Big(\sum_{j=1}^{n} E\Big|X_j^{(n)} - EX_j^{(n)}\Big|^2\Big)^{v/2}\Big\}\\ &\le C\sum_{n=1}^{\infty}\frac{1}{a_n^v}\Big\{\sum_{j=1}^{n} E\Big|X_j^{(n)}\Big|^v + \Big(\sum_{j=1}^{n} E\Big|X_j^{(n)}\Big|^2\Big)^{v/2}\Big\}\\ &\le C\sum_{n=1}^{\infty}\frac{1}{a_n^v}\sum_{j=1}^{n} E|X_{nj}|^v I(|X_{nj}| \le a_n) + C\sum_{n=1}^{\infty}\frac{1}{a_n^v}\Big(\sum_{j=1}^{n} E\Big|X_j^{(n)}\Big|^2\Big)^{v/2}\\ &\le C\sum_{n=1}^{\infty}\sum_{j=1}^{n}\frac{Eg_n(X_{nj})}{g_n(a_n)} + C\sum_{n=1}^{\infty}\Big(\sum_{j=1}^{n} E\Big(\frac{X_{nj}}{a_n}\Big)^2\Big)^{v/2} < \infty.\end{aligned}$$

Hence (2.8) holds for the case $p > 2$.

If $1 < p \le 2$, by $g_n(t)/|t|^p \downarrow$ as $|t| \uparrow$, Markov's inequality, Lemma 1.3, and (2.1), it follows

$$\begin{aligned}\sum_{n=1}^{\infty} P\Big(\max_{1\le j\le n} \Big|T_j^{(n)}\Big| > \frac{\varepsilon}{2}\Big) &\le C\sum_{n=1}^{\infty} E\Big(\max_{1\le j\le n} |T_j^{(n)}|^2\Big)\\ &\le C\sum_{n=1}^{\infty}\frac{1}{a_n^2}\sum_{j=1}^{n} E\Big|X_j^{(n)}\Big|^2\\ &\le C\sum_{n=1}^{\infty}\frac{1}{a_n^2}\sum_{j=1}^{n} E|X_{nj}|^2 I(|X_{nj}| \le a_n)\\ &\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{Eg_n(X_{ni})}{g_n(a_n)} < \infty.\end{aligned}$$

So (2.8) holds for the case $1 < p \le 2$. Consequently, (2.8) holds for the case $p > 2$ and $1 < p \le 2$. By (2.6)–(2.8), we obtain (2.2) immediately.

On the other hand, by Borel-Cantelli Lemma, it follows that

$$\max_{1\le j\le n}\Big|\frac{1}{a_n}\sum_{i=1}^{j}X_{ni}\Big|\to 0 \ \text{ a.s. as } \ n\to\infty.$$

So (1.1) holds.

Theorem 2.2 *Let $\{X_{ni},\ i\ge 1, n\ge 1\}$ be an array of rowwise φ-mixing random variables satisfying $\sum_{n=1}^{\infty}\varphi^{1/2}(n)<\infty$ and $\{a_n,\ n\ge 1\}$ be a sequence of positive real numbers. Let $\{g_n(t), n\ge 1\}$ be a nonnegative sequence of even functions such that $g_n(|t|)$ is an increasing function of $|t|$ for every $n\ge 1$. Assume that there exists a constant $\alpha>0$ such that $g_n(t)\ge \alpha t$ for $0<t\le 1$. If*

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n}Eg_n\Big(\frac{X_{ni}}{a_n}\Big)<\infty, \tag{2.9}$$

then for any $\varepsilon>0$, (2.2) holds.

Proof We use the same notation as that in Theorem 2.1. By the proof of (2.6) in Theorem 2.1, we have to show that (2.5), (2.7) and (2.8) hold under the conditions of Theorem 2.2. Firstly, by the conditions that $g_n(t)\ge\alpha t$ for $0<t\le 1$ and (2.9), we have that

$$\begin{aligned}\max_{1\le j\le n}\Big|\frac{1}{a_n}\sum_{i=1}^{j}EX_i^{(n)}\Big| &\le \sum_{i=1}^{n}E\Big(\frac{|X_{ni}|}{a_n}I(|X_{ni}|\le a_n)\Big)\\ &\le \frac{1}{\alpha}\sum_{i=1}^{n}Eg_n\Big(\frac{X_{ni}}{a_n}\Big)\to 0 \ \text{ as } \ n\to\infty.\end{aligned}$$

So (2.5) holds.

Secondly, for $|X_{ni}|>a_n>0$, we have $g_n\big(\frac{X_{ni}}{a_n}\big)\ge g_n(1)\ge\alpha$. By $g_n(|t|)\uparrow$ as $|t|\uparrow$, Markov's inequality and (2.9), we can get that

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{i=1}^{n}P(|X_{ni}|>a_n) &\le \sum_{n=1}^{\infty}\sum_{i=1}^{n}P\Big(g_n\big(\frac{X_{ni}}{a_n}\big)\ge\alpha\Big)\\ &\le \frac{1}{\alpha}\sum_{n=1}^{\infty}\sum_{i=1}^{n}Eg_n\Big(\frac{X_{ni}}{a_n}\Big)<\infty.\end{aligned}$$

Hence (2.7) holds.

On the other hand, by Markov's inequality, Lemma 1.3 with $q=2$, $g_n(t)\ge\alpha t$ for $0<t\le 1$ and (2.9), we get that

$$\begin{aligned}\sum_{n=1}^{\infty}P\Big(\max_{1\le j\le n}\Big|T_j^{(n)}\Big|>\frac{\varepsilon}{2}\Big) &\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{EX_{ni}^2I(|X_{ni}|\le a_n)}{a_n^2}\\ &\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E|X_{ni}|I(|X_{ni}|\le a_n)}{a_n}\\ &\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n}Eg_n\Big(\frac{X_{ni}}{a_n}\Big)<\infty.\end{aligned}$$

Then we obtain (2.8). By (2.5), (2.7) and (2.8), we have (2.2) finally.

Corollary 2.3 *Let $\{X_{ni},\ i \geq 1, n \geq 1\}$ be an array of rowwise φ-mixing random variables satisfying $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$ and $\{a_n,\ n \geq 1\}$ be a sequence of positive real numbers. If there exists a constant $\beta \in (0,1]$ such that*

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n} E\Big(\frac{|X_{ni}|^\beta}{a_n^\beta + |X_{ni}|^\beta}\Big) < \infty,$$

then (2.2) holds.

Proof In Theorem 2.2, we take

$$g_n(t) = \frac{|t|^\beta}{1+|t|^\beta},\ \ 0 < \beta \leq 1,\ n \geq 1.$$

It is easy to check that $\{g_n(t), n \geq 1\}$ is a sequence of nonnegative, even functions such that $g_n(|t|)$ is an increasing function of $|t|$ for every $n \geq 1$, and

$$g_n(t) \geq \frac{1}{2}t^\beta \geq \frac{1}{2}t,\ \ 0 < t \leq 1,\ 0 < \beta \leq 1.$$

Therefore, we apply Theorem 2.2 and get (2.2) immediately. □

Theorem 2.4 *Let $\{X_{ni},\ i \geq 1, n \geq 1\}$ be an array of rowwise φ-mixing random variables satisfying $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$ and $\{a_n,\ n \geq 1\}$ be a sequence of positive real numbers. $EX_{ni} = 0,\ i \geq 1, n \geq 1$. Let $\{g_n(t), n \geq 1\}$ be a sequence of nonnegative and even functions. Assume that there exist $\beta \in [1,2]$ and $\alpha > 0$ such that $g_n(x) \geq \alpha x^\beta$ for $0 < x \leq 1$ and there exists $\alpha > 0$ such that $g_n(x) \geq \alpha x$ for $x > 1$. If (2.9) is satisfied, then for any $\varepsilon > 0$, (2.2) holds.*

Proof We also use the same notation as that in Theorem 2.1. By the proof of (2.6) in Theorem 2.1, we have to show that (2.5), (2.7) and (2.8) hold under the conditions of Theorem 2.4. Firstly, by the conditions that $EX_{ni} = 0$, $g_n(x) \geq \alpha x$ for $x > 1$ and (2.9), it follows

$$\begin{aligned}\max_{1\leq j\leq n}\Big|\frac{1}{a_n}\sum_{i=1}^{j} EX_i^{(n)}\Big| &\leq \frac{1}{a_n}\sum_{i=1}^{n} E(|X_{ni}|I(|X_{ni}| > a_n))\\ &\leq \frac{1}{\alpha}\sum_{i=1}^{n} Eg_n\Big(\frac{X_{ni}}{a_n}\Big)I(|X_{ni}| > a_n)\\ &\leq \frac{1}{\alpha}\sum_{i=1}^{n} Eg_n\left(\frac{X_{ni}}{a_n}\right) \to 0 \ \text{ as } \ n \to \infty\end{aligned}$$

which implies (2.5).

Obviously, the conditions $g_n(x) \geq \alpha x$ for $x > 1$ and (2.9) yield that

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{i=1}^{n} P(|X_{ni}| > a_n) &\leq \sum_{n=1}^{\infty}\sum_{i=1}^{n} E\Big(\frac{|X_{ni}|}{a_n}I(|X_{ni}| > a_n)\Big)\\ &\leq \frac{1}{\alpha}\sum_{n=1}^{\infty}\sum_{i=1}^{n} Eg_n\Big(\frac{X_{ni}}{a_n}\Big)I(|X_{ni}| > a_n)\\ &\leq \frac{1}{\alpha}\sum_{n=1}^{\infty}\sum_{i=1}^{n} Eg_n\Big(\frac{X_{ni}}{a_n}\Big) < \infty,\end{aligned}$$

which implies (2.7).

Meanwhile, by Markov's inequality, Lemma 1.3 with $q = 2$, $g_n(x) \geq \alpha x^\beta$ for $1 \leq \beta \leq 2$, $0 < x \leq 1$ and (2.9), it follows that

$$\begin{aligned}\sum_{n=1}^\infty P\Big(\max_{1\leq j\leq n}\Big|T_j^{(n)}\Big| > \frac{\varepsilon}{2}\Big) &\leq C\sum_{n=1}^\infty\sum_{i=1}^n \frac{EX_{ni}^2 I(|X_{ni}|\leq a_n)}{a_n^2}\\ &\leq C\sum_{n=1}^\infty\sum_{i=1}^n \frac{E|X_{ni}|^\beta I(|X_{ni}|\leq a_n)}{a_n^\beta}\\ &\leq C\sum_{n=1}^\infty\sum_{i=1}^n Eg_n\Big(\frac{X_{ni}}{a_n}\Big)I(|X_{ni}|\leq a_n)\\ &\leq C\sum_{n=1}^\infty\sum_{i=1}^n Eg_n\Big(\frac{X_{ni}}{a_n}\Big) < \infty,\end{aligned}$$

which implies (2.8). This completes the proof of the theorem. □

Corollary 2.5 *Let $\{X_{ni},\ i \geq 1, n \geq 1\}$ be an array of rowwise φ-mixing random variables satisfying $\sum_{n=1}^\infty \varphi^{1/2}(n) < \infty$ and $\{a_n,\ n \geq 1\}$ be a sequence of positive real numbers. $EX_{ni} = 0,\ i \geq 1,\ n \geq 1$. If there exists a constant $\beta \in (1, 2]$ such that*

$$\sum_{n=1}^\infty\sum_{i=1}^n E\Big(\frac{|X_{ni}|^\beta}{a_n|X_{ni}|^{\beta-1} + a_n^\beta}\Big) < \infty,$$

then (2.2) holds.

Proof In Theorem 2.4, we take

$$g_n(t) = \frac{|t|^\beta}{1 + |t|^{\beta-1}},\quad 1 < \beta \leq 2,\ n \geq 1.$$

It is easy to check that $\{g_n(t), n \geq 1\}$ is a sequence of nonnegative, even functions such that $g_n(|t|)$ is an increasing function of $|t|$ for every $n \geq 1$. And

$$g_n(x) \geq \frac{1}{2}x^\beta,\ \ 0 < x \leq 1,\ 1 < \beta \leq 2\ \text{ and }\ g_n(x) \geq \frac{1}{2}x,\ x > 1.$$

Therefore, by Theorem 2.4, we can easily get (2.2). □

On the other hand, by Corollaries 2.3 and 2.5, we get the following important Chung-type strong law of large numbers for arrays of rowwise φ-mixing random variables.

Corollary 2.6 *Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise φ-mixing random variables satisfying $\sum_{n=1}^\infty \varphi^{1/2}(n) < \infty$ and $\{a_n, n \geq 1\}$ be a sequence of positive real numbers. Assume that there exists some $\beta \in (0, 2]$ such that*

$$\sum_{n=1}^\infty\sum_{i=1}^n \frac{E|X_{ni}|^\beta}{a_n^\beta} < \infty.$$

Let $EX_{ni} = 0$, $i \geq 1$, $n \geq 1$. I f $\beta \in (1, 2]$, then (2.2) holds. Furthermore, $\frac{1}{a_n}\sum_{i=1}^n X_{ni} \to 0$ a.s.

Theorem 2.7 *Let $\{X_{ni},\ i \geq 1, n \geq 1\}$ be an array of rowwise φ-mixing random variables* *satisfying* $\sum_{n=1}^{\infty}\varphi^{1/2}(n) < \infty$ *and* $\{a_n,\ n \geq 1\}$ *be a sequence of positive real numbers. Let* $\{g_n(t), n \geq 1\}$ *be a nonnegative sequence of even functions. Assume that there exists an* $\alpha > 0$ *such that* $g_n(x) \geq \alpha x$ *for* $x > 0$*. If (2.9) satisfies, then for any* $\varepsilon > 0$*, (2.2) holds.*

Proof We also use the same notation as that in Theorem 2.1. By the conditions that $g_n(x) \geq \alpha x$ for $x > 0$ and (2.9), we have

$$\begin{aligned}\max_{1\leq j\leq n}\Big|\frac{1}{a_n}\sum_{i=1}^{j}EX_i^{(n)}\Big| &\leq \frac{1}{a_n}\sum_{i=1}^{n}E|X_{ni}|I(|X_{ni}|\leq a_n)\\ &\leq \frac{1}{\alpha}\sum_{i=1}^{n}Eg_n\Big(\frac{X_{ni}}{a_n}\Big)\to 0 \quad\text{as}\quad n\to\infty,\end{aligned}$$

which implies (2.5).

Obviously, the conditions $g_n(x) \geq \alpha x$ for $x > 0$ and (2.9) yield that

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{i=1}^{n}P(|X_{ni}|>a_n) &\leq \sum_{n=1}^{\infty}\sum_{i=1}^{n}E\Big(\frac{|X_{ni}|}{a_n}I(|X_{ni}|>a_n)\Big)\\ &\leq \frac{1}{\alpha}\sum_{n=1}^{\infty}\sum_{i=1}^{n}Eg_n\Big(\frac{X_{ni}}{a_n}\Big)<\infty,\end{aligned}$$

which implies (2.7).

Meanwhile, by Markov's inequality, Lemma 1.3 with $q = 2$, $g_n(x) \geq \alpha x$ for $x > 0$ and (2.9), it can be checked that

$$\begin{aligned}\sum_{n=1}^{\infty}P\Big(\max_{1\leq j\leq n}\Big|T_j^{(n)}\Big|>\frac{\varepsilon}{2}\Big) &\leq C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E|X_{ni}|^2I(|X_{ni}|\leq a_n)}{a_n^2}\\ &\leq C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E|X_{ni}|I(|X_{ni}|\leq a_n)}{a_n}\\ &\leq C\sum_{n=1}^{\infty}\sum_{i=1}^{n}Eg_n\Big(\frac{X_{ni}}{a_n}\Big)I(|X_{ni}|\leq a_n)\\ &\leq C\sum_{n=1}^{\infty}\sum_{i=1}^{n}Eg_n\Big(\frac{X_{ni}}{a_n}\Big)<\infty,\end{aligned}$$

which implies (2.8). This completes the proof of the theorem. $\Box$

Theorem 2.8 *Let* $\{X_{ni},\ i \geq 1, n \geq 1\}$ *be an array of rowwise* φ*-mixing random variables satisfying* $\sum_{n=1}^{\infty}\varphi^{1/2}(n) < \infty$ *and* $\{a_n,\ n \geq 1\}$ *be a sequence of positive real numbers. Let* $\{g_n(t), n \geq 1\}$ *be a nonnegative sequence of even functions. Assume that there exist* $\beta \in [2,\infty)$ *and* $\alpha > 0$ *such that* $g_n(x) \geq \alpha x^{\beta}$ *for* $x > 0$*. If*

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n}\Big(Eg_n\Big(\frac{X_{ni}}{a_n}\Big)\Big)^{1/\beta}<\infty, \tag{2.10}$$

then for any $\varepsilon > 0$*, (2.2) holds.*

Proof We use the same notation as that in Theorem 2.1. One can see that (2.10) implies that

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n} Eg_n\Big(\frac{X_{ni}}{a_n}\Big) < \infty \tag{2.11}$$

and

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n} \Big(Eg_n\Big(\frac{X_{ni}}{a_n}\Big)\Big)^{2/\beta} < \infty. \tag{2.12}$$

Firstly, by Hölder's inequality, $g_n(x) \geq \alpha x^\beta$ for $\beta \geq 2$, $x > 0$, and (2.10), it follows

$$\begin{aligned}
\max_{1\leq j\leq n}\Big|\frac{1}{a_n}\sum_{i=1}^{j} EX_i^{(n)}\Big| &\leq \frac{1}{a_n}\sum_{i=1}^{n} E|X_{ni}|I(|X_{ni}|\leq a_n)\\
&\leq \sum_{i=1}^{n}\Big(E\Big(\frac{|X_{ni}|^\beta}{a_n^\beta}I(|X_{ni}|\leq a_n)\Big)\Big)^{1/\beta}\\
&\leq C\sum_{i=1}^{n}\Big(Eg_n\Big(\frac{X_{ni}}{a_n}\Big)I(|X_{ni}|\leq a_n)\Big)^{1/\beta}\\
&\leq C\sum_{i=1}^{n}\Big(Eg_n\Big(\frac{X_{ni}}{a_n}\Big)\Big)^{1/\beta} \to 0 \ \text{ as } \ n\to\infty,
\end{aligned}$$

which implies (2.5).

Secondly, by the conditions $g_n(x) \geq \alpha x^\beta$ for $\beta \geq 2$, $x > 0$ and (2.11), it can be seen that

$$\begin{aligned}
\sum_{n=1}^{\infty}\sum_{i=1}^{n} P(|X_{ni}| > a_n) &\leq \sum_{n=1}^{\infty}\sum_{i=1}^{n} E\Big(\frac{|X_{ni}|^\beta}{a_n{}^\beta}I(|X_{ni}| > a_n)\Big)\\
&\leq \frac{1}{\alpha}\sum_{n=1}^{\infty}\sum_{i=1}^{n} Eg_n\Big(\frac{X_{ni}}{a_n}\Big) < \infty,
\end{aligned}$$

which implies (2.7).

Meanwhile, by Markov's inequality, Lemma 1.3 with $q = 2$ and Hölder's inequality, $g_n(x) \geq \alpha x^\beta$ for $\beta \geq 2$, $x > 0$ and (2.12), we obtain that

$$\begin{aligned}
\sum_{n=1}^{\infty} P\Big(\max_{1\leq j\leq n}\Big|T_j^{(n)}\Big| > \frac{\varepsilon}{2}\Big) &\leq C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E|X_{ni}|^2 I(|X_{ni}|\leq a_n)}{a_n^2}\\
&\leq C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\Big(E\frac{|X_{ni}|^\beta}{a_n^\beta}I(|X_{ni}|\leq a_n)\Big)^{2/\beta}\\
&\leq C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\Big(Eg_n\Big(\frac{X_{ni}}{a_n}\Big)I(|X_{ni}|\leq a_n)\Big)^{2/\beta}\\
&\leq C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\Big(Eg_n\Big(\frac{X_{ni}}{a_n}\Big)\Big)^{2/\beta} < \infty,
\end{aligned}$$

which implies (2.8). Therefore, by (2.5), (2.7) and (2.8), we obtain (2.2) finally. □

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