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A Note on (α,β,γ)-Superderivations of Superalgebras

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Abstract This paper is concerned with two results for (α,β,γ)-superderivations of general superalgebras over the field of complex numbers.

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1. Introduction

The search for a new concept of invariant characteristics of Lie algebras led to the definition of (α,β,γ)-derivations which has been studied in connection with degeneration theory of algebras [1, 2]. Recently, twisted cocycles of Lie algebras corresponding to (α,β,γ)-derivations were introduced in [3] and (α,β,γ)-derivations of complex simple Lie algebras were determined by Burde and Dekimpe in [4]. We generalized the definition of (α,β,γ)-derivations to super-versions and investigated some properties of (α,β,γ)-superderivations for finite dimensional complex Lie superalgebras in detail [5]. The aim of this paper is to give two results with respect to (α,β,γ)-superderivations of general superalgebras and the original motivation comes from the researches of Zusmanovich [6].

Throughout this paper we will assume that $\mathbb{C}$ is the field of complex numbers and $\mathbb{Z}_2=\{\bar{0},\bar{1}\}$ is the residue class ring modulo 2.

Let $A=A_{\bar{0}}\oplus A_{\bar{1}}$ be a finite dimensional superalgebra over $\mathbb{C}$. Without being mentioned explicitly, if $\deg(a)$ occurs in some expression in this paper, we always regard a as a $\mathbb{Z}_2$-homogeneous element and $\deg(a)$ as the $\mathbb{Z}_2$-degree of a. The standard Lie super-commutator and Jordan super-product will be written as usual by $[a,b]=ab-(-1)^{\deg(a)\deg(b)}ba$ and by $a\circ b=2^{-1}(ab+(-1)^{\deg(a)\deg(b)}ba)$ for all $a,b\in A$, respectively.

One may argue that the more natural approach would be to generalize the corresponding supernotion. Recall that a $\mathbb{Z}_2$-homogeneous linear map $D:A\to A$ is called a superderivation of a superalgebra $A=A_{\bar{0}}\oplus A_{\bar{1}}$ if

$$D(ab)=D(a)b+(-1)^{\deg(D)\deg(a)}aD(b)$$

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for any two $\mathbb{Z}_2$-homogeneous elements $a, b \in A$.

The supercentroid of A is the set of all $\mathbb{Z}_2$-homogeneous linear maps $\chi : A \to A$ such that

$$\chi(ab) = \chi(a)b = (-1)^{\deg(\chi)\deg(a)} a\chi(b)$$

for any two $\mathbb{Z}_2$-homogeneous elements $a, b \in A$.

Accordingly, a $\mathbb{Z}_2$-homogeneous linear map D is called an (α, β, γ)-superderivation of A if

$$\alpha D(ab) = \beta D(a)b + (-1)^{\deg(D)\deg(x)} \gamma x D(y)$$

for any two $\mathbb{Z}_2$-homogeneous elements $a, b \in A$. Like in the ordinary case, this generalizes superderivations (for $\alpha = \beta = \gamma = 1$), elements of supercentroid (for $\alpha = \beta$ and $\gamma = 0$, for $\alpha = \gamma$ and $\beta = 0$ or for $\frac{\beta}{\alpha} = \frac{\gamma}{\alpha} = \frac{1}{2}$) and elements of δ-superderivations (for $\frac{\beta}{\alpha} = \frac{\gamma}{\alpha} = \delta$).

2. Main results and proofs

Let $\alpha, \alpha', \beta, \beta', \gamma, \gamma'$ be elements of the complex field $\mathbb{C}$.

Theorem 2.1 *Let A be a superalgebra. If D is an (α, β, γ)-superderivation of A and D' is an $(\alpha', \beta', \gamma')$-superderivation of A, then $[D, D']$ and $D \circ D'$ are $(\alpha\alpha', \beta\beta', \gamma\gamma')$-superderivations of A.*

Proof Let x and y be arbitrary elements of A. Then

$$\begin{aligned}
\alpha\alpha'[D, D'](xy) &= \alpha\alpha' DD'(xy) - (-1)^{\deg(D)\deg(D')} \alpha\alpha' D'D(xy) \\
&= \alpha\beta' D(D'(x)y) + (-1)^{\deg(D)\deg(x)} \alpha\gamma' D(xD'(y)) - \\
&\quad (-1)^{\deg(D)\deg(D')} \alpha'\beta D'(D(x)y) - (-1)^{\deg(D)(\deg(D')+\deg(x))} \alpha'\gamma D'(xD(y)) \\
&= \beta\beta' DD'(x)y + (-1)^{\deg(D)(\deg(D')+\deg(x))} \beta'\gamma D'(x)D(y) + \\
&\quad (-1)^{\deg(D')\deg(x)} \beta\gamma' D(x)D'(y) + (-1)^{\deg(x)(\deg(D')+\deg(D))} \gamma\gamma' xDD'(y) - \\
&\quad (-1)^{\deg(D)\deg(D')} \beta\beta' D'D(x)y - (-1)^{\deg(D')\deg(x)} \beta\gamma' D(x)D'(y) - \\
&\quad (-1)^{\deg(D)(\deg(D')+\deg(x))} \beta'\gamma D'(x)D(y) - \\
&\quad (-1)^{\deg(D)\deg(D')+\deg(x)(\deg(D')+\deg(D))} \gamma\gamma' xD'D(y) \\
&= \beta\beta'[D, D'](x)y + (-1)^{\deg(x)(\deg(D')+\deg(D))} \gamma\gamma' x[D, D'](y).
\end{aligned}$$

Therefore, $[D, D']$ is an $(\alpha\alpha', \beta\beta', \gamma\gamma')$-superderivation of A. By the similar method, $D \circ D'$ is also an $(\alpha\alpha', \beta\beta', \gamma\gamma')$-superderivation of A.

Theorem 2.2 *Let A and B be two superalgebras.*

(i) If D is an (α, β, γ)-superderivation of A, then the map $\widehat{D} : (A \otimes B)_{\bar{0}} \to A \otimes B$ defined as

$$a_{\bar{0}} \otimes b_{\bar{0}} + a_{\bar{1}} \otimes b_{\bar{1}} \mapsto D(a_{\bar{0}}) \otimes b_{\bar{0}} + D(a_{\bar{1}}) \otimes b_{\bar{1}}$$

for $a_{\bar{0}} \in A_{\bar{0}}$, $a_{\bar{1}} \in A_{\bar{1}}$, $b_{\bar{0}} \in B_{\bar{0}}$, $b_{\bar{1}} \in B_{\bar{1}}$, is an (α, β, γ)-derivation of $(A \otimes B)_{\bar{0}}$ with values in $A \otimes B$.

(ii) If D is an (α, β, γ)-superderivation of A and χ is an element of the supercentroid of B such that $\deg(D) = \deg(\chi)$, then the map $\widehat{D} : (A \otimes B)_{\bar{0}} \to (A \otimes B)_{\bar{0}}$ defined as

$$a_{\bar{0}} \otimes b_{\bar{0}} + a_{\bar{1}} \otimes b_{\bar{1}} \mapsto D(a_{\bar{0}}) \otimes \chi(b_{\bar{0}}) + (-1)^{\deg(D)} D(a_{\bar{1}}) \otimes \chi(b_{\bar{1}})$$

for $a_{\bar{0}} \in A_{\bar{0}}$, $a_{\bar{1}} \in A_{\bar{1}}$, $b_{\bar{0}} \in B_{\bar{0}}$, $b_{\bar{1}} \in B_{\bar{1}}$, is an (α, β, γ)-derivation of $(A \otimes B)_{\bar{0}}$.

Proof (i) A direct verification shows that

$$\begin{aligned}
&\alpha\widehat{D}\left((a_{\bar{0}} \otimes b_{\bar{0}} + a_{\bar{1}} \otimes b_{\bar{1}})(c_{\bar{0}} \otimes d_{\bar{0}} + c_{\bar{1}} \otimes d_{\bar{1}})\right) \\
&\quad = \alpha\widehat{D}\left(a_{\bar{0}}c_{\bar{0}} \otimes b_{\bar{0}}d_{\bar{0}} + a_{\bar{0}}c_{\bar{1}} \otimes b_{\bar{0}}d_{\bar{1}} + a_{\bar{1}}c_{\bar{0}} \otimes b_{\bar{1}}d_{\bar{0}} - a_{\bar{1}}c_{\bar{1}} \otimes b_{\bar{1}}d_{\bar{1}}\right) \\
&\quad = \alpha D(a_{\bar{0}}c_{\bar{0}}) \otimes b_{\bar{0}}d_{\bar{0}} + \alpha D(a_{\bar{0}}c_{\bar{1}}) \otimes b_{\bar{0}}d_{\bar{1}} + \\
&\qquad (-1)^{\deg(D)}\alpha D(a_{\bar{1}}c_{\bar{0}}) \otimes b_{\bar{1}}d_{\bar{0}} - \alpha D(a_{\bar{1}}c_{\bar{1}}) \otimes b_{\bar{1}}d_{\bar{1}} \\
&\quad = \beta D(a_{\bar{0}})c_{\bar{0}} \otimes b_{\bar{0}}d_{\bar{0}} + \gamma a_{\bar{0}}D(c_{\bar{0}}) \otimes b_{\bar{0}}d_{\bar{0}} + \\
&\qquad \beta D(a_{\bar{0}})c_{\bar{1}} \otimes b_{\bar{0}}d_{\bar{1}} + \gamma a_{\bar{0}}D(c_{\bar{1}}) \otimes b_{\bar{0}}d_{\bar{1}} + \\
&\qquad \beta D(a_{\bar{1}})c_{\bar{0}} \otimes b_{\bar{1}}d_{\bar{0}} + (-1)^{\deg(D)}\gamma a_{\bar{1}}D(c_{\bar{0}}) \otimes b_{\bar{1}}d_{\bar{0}} - \\
&\qquad \beta D(a_{\bar{1}})c_{\bar{1}} \otimes b_{\bar{1}}d_{\bar{1}} - (-1)^{\deg(D)}\gamma a_{\bar{1}}D(c_{\bar{1}}) \otimes b_{\bar{1}}d_{\bar{1}} \\
&\quad = \beta((D(a_{\bar{0}}) \otimes b_{\bar{0}})(c_{\bar{0}} \otimes d_{\bar{0}}) + (D(a_{\bar{0}}) \otimes b_{\bar{0}})(c_{\bar{1}} \otimes d_{\bar{1}}) + \\
&\qquad (D(a_{\bar{1}}) \otimes b_{\bar{1}})(c_{\bar{0}} \otimes d_{\bar{0}}) + (D(a_{\bar{1}}) \otimes b_{\bar{1}})(c_{\bar{1}} \otimes d_{\bar{1}})) + \\
&\qquad \gamma((a_{\bar{0}} \otimes b_{\bar{0}})(D(c_{\bar{0}}) \otimes d_{\bar{0}}) + (a_{\bar{0}} \otimes b_{\bar{0}})(D(c_{\bar{1}}) \otimes d_{\bar{1}}) + \\
&\qquad (a_{\bar{1}} \otimes b_{\bar{1}})(D(c_{\bar{0}}) \otimes d_{\bar{0}}) + (a_{\bar{1}} \otimes b_{\bar{1}})(D(c_{\bar{1}}) \otimes d_{\bar{1}})) \\
&\quad = \beta(D(a_{\bar{0}}) \otimes \chi(b_{\bar{0}}) + D(a_{\bar{1}}) \otimes \chi(b_{\bar{1}}))(c_{\bar{0}} \otimes d_{\bar{0}} + c_{\bar{1}} \otimes d_{\bar{1}}) + \\
&\qquad \gamma(a_{\bar{0}} \otimes b_{\bar{0}} + a_{\bar{1}} \otimes b_{\bar{1}})(D(c_{\bar{0}}) \otimes \chi(d_{\bar{0}}) + D(c_{\bar{1}}) \otimes \chi(d_{\bar{1}})) \\
&\quad = \beta\widehat{D}\left(a_{\bar{0}} \otimes b_{\bar{0}} + a_{\bar{1}} \otimes b_{\bar{1}}\right)\left(c_{\bar{0}} \otimes d_{\bar{0}} + c_{\bar{1}} \otimes d_{\bar{1}}\right) + \\
&\qquad \gamma(a_{\bar{0}} \otimes b_{\bar{0}} + a_{\bar{1}} \otimes b_{\bar{1}})\widehat{D}\left(c_{\bar{0}} \otimes d_{\bar{0}} + c_{\bar{1}} \otimes d_{\bar{1}}\right)
\end{aligned}$$

for all $c_{\bar{0}} \in A_{\bar{0}}$, $c_{\bar{1}} \in A_{\bar{1}}$, $d_{\bar{0}} \in B_{\bar{0}}$, $d_{\bar{1}} \in B_{\bar{1}}$. Thus, the desired result is obtained.

(ii) For all $c_{\bar{0}} \in A_{\bar{0}}$, $c_{\bar{1}} \in A_{\bar{1}}$, $d_{\bar{0}} \in B_{\bar{0}}$, $d_{\bar{1}} \in B_{\bar{1}}$, we have

$$\begin{aligned}
&\alpha\widehat{D}\left((a_{\bar{0}} \otimes b_{\bar{0}} + a_{\bar{1}} \otimes b_{\bar{1}})(c_{\bar{0}} \otimes d_{\bar{0}} + c_{\bar{1}} \otimes d_{\bar{1}})\right) \\
&\quad = \alpha\widehat{D}\left(a_{\bar{0}}c_{\bar{0}} \otimes b_{\bar{0}}d_{\bar{0}} + a_{\bar{0}}c_{\bar{1}} \otimes b_{\bar{0}}d_{\bar{1}} + a_{\bar{1}}c_{\bar{0}} \otimes b_{\bar{1}}d_{\bar{0}} - a_{\bar{1}}c_{\bar{1}} \otimes b_{\bar{1}}d_{\bar{1}}\right) \\
&\quad = \alpha D(a_{\bar{0}}c_{\bar{0}}) \otimes \chi(b_{\bar{0}}d_{\bar{0}}) + (-1)^{\deg(D)}\alpha D(a_{\bar{0}}c_{\bar{1}}) \otimes \chi(b_{\bar{0}}d_{\bar{1}}) + \\
&\qquad (-1)^{\deg(D)}\alpha D(a_{\bar{1}}c_{\bar{0}}) \otimes \chi(b_{\bar{1}}d_{\bar{0}}) - \alpha D(a_{\bar{1}}c_{\bar{1}}) \otimes \chi(b_{\bar{1}}d_{\bar{1}}) \\
&\quad = \beta D(a_{\bar{0}})c_{\bar{0}} \otimes \chi(b_{\bar{0}})d_{\bar{0}} + \gamma a_{\bar{0}}D(c_{\bar{0}}) \otimes \chi(b_{\bar{0}})d_{\bar{0}} + \\
&\qquad (-1)^{\deg(D)}\beta D(a_{\bar{0}})c_{\bar{1}} \otimes \chi(b_{\bar{0}})d_{\bar{1}} + (-1)^{\deg(D)}\gamma a_{\bar{0}}D(c_{\bar{1}}) \otimes \chi(b_{\bar{0}})d_{\bar{1}} + \\
&\qquad (-1)^{\deg(D)}\beta D(a_{\bar{1}})c_{\bar{0}} \otimes \chi(b_{\bar{1}})d_{\bar{0}} + \gamma a_{\bar{1}}D(c_{\bar{0}}) \otimes \chi(b_{\bar{1}})d_{\bar{0}} - \\
&\qquad \beta D(a_{\bar{1}})c_{\bar{1}} \otimes \chi(b_{\bar{1}})d_{\bar{1}} - (-1)^{\deg(D)}\gamma a_{\bar{1}}D(c_{\bar{1}}) \otimes \chi(b_{\bar{1}})d_{\bar{1}}.
\end{aligned}$$

On the other side,

$$\begin{aligned}
&\beta\widehat{D}\,(a_{\bar 0}\otimes b_{\bar 0}+a_{\bar 1}\otimes b_{\bar 1})\,(c_{\bar 0}\otimes d_{\bar 0}+c_{\bar 1}\otimes d_{\bar 1})+\\
&\quad\gamma(a_{\bar 0}\otimes b_{\bar 0}+a_{\bar 1}\otimes b_{\bar 1})\widehat{D}\,(c_{\bar 0}\otimes d_{\bar 0}+c_{\bar 1}\otimes d_{\bar 1})\\
&\ =\beta(D(a_{\bar 0})\otimes\chi(b_{\bar 0})+(-1)^{\deg(D)}D(a_{\bar 1})\otimes\chi(b_{\bar 1}))(c_{\bar 0}\otimes d_{\bar 0}+c_{\bar 1}\otimes d_{\bar 1})+\\
&\quad\gamma(a_{\bar 0}\otimes b_{\bar 0}+a_{\bar 1}\otimes b_{\bar 1})(D(c_{\bar 0})\otimes\chi(d_{\bar 0})+(-1)^{\deg(D)}D(c_{\bar 1})\otimes\chi(d_{\bar 1}))\\
&\ =\beta(D(a_{\bar 0})c_{\bar 0}\otimes\chi(b_{\bar 0})d_{\bar 0}+(-1)^{\deg(D)}D(a_{\bar 0})c_{\bar 1}\otimes\chi(b_{\bar 0})d_{\bar 1}+\\
&\quad(-1)^{\deg(D)}D(a_{\bar 1})c_{\bar 0}\otimes\chi(b_{\bar 1})d_{\bar 0}-D(a_{\bar 1})c_{\bar 1}\otimes\chi(b_{\bar 1})d_{\bar 1})+\\
&\quad\gamma(a_{\bar 0}D(c_{\bar 0})\otimes b_{\bar 0}\chi(d_{\bar 0})+(-1)^{\deg(D)}a_{\bar 0}D(c_{\bar 1})\otimes b_{\bar 0}\chi(d_{\bar 1})+\\
&\quad(-1)^{\deg(D)}a_{\bar 1}D(c_{\bar 0})\otimes b_{\bar 1}\chi(d_{\bar 0})-a_{\bar 1}D(c_{\bar 1})\otimes b_{\bar 1}\chi(d_{\bar 1}))\\
&\ =\beta D(a_{\bar 0})c_{\bar 0}\otimes\chi(b_{\bar 0})d_{\bar 0}+(-1)^{\deg(D)}\beta D(a_{\bar 0})c_{\bar 1}\otimes\chi(b_{\bar 0})d_{\bar 1}+\\
&\quad(-1)^{\deg(D)}\beta D(a_{\bar 1})c_{\bar 0}\otimes\chi(b_{\bar 1})d_{\bar 0}-\beta D(a_{\bar 1})c_{\bar 1}\otimes\chi(b_{\bar 1})d_{\bar 1}+\\
&\quad\gamma a_{\bar 0}D(c_{\bar 0})\otimes\chi(b_{\bar 0})d_{\bar 0}+(-1)^{\deg(D)}\gamma a_{\bar 0}D(c_{\bar 1})\otimes\chi(b_{\bar 0})d_{\bar 1}+\\
&\quad\gamma a_{\bar 1}D(c_{\bar 0})\otimes\chi(b_{\bar 1})d_{\bar 0}-(-1)^{\deg(D)}\gamma a_{\bar 1}D(c_{\bar 1})\otimes\chi(b_{\bar 1})d_{\bar 1}.
\end{aligned}$$

Therefore, $\widehat{D}$ is an (α,β,γ)-derivation of $(A\otimes B)_{\bar 0}$.

Remark 2.3 Theorem 2.2 is a generalization of Lemma 4.2 in [6]. But there is an error in [6, Lemma 4.2 (ii)]. Recently, Prof. Pasha Zusmanovich has given an erratum [7] to correct this error.

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