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Coquasitriangular Weak Hopf Group Algebras and Braided Monoidal Categories

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Abstract In this paper, we first give the definitions of a crossed left π-H-comodules over a crossed weak Hopf π-algebra H, and show that the category of crossed left π-H-comodules is a monoidal category. Finally, we show that a family $\sigma = \{\sigma_{\alpha,\beta} : H_\alpha \otimes H_\beta \to k\}_{\alpha,\beta\in\pi}$ of k-linear maps is a coquasitriangular structure of a crossed weak Hopf π-algebra H if and only if the category of crossed left π-H-comodules over H is a braided monoidal category with braiding defined by σ.

Keywords π-H-comodules; braided monoidal category; coquasitriangular structure.

MR(2010) Subject Classification 16T05

1. Introduction

The notion of a quasitriangular Hopf algebra was introduced by Drinfel'd [2] when he studied the Yang-Baxter equation. Because of their close connections with varied, a priori remote areas of mathematics and physics, this theory has got fast development and many fundamental achievements, see, for example, [5]. Recently, Turaev [7] introduced a Hopf π-coalgebra, which generalizes the notion of a Hopf algebra. Van Daele and Wang studied algebraic properties of weak Hopf group coalgebras and generalized many of the properties of quasitriangular weak Hopf algebras in [1] to the setting of quasitriangular weak Hopf group coalgebras in [8]. Wang also investigated properties of coquasitriangular Hopf group algebras in [9].

In this paper, we give the definitions of a crossed left π-H-comodules over a crossed weak Hopf π-algebra H, and show that the categories of crossed left π-H-comodules is a monoidal category. Finally, we show that a family $\sigma = \{\sigma_{\alpha,\beta} : H_\alpha \otimes H_\beta \to k\}_{\alpha,\beta\in\pi}$ is a coquasitriangular structure of a crossed weak Hopf π-algebra H if and only if the category of crossed left π-H-comodules over H is a braided monoidal category with braiding defined by σ.

2. Preliminaries

Throughout the paper, we let π be a discrete group (with neutral element 1) and k be a

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fixed field. All algebras and coalgebras, π-algebras, and Hopf π-algebras are defined over k. The definitions and properties of algebras, coalgebras, Hopf algebras and categories can be found in [3, 4, 6]. We use the standard Sweedler notation for comultiplication. The tensor product $\otimes = \otimes_k$ is always assumed to be over k. The following definitions and notations in this section can be found in [9].

2.1. π-algebras

A π-algebra is a family $H = \{H_\alpha\}_{\alpha\in\pi}$ of k-spaces together with a family of k-linear maps $m = \{m_{\alpha,\beta} : H_\alpha \otimes H_\beta \longrightarrow H_{\alpha\beta}\}_{\alpha,\beta\in\pi}$ (called a multiplication) and a k-linear map $\eta : k \longrightarrow H_1$ (called a unit), such that m is associative in the sense that, for any $\alpha, \beta, \gamma \in \pi$,

$$m_{\alpha\beta,\gamma}(m_{\alpha,\beta} \otimes \mathrm{id}_{H_\gamma}) = m_{\alpha,\beta\gamma}(\mathrm{id}_{H_\alpha} \otimes m_{\beta,\gamma}),$$
$$m_{\alpha,1}(\mathrm{id}_{H_\alpha} \otimes \eta) = \mathrm{id}_{H_\alpha} = m_{1,\alpha}(\eta \otimes \mathrm{id}_{H_\alpha}).$$

2.2. Hopf π-algebras

A Hopf π-algebra H is a family $\{(H_\alpha, \Delta_\alpha, \varepsilon_\alpha)\}_{\alpha\in\pi}$ of k-coalgebras, here H_α is called the αth component of H, endowed with the following data.

• A family of k-linear maps $m = \{m_{\alpha,\beta} : H_\alpha \otimes H_\beta \longrightarrow H_{\alpha\beta}\}_{\alpha,\beta\in\pi}$, called multiplication, that is associative, in the sense that, for any $\alpha, \beta, \gamma \in \pi$,

$$m_{\alpha\beta,\gamma}(m_{\alpha,\beta} \otimes \mathrm{id}_\gamma) = m_{\alpha,\beta\gamma}(\mathrm{id}_\alpha \otimes m_{\beta,\gamma}). \tag{2.1}$$
$$m_{\alpha,1}(\mathrm{id}_{H_\alpha} \otimes \eta) = \mathrm{id}_{H_\alpha} = m_{1,\alpha}(\eta \otimes \mathrm{id}_{H_\alpha}). \tag{2.2}$$

Given $h \in H_\alpha$ and $g \in H_\beta$, with $\alpha, \beta \in \pi$, we set $hg = m_{\alpha,\beta}(h \otimes g)$. With this notation, Eq. (2.1) can be simply rewritten as $(hg)l = h(gl)$ for any $h \in H_\alpha, g \in H_\beta, l \in H_\gamma$ and $\alpha, \beta, \gamma \in \pi$.

• The map $m_{\alpha,\beta} : H_\alpha \otimes H_\beta \longrightarrow H_{\alpha\beta}$ is a morphism of coalgebras such that

$$\Delta_{\alpha\beta} m_{\alpha,\beta} = (m_\alpha \otimes m_\beta)\Delta_{\alpha\beta}, \tag{2.3}$$
$$(\varepsilon_\alpha \otimes \xi_\beta) = \xi_{\alpha\beta} m_{\alpha,\beta}, \tag{2.4}$$

where we used Sweedler's notation: $\Delta_\beta(g) = g_{(1,\beta)} \otimes g_{(2,\beta)}$ for any $h \in H_\alpha, g \in H_\beta, l \in H_\gamma$ and $\alpha, \beta, \gamma \in \pi$.

• A set of k-linear maps $S = \{S_\alpha : H_\alpha \longrightarrow H_{\alpha^{-1}}\}_{\alpha\in\pi}$, the antipode, such that,

$$m_{\alpha^{-1},\alpha}(S_\alpha \otimes \mathrm{id}_{H_\alpha})\Delta_\alpha = \varepsilon_\alpha 1_1 = m_{\alpha,\alpha^{-1}}(\mathrm{id}_{H_\alpha} \otimes S_\alpha)\Delta_\alpha, \tag{2.5}$$

for any $h \in H_\alpha$ and $\alpha \in \pi$.

Furthermore, the Hopf π-algebra H is called crossed if the following condition holds: There exists a family of coalgebra isomorphisms $\xi = \{\xi_\beta : H_\alpha \longrightarrow H_{\beta\alpha\beta^{-1}}\}$, called conjugation, such that

– ξ is multiplicative, i.e., for any α, β and $\gamma \in \pi$, one has $\xi_\beta \xi_\gamma = \xi_{\beta\gamma} : H_\alpha \longrightarrow H_{(\beta\gamma)\alpha(\beta\gamma)^{-1}}$, in particular, $\xi_1 | H_\alpha = id_\alpha$.

– ξ is compatible with m, i.e., for any $\beta \in \pi$, we have $\xi_\beta(hg) = \xi_\beta(h)\psi_\beta(g)$.

– ξ is compatible with 1, i.e., for any $\beta \in \pi$, we have $\xi_\beta(1) = 1$.

– ξ preserves the antipode, i.e., $\xi_\beta S_\alpha = S_{\beta\alpha\beta^{-1}}\xi_\beta$.

The weak Hopf π-algebra H is said to be of finite type if, for all $\alpha \in \pi$, H_α is finite-dimensional as k-space. Note that it does not mean that $\bigoplus_{\alpha\in\pi} H_\alpha$ is finite dimensional (unless $H_\alpha = 0$ for all but a finite number of $\alpha \in \pi$). Hence, in this case the dual of weak Hopf π-algebra is not a weak Hopf π-coalgebra. The antipode $S = \{S_\alpha\}_{\alpha\in\pi}$ of H is called bijective if each S_α is bijective.

2.3. Left π-H-comodules

Assume that $H = \{H_\alpha\}_{\alpha\in G}$ is a family of coalgebras. A left H-π-comodule over H is a family $M = \{M_\alpha\}_{\alpha\in\pi}$ of k-spaces such that M_α is a left H_α-comodule for any $\alpha \in \pi$. We denote the structure maps of left H_α-comodule M_α and left π-H-comodule M by $\rho^{M_\alpha} : M_\alpha \to H_\alpha \otimes M_\alpha$ and $\rho^M = \{\rho^{M_\alpha}\}_{\alpha\in\pi}$, respectively.

We use the Sweedler's notation in the following way; for $m \in M_\alpha$, we write

$$\rho^{M_\alpha}(m) = m_{(-1,\alpha)} \otimes m_{(0,\alpha)}.$$

2.4. Left π-H-comodule maps

Assume that $H = \{H_\alpha\}_{\alpha\in G}$ is a family of coalgebras. Let $M = \{M_\alpha\}_{\alpha\in\pi}$, $N = \{N_\alpha\}_{\alpha\in\pi}$ be two left π-comodules over H. A left π-H-comodule map $f : M \to N$ is a family $f = \{f_\alpha : M_\alpha \to N_\alpha\}_{\alpha\in\pi}$ of k-linear maps such that $\rho^{N_\alpha} f_\alpha = (\mathrm{id}_{H_\alpha} \otimes f_\alpha)\rho^{M_\alpha}$ for all $\alpha \in \pi$.

3. Weak Hopf π-algebras

In this section, we mainly study some structure properties of weak Hopf π-algebras.

Definition 3.1 *A weak Hopf π-algebra H is a family $\{(H_\alpha, \Delta_\alpha, \varepsilon_\alpha)\}_{\alpha\in\pi}$ of k-coalgebras, here H_α is called the αth component of H, endowed with the following data.*

• *A family of k-linear maps $m = \{m_{\alpha,\beta} : H_\alpha \otimes H_\beta \longrightarrow H_{\alpha\beta}\}_{\alpha,\beta\in\pi}$, called multiplication, that is associative, in the sense that, for any $\alpha, \beta, \gamma \in \pi$,*

$$m_{\alpha\beta,\gamma}(m_{\alpha,\beta} \otimes \mathrm{id}_\gamma) = m_{\alpha,\beta\gamma}(\mathrm{id}_\alpha \otimes m_{\beta,\gamma}). \tag{3.1}$$

Given $h \in H_\alpha$ and $g \in H_\beta$, with $\alpha, \beta \in \pi$, we set $hg = m_{\alpha,\beta}(h \otimes g)$. With this notation, Eq. (3.1) can be simply rewritten as $(hg)l = h(gl)$ for any $h \in H_\alpha, g \in H_\beta, l \in H_\gamma$ and $\alpha, \beta, \gamma \in \pi$.

• *The map $m_{\alpha,\beta} : H_\alpha \otimes H_\beta \longrightarrow H_{\alpha\beta}$ is a (not necessary counit-preserving) morphism of coalgebras such that*

$$\varepsilon_{\alpha\beta\gamma}(hgl) = \varepsilon_{\alpha\beta}(hg_{(1,\beta)})\varepsilon_{\beta\gamma}(g_{(2,\beta)}l) = \varepsilon_{\alpha\beta}(hg_{(2,\beta)})\varepsilon_{\beta\gamma}(g_{(1,\beta)}l) \tag{3.2}$$

where we used Sweedler's notation: $\Delta_\beta(g) = g_{(1,\beta)} \otimes g_{(2,\beta)}$ for any $h \in H_\alpha, g \in H_\beta, l \in H_\gamma$ and $\alpha, \beta, \gamma \in \pi$.

• *An algebra morphism $\eta : k \longrightarrow H_1$, called unit, such that, if we set $1 = \eta(1_k)$, then,*

$$1h = h = h1, \quad \textit{for any} \quad h \in H_\alpha \quad \textit{with} \quad \alpha \in \pi, \tag{3.3}$$

$$(\Delta_1 \otimes \mathrm{id})\Delta_1(1,1) = 1_{(1,1)} \otimes 1_{(2,1)}1'_{(1,1)} \otimes 1'_{(2,1)} = 1_{(1,1)} \otimes 1'_{(1,1)}1_{(2,1)} \otimes 1'_{(2,1)} \tag{3.4}$$

where $1 = 1'$.

- *A set of k-linear maps $S = \{S_\alpha : H_\alpha \longrightarrow H_{\alpha^{-1}}\}_{\alpha\in\pi}$, the antipode, such that,*

$$m_{\alpha^{-1},\alpha}(S_\alpha \otimes \mathrm{id}_\alpha)\Delta_\alpha(h) = 1_{(1,\alpha^{-1})}\varepsilon_\alpha(h1_{(2,\alpha)}), \tag{3.5}$$

$$m_{\alpha,\alpha^{-1}}(\mathrm{id}_\alpha \otimes S_\alpha)\Delta_\alpha(h) = \varepsilon_\alpha(1_{(1,\alpha)}h)1_{(2,\alpha^{-1})}, \tag{3.6}$$

$$S_\alpha(h_{(1,\alpha)})h_{(2,\alpha^{-1})}S_\alpha(h_{(3,\alpha)}) = S_\alpha(h) \tag{3.7}$$

for any $h \in H_\alpha$ and $\alpha \in \pi$.

Definition 3.2 *A weak Hopf π-algebra H is called crossed if the following condition holds: There exists a family of coalgebra isomorphisms $\xi = \{\xi_\beta : H_\alpha \longrightarrow H_{\beta\alpha\beta^{-1}}\}$, called conjugation, such that*

– ξ is multiplicative, i.e., for any α, β and $\gamma \in \pi$, one has $\xi_\beta\xi_\gamma = \xi_{\beta\gamma} : H_\alpha \longrightarrow H_{(\beta\gamma)\alpha(\beta\gamma)^{-1}}$, in particular, $\xi_1|H_\alpha = \mathrm{id}_\alpha$.

– ξ is compatible with m, i.e., for any $\beta \in \pi$, we have $\xi_\beta(hg) = \xi_\beta(h)\xi_\beta(g)$.

– ξ is compatible with 1, i.e., for any $\beta \in \pi$, we have $\xi_\beta(1) = 1$.

Example 3.3 Recall that a finite groupoid G is a category, in which every morphism is an isomorphism, with a finite number of objects. The set of objects of G will be denoted by G_0, and the set of morphisms by G_1. The identity morphism on $x \in G_0$ will also be denoted by x. The source and target maps will be denoted by s and t respectively, i.e., for $\alpha : x \longrightarrow y$ in G_1, we have $s(\alpha) = x$ and $t(\alpha) = y$. For every $x \in G$, $G_x = \{\alpha \in G | s(\alpha) = t(\alpha) = x\}$ is a group.

Let G be a groupoid. The groupoid algebra is the direct product $k[G] = \bigoplus_{\alpha\in G_1} ku_\alpha$, with multiplication defined by the rule $u_\alpha u_\beta = u_{\alpha\beta}$ if $s(\alpha) = t(\beta)$ and $u_\alpha u_\beta = 0$ if $s(\alpha) \neq t(\beta)$. The unit is $1 = \sum_{x\in G_0} u_x$. $k[G]$ is a weak Hopf algebra, with comultiplication, counit and antipode given by the formulas

$$\Delta(u_\alpha) = u_\alpha \otimes u_\alpha, \quad \varepsilon(u_\alpha) = 1 \text{ and } S(u_\alpha) = u_{\alpha^{-1}}.$$

Using $\Delta(1) = \bigoplus_{x\in G_0} u_x \otimes u_x$, we have that $\varepsilon^t : kG \longrightarrow kG$ is given by $\varepsilon^t(u_\alpha) = \sum_{x\in G_0} \varepsilon(u_x u_\alpha) = u_{t(\alpha)}$. Similarly, we have that $\varepsilon^s : kG \longrightarrow kG$ is given by $\varepsilon^s(u_\alpha) = \sum_{x\in G_0} \varepsilon(u_\alpha u_x) = u_{s(\alpha)}$.

The dual of kG is the weak Hopf algebra $k(G) = k^G$ of functions $G \longrightarrow k$. It has a basis $(e_g : G \longrightarrow k)_{g\in G_1}$ defined by $\langle e_g, h\rangle = \delta_{g,h}$. That is, as a k-space we have $k[G] = \sum_{g\in G_1} ke_g$. The weak Hopf algebra structure of k(G) are given by

$$e_g e_h = \delta_{g,h}e_g; \quad 1 = \sum_{g\in G_1} e_g;$$

$$\Delta(e_g) = \sum_{xy=g} e_x \otimes e_y = \sum_{t(x)=t(g)} e_x \otimes e_{x^{-1}g}; \quad \varepsilon(\sum_{g\in G_1} a_g e_g) = \sum_{x\in G_0} a_x e_x;$$

$$S(e_g) = e_{g^{-1}}; \quad \Delta(1) = 1_{(1)} \otimes 1_{(2)} = \sum_{t(g)=s(h)} e_g \otimes e_h$$

for any $g, h \in G_1$.

Set $\phi : k[G] \longrightarrow \mathrm{Aut}(k[G])$ defined by $\phi_g(h) = ghg^{-1}$. It is a well defined group homomorphism. This data leads to a quasi-triangular weak Hopf G_1-coalgebra $\overline{D(k[G], k(G))} = \{D(k[G], k(G))_{(\alpha,\beta)} = D(k[G], k(G), \langle\, ,\, \rangle, \phi)/I_{(\alpha,\beta)}\}_{(\alpha,\beta)\in\mathscr{S}(G_1)}$ which will be denoted by $\overline{D_G(G)} = \{\overline{D}_{(\alpha,\beta)}(G)\}_{(\alpha,\beta)\in G_1}$. More explicitly, $\overline{D_G(G)}$ is described as follows:

For any $\alpha, \beta \in G_1$, the algebra structure of $\overline{D_{(\alpha,\beta)}(G)}$, which is equal to $k[G] \otimes k(G)$ as a k-space, is given by

$$[g \otimes e_h][g' \otimes e_{h'}] = \delta_{\alpha g' \alpha^{-1}, h^{-1}\beta g' \beta^{-1} h'} gg' \otimes e_{h'} \quad \text{for all} \quad g, g', h, h' \in G_1,$$
$$1_{\overline{D_{(\alpha,\beta)}(G)}} = \sum_{x\in G_0, g\in G_1} [u_x \otimes e_g].$$

The crossed weak Hopf G-coalgebra structures of $D_G(G)$ are given, for any $\alpha, \beta, \lambda, \gamma \in G_1$ and $g, h \in G_1$, by

$$\overline{\Delta}_{(\alpha,\beta),(\lambda,\gamma)}([g \otimes e_h]) = \sum_{xy=h} [g \otimes e_{\gamma x \gamma^{-1}}] \otimes [g \otimes e_{\gamma\alpha\gamma^{-1} y \gamma \alpha^{-1} \gamma^{-1}}],$$
$$\overline{\varepsilon}([g \otimes e_h]_{(1,1)}) = \delta_{h,1},$$
$$S_{(\alpha,\beta)}([g \otimes e_h]) = [g^{-1} \otimes e_{\alpha\beta\alpha^{-1} g \alpha h^{-1} \beta g^{-1} \beta^{-1} \alpha^{-1}}],$$
$$\varphi^{(\lambda,\gamma)}_{(\alpha,\beta)}([g \otimes e_h]) = [\beta^{-1}\alpha g \alpha^{-1} \beta \otimes e_{\gamma\alpha^{-1}\gamma^{-1}\beta h \beta^{-1}\gamma\alpha\gamma^{-1}}].$$

Then $D_G(G)^* = \bigoplus_{\alpha\in G} D_G(G)^*_\alpha$ is a crossed weak Hopf G-algebra.

Lemma 3.4 *It is easy to get the following identities:*

(a) $\xi_1 \mid H_\alpha = \mathrm{id}_{H_\alpha}$ *for all* $\alpha \in \pi$.

(b) $\xi_\alpha^{-1} = \xi_{\alpha^{-1}}$ *for all* $\alpha \in \pi$.

(c) ξ *preserves the antipode, i.e.,* $\xi_\beta \circ S_\alpha = S_{\beta\alpha\beta^{-1}} \circ \xi_\beta$ *for all* $\alpha, \beta \in \pi$.

Let H be a weak Hopf π-algebra. Define a family of linear maps $\varepsilon^t = \{\varepsilon^t_\alpha : H_\alpha \to H_1\}_{\alpha\in\pi}$ by $\varepsilon^t_\alpha(h) = \varepsilon_\alpha(1_{(1,1)}h)1_{(2,1)}$ and $\varepsilon^s = \{\varepsilon^s_\alpha : H_\alpha \to H_1\}_{\alpha\in\pi}$ by $\varepsilon^s_\alpha(h) = 1_{(1,1)}\varepsilon_\alpha(h1_{(2,1)})$ for all $h \in H_\alpha$, where $\varepsilon^t, \varepsilon^s$ are called the π-target and π-source counital maps. Introduce the notations $H^t := \varepsilon^t(H) = \{H^t_1 = \varepsilon^t_\alpha(H_\alpha)\}_{\alpha\in\pi}$ and $H^s := \varepsilon^s(H) = \{H^s_1 = \varepsilon^s_\alpha(H_\alpha)\}_{\alpha\in\pi}$ for their images.

By Eq. (3.2), one immediately obtains the following identities:

$$\varepsilon_{\alpha\beta}(gh) = \varepsilon_\alpha(g\varepsilon^t_\beta(h)), \quad \varepsilon_{\alpha\beta}(gh) = \varepsilon_\beta(\varepsilon^s_\alpha(g)h), \tag{3.8}$$
$$\varepsilon^t_1 \circ \varepsilon^t_\alpha = \varepsilon^t_\alpha, \quad \varepsilon^s_1 \circ \varepsilon^s_\alpha = \varepsilon^s_\alpha. \tag{3.9}$$

Lemma 3.5 *Let* H *be a weak Hopf* π*-algebra. Then we have, for all* $x \in H_\alpha, y \in H_\beta$ *and* $\alpha, \beta \in \pi$

(i) $x_{(1,\alpha)} \otimes \varepsilon^t_\alpha(x_{(2,\alpha)}) = 1_{(1,1)}x \otimes 1_{(2,1)}$, (3.10)

(ii) $\varepsilon^s_\alpha(x_{(1,\alpha)}) \otimes x_{(2,\alpha)} = 1_{(1,1)} \otimes x1_{(2,1)}$, (3.11)

(iii) $x\varepsilon^t_\beta(y) = \varepsilon_{\alpha\beta}(x_{(1,\alpha)}y)x_{(2,\alpha)}$, (3.12)

(iv) $\varepsilon^s_\beta(y)x = x_{(1,\alpha)}\varepsilon_{\beta\alpha}(yx_{(2,\alpha)})$, (3.13)

(v) H^t_1 *and* H^s_1 *are subalgebras of* H_1 *containing the unit 1 and we have*

$$h^t g^s = g^s h^t \text{ for all } h^t \in H_1^t \text{ and } g^s \in H_1^s. \tag{3.14}$$

Proof (i) We compute as follows

$$\begin{aligned} x_{(1,\alpha)} \otimes \varepsilon_\alpha^t(x_{(2,\alpha)}) &= x_{(1,\alpha)} \otimes \varepsilon_\alpha(1_{(1,1)}x_{(2,1)})1_{(2,1)} = \widetilde{1}_{(1,1)}x_{(1,\alpha)} \otimes \varepsilon(1_{(1,1)}\widetilde{1}_{(2,1)}x_{(2,\alpha)})1_{(2,1)} \\ &= 1_{(1,1)}x_{(1,\alpha)} \otimes \varepsilon(1_{(2,1)}x_{(2,\alpha)})1_{(3,1)} = 1_{(1,1)}x \otimes 1_{(2,1)}. \end{aligned}$$

(ii) is similar to (i).

(iii) and (iv) are immediate consequence of (ii) and (i).

(v) Obviously, $1 \in H_1^t \cap H_1^s$ since $\varepsilon_\alpha^t(1_\alpha) = \varepsilon_\alpha^s(1_\alpha) = 1$, and H_1^t and H_1^s commute with each other. Finally, the fact that H_1^t and H_1^s are subalgebras of H_1 follows from the formulae:

$$1_{(1,\alpha)} \otimes \varepsilon_\beta^t(1_{(2,\beta)}) \otimes 1_{(3,\gamma)} = \widetilde{1}_{(1,1)}1_{(1,\alpha)} \otimes \widetilde{1}_{(2,1)} \otimes 1_{(2,\gamma)}, \tag{3.15}$$

$$1_{(1,\gamma)} \otimes \varepsilon_\beta^s(1_{(2,\beta)}) \otimes 1_{(3,\alpha)} = 1_{(1,\gamma)} \otimes \widetilde{1}_{(1,1)} \otimes 1_{(2,\alpha)}\widetilde{1}_{(2,1)}, \tag{3.16}$$

for all $\alpha, \beta, \gamma \in \pi$. We also give a direct proof as follows

$$\begin{aligned} \varepsilon_\alpha^t(h)\varepsilon_\beta^t(g) &\overset{(3.12)}{=} \varepsilon_\beta(\varepsilon_\alpha^t(h)_{(1,1)}g)\varepsilon_\alpha^t(h)_{(2,1)} \\ &= \varepsilon_\beta(1_{(1,1)}\varepsilon_\alpha^t(h)g)1_{(2,1)} = \varepsilon_\beta^t(\varepsilon_\alpha^t(h)g). \end{aligned}$$

A statement about H_1^s is proven similarly. $\square$

Lemma 3.6 *Let H be a weak Hopf π-algebra. Then we have*

(i) The kernel $Ker\varepsilon_\alpha^t$ is a left ideal of H_α and $\mathrm{Ker}\varepsilon_\alpha^s$ is a right ideal of H_α for all $\alpha \in \pi$;

(ii) We have the following formulae

$$\varepsilon_\beta^t(\varepsilon_\alpha^t(x)y) = \varepsilon_\alpha^t(x)\varepsilon_\beta^t(y), \quad \varepsilon_\alpha^s(x\varepsilon_\beta^s(y)) = \varepsilon_\alpha^s(x)\varepsilon_\beta^s(y); \tag{3.17}$$

(iii) Furthermore, if H is crossed with the crossing $\xi = \{\xi_\alpha\}_{\alpha\in\pi}$, then we have

$$\xi_\beta \circ \varepsilon_\alpha^s = \varepsilon_{\beta\alpha\beta^{-1}}^s \circ \xi_\beta, \quad \xi_\beta \circ \varepsilon_\alpha^t = \varepsilon_{\beta\alpha\beta^{-1}}^t \circ \xi_\beta$$

for any $\alpha, \beta \in \pi$.

Proof (i) Easy. (ii) One has

$$\begin{aligned} \varepsilon_\beta^t(\varepsilon_\alpha^t(x)y) &= \varepsilon_\beta(1_{(1,1)}\varepsilon_\alpha^t(x)y)1_{(2,1)} \overset{(3.9)}{=} \varepsilon_1(1_{(1,1)}\varepsilon_\alpha^t(x)\varepsilon_\beta^t(y))1_{(2,1)} \\ &\overset{(3.10)}{=} \varepsilon_\alpha^t(z) = \varepsilon_\alpha^t(x)\varepsilon_\beta^t(y). \end{aligned}$$

(iii) We just check that the first formula holds. The second one can be proved similarly. For any $h \in H_\alpha$ and $\alpha, \beta \in \pi$, one has

$$\begin{aligned} \varepsilon_{\beta\alpha\beta^{-1}}^s\xi_\beta(h) &= 1_{(1,1)}\varepsilon_{\beta\alpha\beta^{-1}}(\xi_\beta(h)1_{(2,1)}) = 1_{(1,1)}\varepsilon_\alpha(h\xi_{\beta^{-1}}(1_{(2,1)})) \\ &= \xi_\beta(1_{(1,1)})\varepsilon_\alpha(h1_{(2,1)}) = \xi_\beta\varepsilon_\alpha^s(h). \end{aligned}$$

This finishes the proof. $\square$

By Eqs. (3.5)–(3.7), we have $S_\alpha(x) = S_\alpha(x_{(1,\alpha)})\varepsilon_\alpha^t(x_{(2,\alpha)}) = \varepsilon_\alpha^s(x_{(1,\alpha)})S_\alpha(x_{(2,\alpha)})$.

Theorem 3.7 *Let H be a weak Hopf π-algebra. Then* *(i)* $S_{\alpha\beta}(xy) = S_\beta(y)S_\alpha(x)$ *for any* $\alpha \in \pi$ *and* $x \in H_\alpha, y \in H_\beta$*;*

(ii) $S_\alpha(1_\alpha) = 1_{\alpha^{-1}}$ *for any* $\alpha \in \pi$.

Furthermore if H *is of finite type then* $S : H \longrightarrow H$ *is bijective, i.e.,* $S_\alpha : H_\alpha \longrightarrow H_{\alpha^{-1}}$ *is bijective for any* $\alpha \in \pi$.

Proof Similar to [1]. □

Proposition 3.8 *(i) We have the following formulae:*

$$\varepsilon^t_\alpha(x) = \varepsilon_{\alpha^{-1}}(S_\alpha(x)1_{(1,1)})1_{(2,1)}, \quad \varepsilon^s_\alpha(x) = 1_{(1,1)}\varepsilon_{\alpha^{-1}}(1_{(2,1)}S_\alpha(x)),$$
$$\varepsilon^t_\alpha(x) = S_1(1_{(1,1)})\varepsilon_\alpha(1_{(2,1)}x), \quad \varepsilon^s_\alpha(x) = \varepsilon_\alpha(x1_{(1,1)})S_1(1_{(2,1)})$$

for any $x \in H_\alpha$.

(ii) the following identities hold

$$\varepsilon^t_\alpha \circ S_{\alpha^{-1}} = \varepsilon^t_1 \circ \varepsilon^s_{\alpha^{-1}} = S_1 \circ \varepsilon^s_{\alpha^{-1}}, \quad \varepsilon^s_\alpha \circ S_{\alpha^{-1}} = \varepsilon^s_1 \circ \varepsilon^t_{\alpha^{-1}} = S_1 \circ \varepsilon^t_{\alpha^{-1}}.$$

Proof Similar to [1]. □

4. The category of crossed left π-H comodules

Definition 4.1 *Let* H *be a crossed weak Hopf* π*-algebra. A left* π*-*H*-comodule* M *is called crossed if it is endowed with a family* $\xi_M = \{\xi_{M,\beta} : M_\alpha \to M_{\beta\alpha\beta^{-1}}\}_{\alpha,\beta\in\pi}$ *of* k*-linear maps such that the following conditions are satisfied*

(i) Each $\xi_{M,\beta} : M_\alpha \to M_{\beta\alpha\beta^{-1}}$ *is a vector space isomorphism;*

(ii) Each $\xi_{M,\beta}$ *preserves the coaction, i.e., for all* $\alpha, \beta \in \pi, \rho_{\beta\alpha\beta^{-1}} \circ \xi_{M,\beta} = (\xi_\beta \otimes \xi_{M,\beta}) \circ \rho_\alpha$*;*

(iii) Each ξ_M *is multiplicative in the sense that* $\xi_{M,\beta}\xi_{M,\gamma} = \xi_{M,\beta\gamma}$ *for all* $\beta, \gamma \in \pi$.

Definition 4.2 *Let* $M = \{M_\alpha\}_{\alpha\in\pi}, N = \{N_\alpha\}_{\alpha\in\pi}$ *be two crossed left* π*-*H*-comodules. A crossed left* π*-*H*-comodule morphism is a left* π*-*H*-comodule morphism* $f = \{f_\alpha\}_{\alpha\in\pi} : M \to N$ *such that* $\xi_{N,\beta} \circ f_\alpha = f_{\beta\alpha\beta^{-1}} \circ \xi_{M,\beta}$.

Let $H = (\{H_\alpha\}, m, \eta)$ be a crossed weak Hopf π-algebra. We denote by ${}^H\mathcal{M}_{\mathrm{crossed}}$ the category of all left π-H-comodules, whose morphisms are crossed left π-H-comodule morphisms.

Suppose that $M = \{M_\alpha\}_{\alpha\in\pi}$ and $N = \{N_\alpha\}_{\alpha\in\pi}$ are crossed left π-H-comodules. Now define $M_\beta \boxtimes N_\gamma$, which is the submodule of $M_\beta \otimes N_\gamma$ generated by elements of the form $\varepsilon_{\beta\gamma}(m_{(-1,\beta)}n_{(-1,\gamma)})m_{(0,\beta)} \otimes n_{(0,\gamma)}$ for any $\beta, \gamma \in \pi$ and $m \in M_\beta, n \in N_\gamma$. It is easy to show that $M_\beta \boxtimes N_\gamma$ is left π-H-subcomodule of $M_\beta \otimes N_\gamma$ given by $\rho^{M_\beta\boxtimes N_\gamma}(m \boxtimes n) = m_{(-1,\beta)}n_{(-1,\gamma)} \boxtimes m_{(0,\beta)} \boxtimes n_{(0,\gamma)}$ for any $m \in M_\beta, n \in N_\gamma$. So $(M \boxtimes N)_\alpha := \bigoplus_{\beta\gamma=\alpha} M_\beta \boxtimes N_\gamma$ is a left H_α-comodule. Thus $M \boxtimes N = \{(M \boxtimes N)_\alpha\}_{\alpha\in\pi}$ is a left π-H-comodule, where the structure maps $\rho^{M\boxtimes N} = \{\rho^{(M\boxtimes N)_\alpha}\}_{\alpha\in\pi}$ are given by

$$\rho^{(M\boxtimes N)_\alpha} = \bigoplus_{\beta\gamma=\alpha} (m_{\beta,\gamma} \otimes \mathrm{id}_{M_\beta} \otimes \mathrm{id}_{N_\gamma})(\mathrm{id}_{H_\beta} \otimes \tau_{M_\beta,H_\gamma} \otimes \mathrm{id}_{N_\gamma})(\rho^{M_\beta} \otimes \rho^{N_\gamma}).$$

Now let $g = \{g_\alpha\}_{\alpha\in\pi} : M \to M'$ and $f = \{f_\beta\}_{\beta\in\pi} : N \to N'$ be left π-H-comodule morphisms. Now we define the monoidal product of g and f given by $g \otimes f = \{g_\alpha \otimes f_\beta\}_{\alpha,\beta\in\pi}$:

$M \otimes N \to M' \otimes N'$.

Suppose $P = \{P_\alpha\}_{\alpha\in\pi}$ is also a crossed left π-H-comodule. Then we have two left π-H-comodules $(M \boxtimes N) \boxtimes P$ and $M \boxtimes (N \boxtimes P)$. By definition, for any $\alpha \in \pi$, we have

$$\begin{aligned}((M \boxtimes N) \boxtimes P)_\alpha &= \bigoplus_{\beta\gamma=\alpha} (M \boxtimes N)_\beta \boxtimes P_\gamma = \bigoplus_{\beta\gamma=\alpha} (\bigoplus_{\theta z=\beta} (M_\theta \boxtimes N_z) \boxtimes P_\gamma)\\ &= \bigoplus_{\theta z\gamma=\alpha} (M_\theta \boxtimes N_z) \boxdot P_\gamma\end{aligned}$$

and

$$\begin{aligned}(M \boxtimes (N \boxtimes P))_\alpha &= \bigoplus_{\theta\beta=\alpha} M_\theta \boxtimes (N \boxtimes P)_\beta = \bigoplus_{\theta\beta=\alpha} M_\theta \boxtimes (\bigoplus_{z\gamma=\beta} (N_z \boxtimes P_\gamma))\\ &= \bigoplus_{\theta z\gamma=\alpha} (M_\theta \boxtimes N_z) \boxtimes P_\gamma.\end{aligned}$$

Let $\theta, z, \gamma \in \pi$. One knows that $a_{\theta,z,\gamma} : (M_\theta \boxtimes N_z) \boxtimes P_\gamma \to M_\theta \boxtimes (N_z \boxtimes P_\gamma)$, $(m \otimes n) \otimes p \mapsto m \otimes (n \otimes p)$, where $m \in M_\theta$, $n \in N_z$, $p \in P_\gamma$, is an isomorphism of $H_{\theta z\gamma}$ comodule. Hence, for any $\alpha \in \pi$, $a_\alpha = \bigoplus_{\theta z\gamma=\alpha} a_{\theta,z,\gamma}$ is an isomorphism of H_α comodule from $((M \boxtimes N) \boxtimes P)_\alpha$ to $(M \boxtimes (N \boxtimes P))_\alpha$, and $a = \{a_\alpha\}_{\alpha\in\pi} : (M \boxtimes N) \boxtimes P \to M \boxtimes (N \boxtimes P)$ is a left π-H-comodule isomorphism, it is a family of natural isomorphisms.

Let M, N be any crossed left π-H-comodules. We have proved that $M \boxtimes N$ is also a crossed left π-H-comodule.

Definition 4.3 *With the above notations. A left π-H-comodule $M \boxtimes N$ is called crossed if it is endowed with a family $\xi_{M\boxtimes N} = \{\xi_{M\boxtimes N,z} : (M \boxtimes N)_\alpha \to (M \boxtimes N)_{z\alpha z^{-1}}\}_{\alpha,z\in\pi}$ of k-linear maps such that the following conditions are satisfied:*

(i) Each $\xi_{M\boxtimes N,\beta} : (M \boxtimes N)_\alpha \to (M \boxtimes N)_{z\alpha z^{-1}}$ is a vector space isomorphism;

(ii) Each $\xi_{M\boxtimes N,z|M_\beta\boxtimes N_\gamma} := \xi_{M,z|M_\beta} \boxtimes \xi_{N,z|N_\gamma}$, where for any $\alpha, \beta, \gamma, z \in \pi$.

Since $(M \boxtimes N)_\alpha = \bigoplus_{\beta\gamma=\alpha} M_\beta \boxtimes N_\gamma$ and

$$(M \boxtimes N)_{z\alpha z^{-1}} = \bigoplus_{z\beta\gamma z^{-1}=z\alpha z^{-1}} M_{z\beta z^{-1}} \boxtimes N_{z\gamma z^{-1}} = \bigoplus_{\beta\gamma=\alpha} M_{z\beta z^{-1}} \boxtimes N_{z\gamma z^{-1}}.$$

$\xi_{M\otimes N,z}$ is well defined k-linear isomorphism from $(M \boxtimes N)_\alpha$ to $(M \boxtimes N)_{z\alpha z^{-1}}$ for any $\alpha, z \in \pi$. Moreover, for any $m \in M_\beta$ and $n \in N_\gamma$, we have

$$\begin{aligned}&\rho^{(M\boxtimes N)_{z\alpha z^{-1}}} \circ (\xi_{M\boxtimes N,z})(m \otimes n)\\ &= \rho^{(M\boxtimes N)_{z\alpha z^{-1}}} \circ (\xi_{M,z} \otimes \xi_{N,z})(m \otimes n)\\ &= \rho^{(M\boxtimes N)_{z\alpha z^{-1}}} (\xi_{M,\gamma}(m) \otimes \xi_{N,\gamma}(n))\\ &= \xi_z(m_{(-1,\beta)})\xi_z(n_{(-1,\gamma)}) \otimes \xi_{M,z}(m_{(0,\beta)}) \otimes \xi_{N,z}(n_{(0,\gamma)})\\ &= (\xi_z \otimes \xi_{M\otimes N,z})\rho^{(M\otimes N)_\alpha}(m \otimes n).\end{aligned}$$

Now let M, N and P be crossed left π-H-comodules. Then one can easily check that $\xi_{M\boxtimes(N\boxtimes P),z} a_\alpha = a_{z\alpha z^{-1}} \xi_{(M\boxtimes N)\boxtimes P,z}$ for any $\alpha, z \in \pi$, and hence $a = \{a_\alpha\}_{\alpha\in\pi} : (M \boxtimes N) \boxtimes P \to M \boxtimes (N \boxtimes P)$ is a crossed left π-H-comodule morphism.

Since $H_1^t=\varepsilon_\alpha^t(H_\alpha)$ for every $\alpha\in\pi$, let $\rho^{H_1^t}: H_1^t\to H_1^t\otimes H_1^t, \lambda\mapsto \Delta_{1,1}(\lambda)$. Hence, H_1^t is a left π-H-comodule. For any left π-H-comodule M, we have $(H^t\boxtimes M)_\alpha = H_1^t\boxtimes M_\alpha$ and $(M\boxtimes H^t)_\alpha = M_\alpha\boxtimes H_1^t$, $\alpha\in\pi$. Define isomorphisms $l_M: H^t\boxtimes M\to M$ and $r_M: M\boxtimes H^t\to M$ by

$$(l_M)_\alpha: H_1^t\boxtimes M_\alpha\to M_\alpha, \lambda\otimes m\mapsto \varepsilon(\lambda)m,$$

$$(r_M)_\alpha: M_\alpha\boxtimes H_1^t\to M_\alpha, m\otimes\lambda\mapsto m\varepsilon(\lambda),$$

and

$$(l_M)_\alpha^{-1}: M_\alpha\to H_1^t\boxtimes M_\alpha, m\mapsto \varepsilon_\alpha^t(m_{(1,\alpha)})\otimes m_{(0,\alpha)},$$

$$(r_M)_\alpha^{-1}: M_\alpha\to M_\alpha\boxtimes H_1^t, m\mapsto m_{(0,\alpha)}\otimes S^{-1}\varepsilon_\alpha^s(m_{(1,\alpha)}).$$

Then $l=\{l_M\}$ and $r=\{r_M\}$ are two families of natural isomorphisms of left π-H-comodules.

We summarize the above discussion as follows.

Theorem 4.4 $({}^H\mathcal{M}_{\text{crossed}},\boxtimes, H_1^t, a, l, r)$ *is a monoidal category, where* H_1^t *is the unit object.*

5. The Braided monoidal category

Throughout this section, assume that $H=(\{H_\alpha\}, m,\eta)$ is a crossed weak Hopf π-algebra with a crossing ξ.

Definition 5.1 *A coquasitriangular weak Hopf π-algebra is a crossed weak Hopf π-algebra (with crossing ξ) endowed with a family $\sigma=\{\sigma_{\beta,\gamma}: H_\beta\otimes H_\gamma\to k\}_{\beta,\gamma\in\pi}$ of k-linear maps such that $\sigma_{\beta,\gamma}$ is weak convolution invertible for any $\beta,\gamma\in\pi$ and the following conditions are satisfied:*

(i) For any $\beta,\gamma,\theta\in\pi$ and $x\in H_\beta, y\in H_\gamma, p\in H_\theta$,

$$\sigma_{\beta,\gamma\theta}(x,yp)=\sigma_{\beta,\gamma}(x_{(1,\beta)},y)\sigma_{\gamma^{-1}\beta\gamma,\theta}(\xi_{\gamma^{-1}}(x_{(2,\beta)}),p); \tag{5.1}$$

(ii) For any $\beta,\gamma,z\in\pi$ and $x\in H_\beta, y\in H_\gamma, p\in H_z$

$$\sigma_{\beta\gamma,z}(xy,p)=\sigma_{\beta,z}(x,p_{(2,z)})\sigma_{\gamma,z}(y,p_{(1,z)}); \tag{5.2}$$

(iii) For any $\beta,\gamma\in\pi$ and $x\in H_\beta, y\in H_\gamma$,

$$\sigma_{\beta,\gamma}(x_{(1,\beta)},y_{(1,\gamma)})y_{(2,\gamma)}\xi_{\gamma^{-1}}(x_{(2,\beta)})=x_{(1,\beta)}y_{(1,\gamma)}\sigma_{\beta,\gamma}(x_{(2,\beta)},y_{(2,\gamma)}); \tag{5.3}$$

(iv) For any $\beta,\gamma,z\in\pi$ and $x\in H_\beta, y\in H_\gamma$,

$$\sigma_{\beta,\gamma}(x,y)=\sigma_{z\beta z^{-1},z\gamma z^{-1}}(\xi_z(x),\xi_z(y)); \tag{5.4}$$

(v) For any $\beta,\gamma\in\pi$ and $x\in H_\beta, y\in H_\gamma$,

$$\sigma_{\gamma,\beta}(y,x)=\varepsilon_{\beta\gamma}(x_{(1,\beta)}y_{(1,\gamma)})\sigma_{\gamma,\beta}(y_{(2,\gamma)},x_{(2,\beta)})\varepsilon_{\gamma\beta}(y_{(3,\gamma)}x_{(3,\beta)}). \tag{5.5}$$

Here weak convolution invertible means that there exist a family of k-linear maps $\sigma^{-1}=\{\sigma_{\beta,\gamma}^{-1}: H_\beta\boxtimes H_\gamma\to k\}_{\beta,\gamma\in\pi}$ such that:

(vi) For any $\beta,\gamma\in\pi$ and $x\in H_\beta, y\in H_\gamma$,

$$\sigma_{\beta,\gamma}(x_{(1,\beta)},y_{(1,\gamma)})\sigma_{\beta,\gamma}^{-1}(x_{(2,\beta)},y_{(2,\gamma)})=\varepsilon_{\beta\gamma}(xy); \tag{5.6}$$

(vii) *For any* $\beta, \gamma \in \pi$ *and* $x \in H_\beta, y \in H_\gamma$,

$$\sigma^{-1}_{\beta,\gamma}(x_{(1,\beta)}, y_{(1,\gamma)})\sigma_{\beta,\gamma}(x_{(2,\beta)}, y_{(2,\gamma)}) = \varepsilon_{\gamma\beta}(yx); \tag{5.7}$$

(viii) *For any* $\beta, \gamma \in \pi$ *and* $x \in H_\beta, y \in H_\gamma$,

$$\sigma^{-1}_{\gamma,\beta}(y, x) = \varepsilon_{\beta\gamma}(x_{(1,\beta)}y_{(1,\gamma)})\sigma_{\gamma,\beta}(y_{(2,\gamma)}, x_{(2,\beta)})\varepsilon_{\gamma\beta}(y_{(3,\gamma)}x_{(3,\beta)}) \tag{5.8}$$

where $\sigma^{-1} = \{\sigma^{-1}_{\beta,\gamma}\}_{\beta,\gamma\in\pi}$ *is called a weak convolution inverse of* $\sigma = \{\sigma_{\beta,\gamma}\}_{\beta,\gamma\in\pi}$.

Let $\sigma = \{\sigma_{\beta,\gamma} : H_\beta \otimes H_\gamma \to k\}_{\beta,\gamma\in\pi}$ be a family of linear maps such that $\sigma_{\beta,\gamma}$ is weak convolution invertible for any $\beta, \gamma \in \pi$. Let M and N be any crossed left π-H-comodules. For any $\beta, \gamma \in \pi$, define $c_{M_\beta,N_\gamma} : M_\beta \boxtimes N_\gamma \to N_\gamma \boxtimes M_{\gamma^{-1}\beta\gamma}$ by

$$c_{M_\beta,N_\gamma}(m \otimes n) = \sigma_{\beta,\gamma}(m_{(-1,\beta)}, n_{(-1,\gamma)})(n_{(0,\gamma)} \otimes \xi_{M,\gamma^{-1}}(m_{(0,\beta)})),$$

where $m \in M_\beta$ and $n \in N_\gamma$. For any $\alpha \in \pi$, define

$$(c_{M,N})_\alpha : (M \boxtimes N)_\alpha = \bigoplus_{\beta\gamma=\alpha} M_\beta \boxtimes N_\gamma \to (N \boxtimes M)_\alpha = \bigoplus_{\beta\gamma=\alpha} N_\gamma \boxtimes M_{\gamma^{-1}\beta\gamma}$$

by $(c_{M,N})_\alpha = \bigoplus_{\beta\gamma=\alpha} c_{M_\beta,N_\gamma}$. Then it is obvious that $(c_{M,N})_\alpha$ is a k-linear isomorphism for any $\alpha \in \pi$ if and only if so is c_{M_β,N_γ} for any $\beta, \gamma \in \pi$.

Lemma 5.2 *With the above notations, we have*

(i) $(c_{M,N})_\alpha$ *is a k-linear isomorphism for any crossed left π-H-comodules M and N, and $\alpha \in \pi$ if and only if σ is a family of weak convolution invertible k-linear maps.*

(ii) $c_{M,N} : M \boxtimes N \to N \boxtimes M$ *is a left π-H-comodule morphism for any crossed left π-H-comodules M and N if and only if*

$$\sigma_{\beta,\gamma}(x_{(1,\beta)}, y_{(1,\gamma)})y_{(2,\gamma)}\xi_{\gamma^{-1}}(x_{(2,\beta)}) = x_{(1,\beta)}y_{(1,\gamma)}\sigma_{\beta,\gamma}(x_{(2,\beta)}, y_{(2,\gamma)})$$

for all $\beta, \gamma \in \pi$ *and* $x \in H_\beta, y \in H_\gamma$.

Proof (i) Assume that $\sigma = \{\sigma_{\beta,\gamma} : H_\beta \otimes H_\gamma \to k\}_{\beta,\gamma\in\pi}$ is a family of weak convolution invertible k-linear maps. Then define $c^{-1}_{N_\gamma,M_{\gamma^{-1}\beta\gamma}} : N_\gamma \boxtimes M_{\gamma^{-1}\beta\gamma} \to M_\beta \boxtimes N_\gamma$ by

$$c^{-1}_{N_\gamma,M_{\gamma^{-1}\beta\gamma}}(n \otimes p) = \sigma^{-1}_{\beta,\gamma}(\xi_\gamma(p_{(-1,\gamma^{-1}\beta\gamma)}), n_{(-1,\gamma)})\xi_{M,\gamma}(p_{(0,\gamma^{-1}\beta\gamma)}) \otimes n_{(0,\gamma)},$$

where $p \in M_{\gamma^{-1}\beta\gamma}$ and $n \in N_\gamma$. Then c_{M_β,N_γ} is a k-linear isomorphism as follows:

$$\begin{aligned}
&c^{-1}_{N_\gamma,M_{\gamma^{-1}\beta\gamma}}c_{M_\beta,N_\gamma}(m \otimes n)\\
&\quad= c^{-1}_{N_\gamma,M_{\gamma^{-1}\beta\gamma}}(\sigma_{\beta,\gamma}(m_{(-1,\beta)}, n_{(-1,\gamma)})(n_{(0,\gamma)} \otimes \xi_{M,\gamma^{-1}}(m_{(0,\beta)})))\\
&\quad= \sigma_{\beta,\gamma}(m_{(-1,\beta)}, n_{(-1,\gamma)})\sigma^{-1}_{\beta,\gamma}(\xi_\gamma(\xi_{M,\gamma^{-1}}(m_{(0,\beta)})_{(-1,\gamma^{-1}\beta\gamma)}), n_{(-1,\gamma)})\\
&\qquad\xi_{M,\gamma}(\xi_{M,\gamma^{-1}}(m_{(0,\beta)})_{(0,\gamma^{-1}\beta\gamma)}) \otimes n_{(0,\gamma)}\\
&\quad= \sigma_{\beta,\gamma}(m_{(-1,\beta)}, n_{(-1,\gamma)})\sigma^{-1}_{\beta,\gamma}(m_{(0,\beta)(-1,\beta)}, n_{(0,\gamma)(-1,\gamma)})(m_{(0,\beta)(0,\beta)} \otimes n_{(0,\gamma)(0,\gamma)})\\
&\quad= \varepsilon_{\beta\gamma}(m_{(-1,\beta)}n_{(-1,\gamma)})(m_{(0,\beta)} \otimes n_{(0,\gamma)}) = m \otimes n.
\end{aligned}$$

Conversely, let $M = N = H$. Then $c_{H_\beta,H_\gamma} : H_\beta \boxtimes H_\gamma \to H_\gamma \boxtimes H_{\gamma^{-1}\beta\gamma}$ is a left π-H-comodule isomorphism. Then $\sigma = \{\sigma_{\beta,\gamma} : H_\beta \otimes H_\gamma \to k\}_{\beta,\gamma\in\pi}$ by $\sigma_{\beta,\gamma}(x, y) = (\varepsilon_\beta \otimes \varepsilon_{\beta^{-1}\gamma\beta})c_{H_\gamma,H_\beta}(y \otimes$

$x)$, $x \in H_\beta, y \in H_\gamma$. Define a family of k-linear maps $\tau = \{\tau_{\beta,\gamma} : H_\beta \otimes H_\gamma \to k\}_{\beta,\gamma\in\pi}$ by

$$\tau_{\beta,\gamma}(x \otimes y) = (\varepsilon_{\beta\gamma\beta^{-1}} \otimes \varepsilon_\beta)c^{-1}_{H_\beta,H_\gamma}(x \otimes y),\ x \in H_\beta, y \in H_\gamma.$$

Then

$$c^{-1}_{H_\beta,H_\gamma}(x \otimes y) = (\xi_\beta(y_{(2,\gamma)}) \otimes x_{(2,\beta)})\tau_{\beta\gamma\beta^{-1},\beta}(\xi_\beta(y_{(2,\gamma)}), x_{(1,\beta)}),\ x \in H_\beta, y \in H_\gamma.$$

Thus for any $x \in H_\beta, y \in H_\gamma$, we have

$$\begin{aligned} x \otimes y &= c_{H_{\beta\gamma\beta^{-1}},H_\beta} c^{-1}_{H_\beta,H_\gamma}(x \otimes y) \\ &= c_{H_{\beta\gamma\beta^{-1}},H_\beta}((\xi_\beta(y_{(2,\gamma)}) \otimes x_{(2,\beta)})\tau_{\gamma,\beta}(y_{(1,\gamma)}, x_{(1,\beta)})) \\ &= x_{(3,\beta)} \otimes y_{(3,\gamma)}\sigma_{\beta\gamma\beta^{-1},\beta}(\xi_\beta(y_{(2,\gamma)}), x_{(2,\beta)})\tau_{\beta\gamma\beta^{-1},\beta}(\xi_\beta(y_{(1,\gamma)}), x_{(1,\beta)}) \end{aligned}$$

and

$$x \otimes y = \varepsilon_{\beta\gamma}(x_{(1,\beta)}y_{(1,\gamma)})x_{(2,\beta)} \otimes_k y_{(2,\gamma)}.$$

Applying $\varepsilon_\beta \otimes_k \varepsilon_\gamma$ to the above two equations, one gets

$$\sigma_{\gamma,\beta}(y_{(2,\gamma)}, x_{(2,\beta)})\tau_{\gamma,\beta}(y_{(1,\gamma)}, x_{(1,\beta)}) = \varepsilon_{\beta\gamma}(xy).$$

Then an argument similar to the above shows that

$$\sigma_{\beta,\gamma}(x_{(1,\beta)}, y_{(1,\gamma)})\tau_{\beta,\gamma}(x_{(2,\beta)}, y_{(2,\gamma)})) = \varepsilon_{\beta\gamma}(xy).$$

And we have

$$\begin{aligned} \sigma_{\beta,\gamma}(x, y) &= (\varepsilon_\beta \otimes \varepsilon_{\beta^{-1}\gamma\beta})(\sigma_{\gamma,\beta}(y_{(1,\gamma)}, x_{(1,\beta)})(x_{(2,\beta)} \otimes \xi_{\beta^{-1}}(y_{(2,\gamma)}))) \\ &= (\varepsilon_\beta \otimes \varepsilon_{\beta^{-1}\gamma\beta})(\varepsilon_{\gamma\beta}(y_{(1,\gamma)}x_{(1,\beta)})\sigma_{\gamma,\beta}(y_{(2,\gamma)}, x_{(2,\beta)})(x_{(3,\beta)} \otimes \xi_{\beta^{-1}}(y_{(3,\gamma)}))) \\ &= \varepsilon_{\gamma\beta}(y_{(1,\gamma)}x_{(1,\beta)})(\varepsilon_\beta \otimes \varepsilon_{\beta^{-1}\gamma\beta})(\sigma_{\gamma,\beta}(y_{(2,\gamma)}, x_{(2,\beta)})(x_{(3,\beta)} \otimes \xi_{\beta^{-1}}(y_{(3,\gamma)}))) \\ &= \varepsilon_{\gamma\beta}(y_{(1,\gamma)}x_{(1,\beta)})(\varepsilon_\beta \otimes \varepsilon_{\beta^{-1}\gamma\beta})(c_{H_\gamma,H_\beta}(y_{(2,\gamma)} \otimes x_{(2,\beta)})) \\ &= \varepsilon_{\gamma\beta}(y_{(1,\gamma)}x_{(1,\beta)})(\varepsilon_\beta \otimes \varepsilon_{\beta^{-1}\gamma\beta})(S_\beta \otimes S_{\beta^{-1}\gamma\beta})(c_{H_\gamma,H_\beta}(y_{(2,\gamma)} \otimes x_{(2,\beta)})) \\ &= \varepsilon_{\gamma\beta}(y_{(1,\gamma)}x_{(1,\beta)})(\varepsilon_\beta \otimes \varepsilon_{\beta^{-1}\gamma\beta})c_{H_{\gamma^{-1}},H_{\beta^{-1}}}(S_\gamma \otimes S_\beta)(y_{(2,\gamma)} \otimes x_{(2,\beta)}) \\ &= \varepsilon_{\gamma\beta}(y_{(1,\gamma)}x_{(1,\beta)})(\varepsilon_\beta \otimes \varepsilon_{\beta^{-1}\gamma\beta})c_{H_{\gamma^{-1}},H_{\beta^{-1}}}(S_\gamma(y_{(2,\gamma)}) \otimes S_\beta(x_{(2,\beta)})) \\ &= \varepsilon_{\gamma\beta}(y_{(1,\gamma)}x_{(1,\beta)})(\varepsilon_\beta \otimes \varepsilon_{\beta^{-1}\gamma\beta})c_{H_{\gamma^{-1}},H_{\beta^{-1}}}(S_\gamma(y_{(2,\gamma)}) \otimes S_\beta(x_{(2,\beta)})) \\ &\qquad \varepsilon_{\gamma^{-1}\beta^{-1}}(S_\gamma(y_{(3,\gamma)})S_\beta(x_{(3,\beta)})) \\ &= \varepsilon_{\gamma\beta}(y_{(1,\gamma)}x_{(1,\beta)})(\varepsilon_\beta \otimes \varepsilon_{\beta^{-1}\gamma\beta})c_{H_\gamma,H_\beta}(y_{(2,\gamma)} \otimes x_{(2,\beta)})\varepsilon_{\beta\gamma}(x_{(3,\beta)}y_{(3,\gamma)}) \\ &= \varepsilon_{\gamma\beta}(y_{(1,\gamma)}x_{(1,\beta)})\sigma_{\beta,\gamma}(x_{(2,\beta)}, y_{(2,\gamma)})\varepsilon_{\beta\gamma}(x_{(3,\beta)}y_{(3,\gamma)}). \end{aligned}$$

Similarly, we have

$$\tau_{\beta,\gamma}(x, y) = \varepsilon_{\beta\gamma}(x_{(1,\beta)}y_{(1,\gamma)})\tau_{\beta\gamma}(x_{(2,\beta)}, y_{(2,\gamma)})\varepsilon_{\gamma\beta}(y_{(3,\gamma)}x_{(3,\beta)}).$$

This shows that $\sigma = \{\sigma_{\beta,\gamma}\}$ is a family of weak convolution invertible k-linear maps with inverse $\tau = \{\tau_{\beta,\gamma}\}$.

(ii) Now we claim that $c_{M,N} = \{(c_{M,N})_\alpha\}_{\alpha\in\pi} : M \boxtimes N \to N \boxtimes M$ is a morphism of left

π-H-comodules. In fact, for $\beta,\gamma\in\pi, m\in M_\beta$ and $n\in N_\gamma$, we have

$$\begin{aligned}&\rho^{(N\boxtimes M)_{\beta\gamma}}c_{M_\beta,N_\gamma}(m\otimes n)\\&=\rho^{(N\otimes M)_{\beta\gamma}}(n_{(0,\gamma)}\otimes\xi_{M,\gamma^{-1}}(m_{(0,\beta)}))\sigma_{\beta,\gamma}(m_{(-1,\beta)},n_{(-1,\gamma)})\\&=n_{(-1,\gamma)}\xi_{\gamma^{-1}}(m_{(-1,\beta)})\otimes n_{(0,\gamma)}\otimes\xi_{M,\gamma^{-1}}(m_{(0,\beta)})\sigma_{\beta,\gamma}(m_{(-2,\beta)},n_{(-2,\gamma)})\end{aligned}$$

and

$$\begin{aligned}&(\mathrm{id}_{H_{\beta\gamma}}\boxtimes c_{M_\beta,N_\gamma})\rho^{(N\boxtimes M)_{\beta\gamma}}(m\otimes n)\\&=m_{(-2,\beta)}n_{(-2,\gamma)}\otimes n_{(0,\gamma)}\otimes\xi_{M,\gamma^{-1}}(m_{(0,\beta)})\sigma_{\beta,\gamma}(m_{(-1,\beta)},n_{(-1,\gamma)}).\end{aligned}$$

Because $\xi_{M,\gamma^{-1}}$ is an isomorphism, if

$$\sigma_{\beta,\gamma}(x_{(1,\beta)},y_{(1,\gamma)})y_{(2,\gamma)}\xi_{\gamma^{-1}}(x_{(2,\beta)})=x_{(1,\beta)}y_{(1,\gamma)}\sigma_{\beta,\gamma}(x_{(2,\beta)},y_{(2,\gamma)}),$$

we have c_{M_β,N_γ} is an isomorphism of left $H_{\beta\gamma}$-comodules. Conversely, let $M=N=H$. Since $c_{H,H}$ is a left π-H-comodule map, $\rho^{(H\boxtimes H)_{\beta\gamma}}(c_{H_\beta,H_\gamma})=(\mathrm{id}_{H_{\beta\gamma}}\boxtimes c_{H_\beta,H_\gamma})\rho^{(H\boxtimes H)_{\beta\gamma}}$ for all $\beta,\gamma\in\pi$. Now let $\beta\in\pi$ and $x\in H_\beta, y\in H_\gamma$. We have

$$\begin{aligned}&\rho^{(H\boxtimes H)_{\beta\gamma}}c_{H_\beta,H_\gamma}(x\otimes y)=\rho^{(H\boxtimes H)_{\beta\gamma}}(y_{(2,\gamma)}\otimes\xi_{\gamma^{-1}}(x_{(2,\beta)}))\sigma_{\beta,\gamma}(x_{(1,\beta)},y_{(1,\beta)})\\&=\sigma_{\beta,\gamma}(x_{(1,\beta)},y_{(1,\gamma)})y_{(2,\gamma)}\xi_{\gamma^{-1}}(x_{(2,\beta)})\otimes y_{(3,\gamma)}\otimes\xi_{\gamma^{-1}}(x_{(3,\beta)}).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&(\mathrm{id}_{H_{\beta\gamma}}\boxtimes c_{H_\beta,H_\gamma})\rho^{(H\boxtimes H)_{\beta\gamma}}(x\otimes y)=(\mathrm{id}_{H_{\beta\gamma}}\boxtimes c_{H_\beta,H_\gamma})(x_{(1,\beta)}y_{(1,\gamma)}\otimes x_{(2,\beta)}\otimes y_{(2,\gamma)})\\&=\sigma_{\beta,\gamma}(x_{(2,\beta)},y_{(2,\gamma)})x_{(1,\beta)}y_{(1,\gamma)}\otimes y_{(3,\gamma)}\otimes\xi_{\gamma^{-1}}(x_{(3,\beta)}).\end{aligned}$$

Hence, we have

$$\begin{aligned}&\sigma_{\beta,\gamma}(x_{(1,\beta)},y_{(1,\gamma)})y_{(2,\gamma)}\xi_{\gamma^{-1}}(x_{(2,\beta)})\otimes y_{(3,\gamma)}\otimes\xi_{\gamma^{-1}}(x_{(3,\beta)})\\&=\sigma_{\beta,\gamma}(x_{(2,\beta)},y_{(2,\gamma)})x_{(1,\beta)}y_{(1,\gamma)}\otimes y_{(3,\gamma)}\otimes\xi_{\gamma^{-1}}(x_{(3,\beta)}).\end{aligned}$$

Applying $\mathrm{id}_{H_{\beta\gamma}}\otimes\varepsilon_\gamma\otimes\varepsilon_{\gamma^{-1}\beta\gamma}$ to the both sides of the above equation, one gets

$$\sigma_{\beta,\gamma}(x_{(1,\beta)},y_{(1,\gamma)})y_{(2,\gamma)}\xi_{\gamma^{-1}}(x_{(2,\beta)})=x_{(1,\beta)}y_{(1,\gamma)}\sigma_{\beta,\gamma}(x_{(2,\beta)},y_{(2,\gamma)}).\ \square$$

Lemma 5.3 *The following two statements are equivalent:*

(i) $\xi_{N\boxtimes M,z}(c_{M,N})_\alpha=(c_{M,N})_{z\alpha z^{-1}}\xi_{M\boxtimes N,z}$ *for any crossed left π-H-comodules M and N, and $\alpha,z\in\pi$.*

(ii) $\sigma_{\beta,\gamma}(x,y)=\sigma_{z\beta z^{-1},z\gamma z^{-1}}(\xi_z(x),\xi_z(y))$ *for any $\beta,\gamma,z\in\pi$ and $x\in H_\beta, y\in H_\gamma$.*

Proof Let M and N be crossed left π-H-comodules. For any $\alpha,\beta,z\in\pi, m\in M_\beta$ and $n\in N_\gamma$, we have

$$\begin{aligned}&\xi_{N\boxtimes M,z}(c_{M,N})_{\beta\gamma}(m\otimes n)=(\xi_{N,z}\otimes\xi_{M,z})(c_{M_\beta,N_\gamma})\\&=(\xi_{N,z}\otimes\xi_{M,z})\sigma_{\beta,\gamma}(m_{(-1,\beta)},n_{(-1,\gamma)})(n_{(0,\gamma)}\otimes\xi_{M,\gamma^{-1}}(m_{(0,\beta)}))\\&=\sigma_{\beta,\gamma}(m_{(-1,\beta)},n_{(-1,\gamma)})(\xi_{N,z}(n_{(0,\gamma)})\otimes\xi_{M,z}\xi_{M,\gamma^{-1}}(m_{(0,\beta)}))\\&=\sigma_{\beta,\gamma}(m_{(-1,\beta)},n_{(-1,\gamma)})(\xi_{N,z}(n_{(0,\gamma)})\otimes\xi_{M,z\gamma^{-1}}(m_{(0,\beta)}))\end{aligned}$$

and

$$
\begin{aligned}
&(c_{M,N})_{z\beta\gamma z^{-1}}\xi_{M\boxtimes N,z}(m\otimes n) = c_{M_{z\beta z^{-1}},N_{z\gamma z^{-1}}}\xi_{M\boxtimes N,z}(m\otimes n)\\
&\quad = c_{M_{z\beta z^{-1}},N_{z\gamma z^{-1}}}(\xi_z(m)\otimes\xi_z(n))\\
&\quad = \sigma_{z\beta z^{-1},z\gamma z^{-1}}(\xi_z(m_{(-1,\beta)}),\xi_z(n_{(-1,\gamma)}))(\xi_{N,z}(n_{(0,\gamma)})\otimes\xi_{M,z\gamma^{-1}z^{-1}}\xi_{M,z}(m_{(0,\beta)})).
\end{aligned}
$$

Then $\xi_{N\boxtimes M,z}(c_{M,N})_{\beta\gamma} = (c_{M,N})_{z\beta\gamma z^{-1}}\xi_{M\boxtimes N,z}$ if and only if $\sigma_{\beta,\gamma}(x,y) = \sigma_{z\beta z^{-1},z\gamma z^{-1}}(\xi_z(x),\xi_z(y))$. □

Lemma 5.4 *The following two statements are equivalent:*

(i) $c_{M,N\boxtimes P} = (\mathrm{id}_N\boxtimes c_{M,P})(c_{M,N}\boxtimes\mathrm{id}_P)$ *for any crossed left π-H-comodules M,N and P, if and only if for any $\alpha,\beta,\gamma\in\pi$ and $x\in H_\alpha, y\in H_\beta, p\in H_\gamma$,*

$$\sigma_{\alpha,\beta\gamma}(x,yp) = \sigma_{\alpha,\beta}(x_{(1,\alpha)},y)\sigma_{\beta^{-1}\beta\alpha,\gamma}(\xi_{\beta^{-1}}(x_{(2,\alpha)}),p);$$

(ii) $c_{M\boxtimes N,P} = (c_{M,P}\boxtimes\mathrm{id}_N)(\mathrm{id}_M\boxtimes c_{N,P})$ *for any crossed left π-H-comodules M,N and P, if and only if for any $\alpha,\beta,\gamma\in\pi$ and $x\in H_\alpha, y\in H_\beta, p\in H_\gamma$*

$$\sigma_{\alpha\beta,\gamma}(xy,p) = \sigma_{\alpha,\gamma}(x,p_{(2,\gamma)})\sigma_{\beta,\gamma}(y,p_{(1,\gamma)}).$$

Proof We only prove Part (2). The proof of Part (1) is similar. Let M,N,P be any crossed left π-H-comodules for $\alpha,\beta,\gamma\in\pi$. Then for any $m\in M_\alpha$, $n\in N_\beta$ and $p\in P_\gamma$, we have

$$
\begin{aligned}
&(c_{M\boxtimes N,P})_{\alpha\beta\gamma}(m\otimes n\otimes p) = c_{M_\alpha\boxtimes N_\beta,P_\gamma}(m\otimes n\otimes p)\\
&\quad = p_{(0,\gamma)}\otimes\xi_{M,\gamma^{-1}}(m_{(0,\alpha)})\otimes\xi_{N,\gamma^{-1}}(n_{(0,\beta)})\sigma_{\alpha\beta,\gamma}(m_{(-1,\alpha)}n_{(-1,\beta)},p_{(-1,\gamma)})\\
&\quad = p_{(0,\gamma)}\otimes\xi_{M,\gamma^{-1}}(m_{(0,\alpha)})\otimes\xi_{N,\gamma^{-1}}(n_{(0,\beta)})\sigma_{\alpha,\gamma}(m_{(-1,\alpha)},p_{(-1,\gamma)(2,\gamma)})\\
&\qquad \sigma_{\beta,\gamma}(n_{(-1,\beta)},p_{(-1,\gamma)(1,\gamma)})
\end{aligned}
$$

and

$$
\begin{aligned}
&((c_{M,P}\boxtimes\mathrm{id}_N)(\mathrm{id}_M\boxtimes c_{N,P}))_{\alpha\beta\gamma}(m\otimes n\otimes p)\\
&\quad = (c_{M_\alpha,P_\gamma}\boxtimes\mathrm{id}_{N_{\gamma^{-1}\beta\gamma}})(\mathrm{id}_{M_\alpha}\boxtimes c_{N_\beta,P_\gamma})(m\otimes n\otimes p)\\
&\quad = (c_{M_\alpha,P_\gamma}\boxtimes\mathrm{id}_{N_{\gamma^{-1}\beta\gamma}})(m\otimes p_{(0,\gamma)}\otimes\xi_{N,\gamma^{-1}}(n_{(0,\beta)}))\sigma_{\beta,\gamma}(n_{(-1,\beta)},p_{(-1,\gamma)}).
\end{aligned}
$$

Thus, if $\sigma_{\alpha\beta,\gamma}(xy,p) = \sigma_{\alpha,\gamma}(x,p_{(2,\gamma)})\sigma_{\beta,\gamma}(y,p_{(1,\gamma)})$ for any $\alpha,\beta,\gamma\in\pi$ and $x\in H_\alpha, y\in H_\beta, p\in H_\gamma$, then $c_{M\boxtimes N,P} = (c_{M,P}\boxtimes\mathrm{id}_N)(\mathrm{id}_M\boxtimes c_{N,P})$ for any crossed left π-H-comodules M,N and P. Conversely, let $M=N=P=H$. Since c is a braiding, we have $c_{H_\alpha\boxtimes H_\beta,H_\gamma} = (c_{H_\alpha,H_\gamma}\boxtimes\mathrm{id}_{H_\beta})(\mathrm{id}_{H_\alpha}\boxtimes c_{H_\beta,H_\gamma})$. Thus, for any $x\in H_\alpha, y\in H_\beta, z\in H_\gamma$, we have

$$c_{H_\alpha\boxtimes H_\beta,H_\gamma}(x\otimes y\otimes z) = z_{(2,\gamma)}\otimes\xi_{\gamma^{-1}}(x_{(2,\alpha)})\otimes\xi_{\gamma^{-1}}(y_{(2,\beta)})\sigma_{\alpha\beta,\gamma}(x_{(1,\alpha)}y_{(1,\beta)},z_{(1,\gamma)})$$

and

$$
\begin{aligned}
&(c_{H_\alpha,H_\gamma}\boxtimes\mathrm{id}_{H_\beta})(\mathrm{id}_{H_\alpha}\boxtimes c_{H_\beta,H_\gamma})(x\otimes y\otimes z)\\
&\quad = (c_{H_\alpha,H_\gamma}\boxtimes\mathrm{id}_{H_\beta})(x\otimes z_{(2,\gamma)}\otimes\xi_{\gamma^{-1}}(y_{(2,\beta)}))\sigma_{\beta,\gamma}(y_{(1,\beta)},z_{(1,\gamma)})\\
&\quad = z_{(2,\gamma)(2,\gamma)}\otimes\xi_{\gamma^{-1}}(x_{(2,\alpha)})\otimes\xi_{\gamma^{-1}}(y_{(2,\beta)})\sigma_{\alpha,\gamma}(x_{(1,\alpha)}\otimes z_{(2,\gamma)(1,\gamma)})\sigma_{\beta,\gamma}(y_{(1,\beta)},z_{(1,\gamma)}).
\end{aligned}
$$

Applying $\varepsilon_\gamma \otimes \varepsilon_{\gamma^{-1}\alpha\gamma} \otimes \varepsilon_{\gamma^{-1}\beta\gamma}$ to the above two equations, one gets

$$\sigma_{\alpha,\beta\gamma}(x, yz) = \sigma_{\alpha,\beta}(x_{(1,\alpha)}, y)\sigma_{\beta^{-1}\alpha\beta,\gamma}(\xi_{\beta^{-1}}(x_{(2,a)}), z). \quad \square$$

Theorem 5.5 *Let $H = (\{H_\alpha\}, m, \eta)$ be a crossed weak Hopf π-algebra and let $\sigma = \{\sigma_{\beta,\gamma} : H_\beta \otimes H_\gamma \to k\}_{\beta,\gamma\in\pi}$ be a family of k-linear maps. Then the monoidal category $({}^H\mathcal{M}_{\text{crossed}}, \boxtimes, H_1^t, a, l, r)$ of crossed left π-H-comodules is a braided monoidal category with the braiding c if and only if $H = (\{H_\alpha\}, m, \eta)$ is a coquasitriangular weak Hopf π-algebra where c is defined by σ as above.*

Proof If c is a braiding of the monoidal category $({}^H\mathcal{M}_{\text{crossed}}, \boxtimes, H_1^t, a, l, r)$, then it follows from Lemmas 5.2, 5.3 and 5.4 that σ is a weak coquasitriangular structure. Conversely, assume that σ is a weak coquasitriangular structure. Then by Lemmas 5.2, 5.3 and 5.4, it suffices to show that $c = \{c_{M,N}\}$ is natural. Now let $g = \{g_\alpha\}_{\alpha\in\pi} : M \to M'$ and $f = \{f_\beta\}_{\beta\in\pi} : N \to N'$ be left π-H-comodule morphisms. Then for any $\alpha, \beta \in \pi$, $m \in M_\alpha$ and $n \in N_\beta$, we have

$$\begin{aligned}
((f \otimes g)c_{M,N})_{\alpha\beta}(m \otimes n) &= (f_\beta \otimes g_{\beta^{-1}\alpha\beta})c_{M_\alpha,N_\beta}(m \otimes n)\\
&= (f_\beta \otimes g_{\beta^{-1}\alpha\beta})(n_{(0,\beta)} \otimes \xi_{\beta^{-1}}(m_{(0,\alpha)})\sigma_{\alpha,\beta}(m_{(-1,\alpha)}, n_{(-1,\beta)}))\\
&= f_\beta(n_{(0,\beta)}) \otimes g_{\beta^{-1}\alpha\beta}(\xi_{\beta^{-1}}(m_{(0,\alpha)}))\sigma_{\alpha,\beta}(m_{(-1,\alpha)}, n_{(-1,\beta)})\\
&= f_\beta(n)_{(0,\beta)} \otimes \xi_{\beta^{-1}}(g_\alpha(m)_{(0,\alpha)})\sigma_{\alpha,\beta}(g_\alpha(m)_{(-1,\alpha)}, f_\beta(n)_{(-1,\beta)})\\
&= c_{M'_\alpha,N'_\beta}(g_\alpha(m) \otimes f_\beta(n))\\
&= c_{M'_\alpha,N'_\beta}(g_\alpha \otimes f_\beta)(m \otimes n)\\
&= (c_{M',N'}(g \otimes f))_{\alpha\beta}(m \otimes n).
\end{aligned}$$

Hence $(f \otimes g)c_{M,N} = c_{M',N'}(g \otimes f)$. The proof is completed. $\square$

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