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Growth of Meromorphic Solutions of Complex Linear Differential-Difference Equations with Coefficients Having the Same Order

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Abstract The main purpose of this paper is to study the growth of meromorphic solutions of complex linear differential-difference equations

$$L(z,f)=\sum_{i=0}^{n}\sum_{j=0}^{m}A_{ij}(z)f^{(j)}(z+c_i)=0 \ \text{ or } \ F(z)$$

with entire or meromorphic coefficients, and $c_i, i=0,\ldots,n$ being distinct complex numbers, where there is only one dominant coefficient.

Keywords linear differential-difference equation; meromorphic solution; order; lower order.

MR(2010) Subject Classification 30D35; 34K06; 34K12

1. Introduction and main results

In this paper, we assume that the readers are familiar with the standard notations and basic results of Nevanlinna theory [1–3]. In the whole paper, let $f(z)$ be a meromorphic function in the whole complex plane. In addition, we use the notations $\sigma(f)$ and $\mu(f)$ to denote the order and the lower order of a meromorphic function $f(z)$ respectively, and the notations

$$\tau(f)=\overline{\lim_{r\to\infty}}\frac{\log M(r,f)}{r^{\sigma(f)}} \quad \text{and} \quad \underline{\tau}(f)=\underline{\lim_{r\to\infty}}\frac{\log M(r,f)}{r^{\mu(f)}}$$

to denote the type and the lower type of an entire function $f(z)$, respectively.

Recently, the research on analytic properties of meromorphic solutions of complex difference equations has become a subject of great interest from the viewpoint of Nevanlinna theory [4–11].

In particular, in 2007, Laine and Yang [10] considered complex linear difference equations and obtained the following theorem.

Theorem 1.1 ([10]) *Let $A_0(z),\ldots,A_n(z)$ be entire functions of finite order such that among*

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those having the maximal order $\sigma = \max_{0\le k\le n}\sigma(A_k)$, exactly one has its type strictly greater than the others. Then for any meromorphic solution $f(z)$ ($\not\equiv 0$) of

$$A_n(z)f(z+\omega_n)+\cdots+A_1(z)f(z+\omega_1)+A_0(z)f(z)=0, \tag{1.1}$$

where $\omega_1,\ldots,\omega_n$ are distinct complex numbers, we have $\sigma(f)\ge\sigma+1$.

In 2008, Tu and Yi [18] investigated the growth of solutions of a class of complex linear differential equations and obtained the following theorem.

Theorem 1.2 ([12]) *Let $A_j(z), j=0,\ldots,k-1$, be entire functions satisfying $\sigma(A_0)=\sigma$, $\tau(A_0)=\tau$, $0<\sigma<\infty, 0<\tau<\infty$, and let $\sigma(A_j)\le\sigma$, $\tau(A_j)<\tau$ if $\sigma(A_j)=\sigma, j\in\{1,\ldots,k-1\}$. Then for every solution $f(z)$ ($\not\equiv 0$) of*

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_0(z)f(z)=0, \tag{1.2}$$

we have $\sigma(f)=\infty, \sigma_2(f)=\sigma(A_0)$.

Note that in Theorems 1.1 and 1.2, when there is exactly one dominant coefficient among those coefficients having the same maximal order, we may get the growth relation between the solutions and the coefficients of complex linear difference equation (1.1) or complex linear differential equation (1.2). We proceed in this way by combining complex differentials with complex differences. In fact, we shall consider complex linear differential-difference equations as follows.

We denote

$$L(z,f)=\sum_{i=0}^{n}\sum_{j=0}^{m}A_{ij}(z)f^{(j)}(z+c_i),$$

where $A_{00}A_{n0}\not\equiv 0$ and consider complex linear differential-difference equations

$$L(z,f)=0 \tag{1.3}$$

and

$$L(z,f)=F(z), \tag{1.4}$$

where $c_i, i=0,\ldots,n$ are distinct complex numbers.

The first main aim of our paper is to investigate the growth of meromorphic solutions of the homogeneous equation (1.3) with some coefficients having the same maximal order or maximal lower order, and we obtain the following results.

Theorem 1.3 *Let $A_{ij}(z), i=0,\ldots,n, j=0,\ldots,m$ be entire functions such that there exists an integer l ($0\le l\le n$) satisfying*

$$\max\{\sigma(A_{ij}):(i,j)\ne(l,0)\}\le\sigma(A_{l0})<\infty, \tag{1.5}$$

and

$$\tau_1=\max\{\tau(A_{ij}):\sigma(A_{ij})=\sigma(A_{l0}),(i,j)\ne(l,0)\}<\tau(A_{l0}). \tag{1.6}$$

If $f(z)$ is a transcendental meromorphic solution of (1.3), then we have $\sigma(f)\ge\sigma(A_{l0})+1$.

Theorem 1.4 *Let $A_{ij}(z), i = 0, \dots, n, j = 0, \dots, m$ be entire functions such that there exists an integer l $(0 \le l \le n)$ satisfying*

$$\max\{\sigma(A_{ij}) : (i,j) \ne (l,0)\} \le \mu(A_{l0}) < \infty, \tag{1.7}$$

and

$$\tau_2 = \max\{\tau(A_{ij}) : \sigma(A_{ij}) = \mu(A_{l0}), (i,j) \ne (l,0)\} < \underline{\tau}(A_{l0}). \tag{1.8}$$

If $f(z)$ is a transcendental meromorphic solution of (1.3), then we have $\mu(f) \ge \mu(A_{l0}) + 1$.

Theorem 1.5 *Let H be a complex set satisfying $\overline{\text{logdens}}\{r = |z| : z \in H\} > 0$, and let $A_{ij}(z), i = 0, \dots, n, j = 0, \dots, m$ be entire functions. If there exist two constants α_1, α_2 $(0 < \alpha_2 < \alpha_1)$ and an integer l $(0 \le l \le n)$ such that for any given ε $(0 < \varepsilon < \alpha_1 - \alpha_2)$,*

$$|A_{l0}(z)| \ge \exp\{r^{\alpha_1 - \varepsilon}\}, \quad z \in H, \tag{1.9}$$

$$|A_{ij}(z)| \le \exp\{r^{\alpha_2}\}, \quad (i,j) \ne (l,0), \ z \in H, \tag{1.10}$$

then every transcendental meromorphic solution $f(z)$ of (1.3) satisfies $\sigma(f) \ge \alpha_1 + 1$.

Theorem 1.6 *Let $A_{ij}(z), i = 0, \dots, n, j = 0, \dots, m$ be entire functions. If there exists an integer l $(0 \le l \le n)$ satisfying $A_{l0}(z)$ is transcendental,*

$$\max\{\sigma(A_{ij}) : i = 0, \dots, n, j = 0, \dots, m\} \le \sigma(A_{l0}) < \infty,$$

and

$$\overline{\lim_{r \to \infty}} \frac{\sum\limits_{(i,j) \ne (l,0)} m(r, A_{ij})}{m(r, A_{l0})} < 1, \tag{1.11}$$

then every meromorphic solution $f(z)(\not\equiv 0)$ of (1.3) satisfies $\sigma(f) \ge \sigma(A_{l0}) + 1$.

Secondly, we consider the growth of meromorphic solutions of (1.3) with meromorphic coefficients and obtain the result as follows.

Theorem 1.7 *Let $A_{ij}(z), i = 0, \dots, n, j = 0, \dots, m$ be meromorphic functions such that there exists an integer l $(0 \le l \le n)$ satisfying*

$$\max\{\sigma(A_{ij}) : (i,j) \ne (l,0)\} < \sigma(A_{l0}) < \infty \quad \text{and} \quad \delta(\infty, A_{l0}) > 0.$$

If $f(z)(\not\equiv 0)$ is a meromorphic solution of (1.3), then we have $\sigma(f) \ge \sigma(A_{l0}) + 1$.

Thirdly, we turn to consider the growth of entire solutions of the non-homogeneous equation (1.4). Note that the above results may not be guaranteed now even if there is only one dominant coefficient. But we can obtain the similar results with some additional conditions.

Theorem 1.8 *Let $A_{ij}(z), i = 0, \dots, n, j = 0, \dots, m, F(z)$ be entire functions such that there exists an integer l $(0 \le l \le n)$ satisfying*

$$b = \max\{\sigma(A_{ij}), (i,j) \ne (l,0), \sigma(F)\} < \sigma(A_{l0}) < \infty. \tag{1.12}$$

If $A_{l0}(z)$ also satisfies one of the following conditions

(i) $\sigma(A_{l0}) < \frac{1}{2}$;

or (ii) $A_{l0}(z) = \sum_{n=0}^{\infty} c_{\lambda_n} z^{\lambda_n}$, *where the sequence of exponents* $\{\lambda_n\}$ *satisfies the Fabry gap condition:*

$$\frac{\lambda_n}{n} \to \infty; \tag{1.13}$$

or (iii)

$$T(r, A_{l0}) \sim \log M(r, A_{l0}), \quad r \to \infty, \ r \notin E, \tag{1.14}$$

where $E \subset (1, +\infty)$ *has finite logarithmic measure, then every transcendental entire solution* $f(z)$ *of (1.4) satisfies* $\sigma(f) \geq \sigma(A_{l0}) + 1$.

Finally, when the coefficients in (1.3) or (1.4) are polynomials, we obtain the following results with the similar method as the one in [5, 6].

Theorem 1.9 *Let* $A_{ij}(z), i = 0, \ldots, n, j = 0, \ldots, m, F(z)$ *be polynomials satisfying* $FA_{00}A_{n0} \not\equiv 0$. *Then every transcendental meromorphic solution* $f(z)$ *of the equation*

$$\sum_{i=0}^{n} \sum_{j=0}^{m} A_{ij}(z) f^{(j)}(z+i) = F(z) \tag{1.15}$$

satisfies $\sigma(f) \geq 1$.

Theorem 1.10 *Let* $A_{ij}(z), i = 0, \ldots, n, j = 0, \ldots, m$ *be polynomials satisfying* $A_{00}A_{n0} \not\equiv 0$ *and* $\sum_{i=0}^{n} A_{i0} \not\equiv 0$. *Then every transcendental meromorphic solution* $f(z)$ *of the equation*

$$\sum_{i=0}^{n} \sum_{j=0}^{m} A_{ij}(z) f^{(j)}(z+i) = 0 \tag{1.16}$$

satisfies $\sigma(f) \geq 1$. *Further, if*

$$\deg\Big(\sum_{i=0}^{n} A_{i0}\Big) = \max\{\deg(A_{ij}), \quad i = 0, \ldots, n, \ j = 0, \ldots, m\}, \tag{1.17}$$

then every meromorphic solution $f(z)(\not\equiv 0)$ *of the equation (1.16) satisfies* $\sigma(f) \geq 1$.

2. Lemmas for proofs of main results

Lemma 2.1 ([13]) *(i) Let* $f(z)$ *be a transcendental meromorphic function,* $\Gamma = \{(k_1, j_1), \ldots, (k_m, j_m)\}$ *be a finite set of distinct pair of integers which satisfy* $k_i > j_i \geq 0$ *for* $i = 1, \ldots, m$, *and let* $\alpha > 1$ *be a given real constant. Then there exists a set* $E \subset (1, +\infty)$ *that has finite logarithmic measure, and there exist constants* $A > 0$ *and* $B > 0$ *that depend only on* α *and* Γ, *such that for all* z *satisfying* $|z| \notin E \cup [0, 1]$ *and for all* $(k, j) \in \Gamma$, *we have*

$$\Big|\frac{f^{(k)}(z)}{f^{(j)}(z)}\Big| \leq B\Big(\frac{T(\alpha r, f)}{r} \log^{\alpha} r \log T(\alpha r, f)\Big)^{k-j}.$$

(ii) Let $f(z)$ *be a transcendental meromorphic function with* $\sigma(f) = \sigma < +\infty$, *and let* $\varepsilon > 0$ *be a given constant. Then there exists a set* $E_1 \subset (1, +\infty)$ *that has finite logarithmic measure, such that for all* z *satisfying* $|z| \notin E_1 \cup [0, 1]$ *and for all* $(k, j) \in \Gamma$, *we have*

$$\Big|\frac{f^{(k)}(z)}{f^{(j)}(z)}\Big| \leq |z|^{(k-j)(\sigma-1+\varepsilon)}.$$

Remark 2.2 It is shown in [14, P.66], that for an arbitrary complex number $c \neq 0$, the following inequalities

$$(1+o(1))T(r-|c|, f(z)) \leq T(r, f(z+c)) \leq (1+o(1))T(r+|c|, f(z))$$

hold as $r \to \infty$ for a general meromorphic function $f(z)$. Therefore, it is easy to obtain that

$$\sigma(f(z+c)) = \sigma(f), \quad \mu(f(z+c)) = \mu(f).$$

Lemma 2.3 ([7]) *Let $f(z)$ be a meromorphic function, $\eta(\neq 0)$, η_1, η_2 $(\eta_1 \neq \eta_2)$ be arbitrary complex numbers, and let $\gamma > 1$ and $\varepsilon > 0$ be given real constants. Then there exists a subset $E_2 \subset (1, +\infty)$ with finite logarithmic measure,*

(i) and a constant A depending only on γ and η, such that for all $|z| = r \notin E_2 \cup [0,1]$, we have

$$\big|\log|\frac{f(z+\eta)}{f(z)}|\big| \leq A\big(\frac{T(\gamma r, f)}{r} + \frac{n(\gamma r)}{r}\log^{\gamma} r \log^{+} n(\gamma r)\big);$$

(ii) and if in addition that $f(z)$ has finite order σ, and such that for all $|z| = r \notin E_2 \cup [0,1]$, we have

$$\exp\{-r^{\sigma-1+\varepsilon}\} \leq |\frac{f(z+\eta_1)}{f(z+\eta_2)}| \leq \exp\{r^{\sigma-1+\varepsilon}\}.$$

Lemma 2.4 ([11]) *Let $f(z)$ be a meromorphic function with $\mu(f) < +\infty$. Then for any given $\varepsilon > 0$, there exists a subset $E_3 \subset (1, +\infty)$ having infinite logarithmic measure such that for all $|z| = r \in E_3$, we have*

$$T(r, f) < r^{\mu(f)+\varepsilon}.$$

By Lemmas 2.1(i), 2.3(i) and 2.4, we obtain the following lemma.

Lemma 2.5 ([11]) *Let $f(z)$ be a transcendental meromorphic function with $\mu = \mu(f) < +\infty$, η_1, η_2 be distinct complex numbers, and let $\varepsilon(> 0)$ be given real constant. Then there exists a subset $E_4 \subset (1, +\infty)$ of infinite logarithmic measure such that for all $|z| = r \in E_4$ and for all $(k, j) \in \Gamma$, we have*

(i) $|\frac{f^{(k)}(z)}{f^{(j)}(z)}| \leq |z|^{(k-j)(\mu-1+\varepsilon)}$;

(ii) $\exp\{-r^{\mu-1+\varepsilon}\} \leq |\frac{f(z+\eta_1)}{f(z+\eta_2)}| \leq \exp\{r^{\mu-1+\varepsilon}\}$.

Lemma 2.6 ([7]) *Let η_1, η_2 be distinct complex numbers, and let $f(z)$ be a finite order meromorphic function. Let σ be the order of $f(z)$. Then for each $\varepsilon > 0$, we have*

$$m(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}) = O(r^{\sigma-1+\varepsilon}).$$

Lemma 2.7 ([15]) *Let $f(z)$ be an entire function of order $\sigma(f) = \sigma < \frac{1}{2}$ and denote $A(r) = \inf_{|z|=r} \log|f(z)|$, $B(r) = \sup_{|z|=r} \log|f(z)|$. If $\sigma < \alpha < \frac{1}{2}$, then*

$$\underline{\log \mathrm{dens}}\{r : A(r) > (\cos \pi\alpha)B(r)\} \geq 1 - \frac{\sigma}{\alpha}.$$

Lemma 2.8 ([16]) *Let $f(z)$ be an entire function with $\mu(f) = \mu < \frac{1}{2}$ and $\mu < \sigma = \sigma(f)$. If $\mu \leq \delta < \min\{\sigma, \frac{1}{2}\}$ and $\delta < \alpha < \frac{1}{2}$, then*

$$\overline{\log \mathrm{dens}}\{r : A(r) > (\cos \pi\alpha)B(r) > r^{\delta}\} > C(\sigma, \delta, \alpha),$$

where $C(\sigma,\delta,\alpha)$ is a positive constant depending only on σ, δ, α.

Lemma 2.9 ([17]) *Let $f(z)=\sum_{n=1}^{\infty} c_{\lambda_n} z^{\lambda_n}$ be an entire function of order $0<\sigma(f)<+\infty$. If the sequence of exponents $\{\lambda_n\}$ satisfies the Fabry gap condition (1.13), then for any $\beta<\sigma(f)$, there exists a set E_5 with positive upper logarithmic density such that for all $|z|=r\in E_5$, we have that $\log L(r,f)>r^{\beta}$, where $L(r,f)=\min_{|z|=r}|f(z)|$.*

Lemma 2.10 ([11]) *Let $f(z)$ be an entire function of order $0<\sigma(f)=\sigma<+\infty$. Then for any $\beta<\sigma$, there exists a set E_6 with positive upper logarithmic density such that for all $|z|=r\in E_6$, we have that $\log M(r,f)>r^{\beta}$, where $M(r,f)=\max_{|z|=r}|f(z)|$.*

Lemma 2.11 ([11]) *Let $g(r)$ and $h(r)$ be monotone nondecreasing functions on $[0,+\infty)$ such that $g(r)\le h(r)$ for all $r\notin E_7\cup[0,1]$, where $E_7\subset(1,+\infty)$ is a set of finite logarithmic measure. Let $\alpha>1$ be a given constant. Then there exists an $r_0=r_0(\alpha)>0$ such that $g(r)\le h(\alpha r)$ for all $r\ge r_0$.*

Lemma 2.12 ([19]) *Let $g(z)$ be a function transcendental and meromorphic in the complex plane of order less than 1. Let $h>0$. Then there exists an ε-set E_8 such that*

$$\frac{g'(z+c)}{g(z+c)}\to 0,\quad \frac{g(z+c)}{g(z)}\to 1 \text{ as } z\to\infty \text{ in } \mathbb{C}\backslash E_8,$$

uniformly in c for $|c|\le h$. Further, E_8 may be chosen so that for large z not in E_8, the function $g(z)$ has no zeros or poles in $|\zeta-z|\le h$.

Remark 2.13 ([11]) Following Hayman [20, P.75-76], we define an ε-set to be a countable union of open discs not containing the origin and satisfying the sum of subtending angles of these discs at the origin is finite. If E is an ε-set, then the set of $r\ge 1$ for which the circle $S(0,r)$ meets E has finite logarithmic measure, and for almost all real θ the intersection of E with the ray $\arg z=\theta$ is bounded.

3. Proofs of Theorems 1.3-1.10

Proof of Theorem 1.3 Suppose that $f(z)$ is a transcendental meromorphic solution of (1.3) satisfying $\sigma(f)<\sigma(A_{l0})+1<\infty$.

Set $\sigma_1=\max\{\sigma(A_{ij}):\sigma(A_{ij})<\sigma(A_{l0}),(i,j)\ne(l,0)\}<\sigma(A_{l0})$. In relation to (1.5) and (1.6), for any given $\varepsilon(>0)$ and sufficiently large r, we have that

$$|A_{ij}(z)|\le\exp\{r^{\sigma_1+\varepsilon}\},\quad \text{if } \ \sigma(A_{ij})<\sigma(A_{l0}), \tag{3.1}$$

and

$$|A_{ij}(z)|\le\exp\{(\tau_1+\varepsilon)r^{\sigma(A_{l0})}\},\quad \text{if } \ \sigma(A_{ij})=\sigma(A_{l0}),(i,j)\ne(l,0). \tag{3.2}$$

By Lemma 2.1 and Remark 2.2, there exists a subset $E_1\subset(1,+\infty)$ of finite logarithmic measure such that for all z satisfying $|z|=r\notin[0,1]\cup E_1$, we have

$$\left|\frac{f^{(j)}(z+c_i)}{f(z+c_i)}\right|\le r^{j(\sigma(f(z+c_i))-1)+\varepsilon}=r^{j(\sigma(f)-1)+\varepsilon},\quad i=0,\dots,n,\ j=1,\dots,m. \tag{3.3}$$

By Lemma 2.3, there exists a subset $E_2 \subset (1, +\infty)$ of finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_2$, we have

$$\left|\frac{f(z+c_i)}{f(z+c_l)}\right| \leq \exp\{r^{\sigma(f)-1+\varepsilon}\}, \quad i = 0, \ldots, n, \ i \neq l. \tag{3.4}$$

Then we can choose $\varepsilon(>0)$ sufficiently small to satisfy

$$\max\{\sigma_1, \sigma(f) - 1\} + 2\varepsilon < \sigma(A_{l0}), \quad \tau_1 + 2\varepsilon < \tau(A_{l0}). \tag{3.5}$$

Now, we divide (1.3) by $f(z+l)$ to get

$$-A_{l0}(z) = \sum_{\substack{i=0 \\ i\neq l}}^{n} \sum_{j=0}^{m} A_{ij}(z) \frac{f^{(j)}(z+c_i)}{f(z+c_i)} \frac{f(z+c_i)}{f(z+c_l)} + \sum_{j=1}^{m} A_{lj}(z) \frac{f^{(j)}(z+c_l)}{f(z+c_l)}. \tag{3.6}$$

Substituting (3.1)–(3.4) into (3.6) results in

$$\begin{aligned} M(r, A_{l0}) \leq & O(\exp\{r^{\sigma_1+\varepsilon}\} + \exp\{(\tau_1+\varepsilon) r^{\sigma(A_{l0})}\}) \cdot \exp\{r^{\sigma(f)-1+\varepsilon}\} \cdot \\ & r^{m(\sigma(f)-1)+\varepsilon}, \end{aligned} \tag{3.7}$$

where $|z| = r \notin [0,1] \cup E_1 \cup E_2$, and $|A_{l0}(z)| = M(r, A_{l0})$. Then, (3.5), (3.7) together with Lemma 2.11 imply that

$$\tau(A_{l0}) \leq \tau_1 + \varepsilon < \tau(A_{l0}) - \varepsilon,$$

a contradiction.

Therefore, $\sigma(f) \geq \sigma(A_{l0}) + 1$ holds. $\square$

Proof of Theorem 1.4 We use the method similar to the one in the proof of Theorem 1.3 here. Suppose that $f(z)$ is a transcendental meromorphic solution of (1.3) satisfying $\mu(f) < \mu(A_{l0}) + 1 < \infty$.

Set $\sigma_2 = \max\{\sigma(A_{ij}) : \sigma(A_{ij}) < \mu(A_{l0})(i,j) \neq (l,0)\} < \mu(A_{l0})$. In relation to (1.7) and (1.8), for any given $\varepsilon(>0)$ and sufficiently large r, we have that

$$|A_{ij}(z)| \leq \exp\{r^{\sigma_2+\varepsilon}\}, \quad \text{if} \ \ \sigma(A_{ij}) < \mu(A_{l0}), \tag{3.8}$$

and

$$|A_{ij}(z)| \leq \exp\{(\tau_2+\varepsilon) r^{\mu(A_{l0})}\}, \quad \text{if} \ \ \sigma(A_{ij}) = \mu(A_{l0}), (i,j) \neq (l,0). \tag{3.9}$$

By Lemma 2.5 and Remark 2.2, there exists a subset $E_3 \subset (1, +\infty)$ having infinite logarithmic measure such that for all $|z| = r \in E_3$,

$$\left|\frac{f^{(j)}(z+c_i)}{f(z+c_i)}\right| \leq r^{j(\mu(f)-1)+\varepsilon}, \quad i = 0, \ldots, n, \ j = 1, \ldots, m \tag{3.10}$$

and

$$\left|\frac{f(z+c_i)}{f(z+c_l)}\right| \leq \exp\{r^{\mu(f)-1+\varepsilon}\}, \quad i = 0, \ldots, n, \ i \neq l \tag{3.11}$$

hold simultaneously. Then we can choose $\varepsilon(>0)$ sufficiently small to satisfy

$$\max\{\sigma_2, \mu(f) - 1\} + 2\varepsilon < \mu(A_{l0}), \quad \tau_2 + 2\varepsilon < \underline{\tau}(A_{l0}). \tag{3.12}$$

Now, substituting (3.8)–(3.11) into (3.6) results in

$$M(r,A_{l0}) \le O(\exp\{r^{\sigma_2+\varepsilon}\}+\exp\{(\tau_2+\varepsilon)r^{\mu(A_{l0})}\})\cdot\exp\{r^{\mu(f)-1+\varepsilon}\}\cdot r^{m(\mu(f)-1)+\varepsilon}, \tag{3.13}$$

where $|z|=r\in E_3$, and $|A_{l0}(z)|=M(r,A_{l0})$. Then, (3.12), (3.13) imply that

$$\underline{\tau}(A_{l0}) \le \varliminf_{\substack{r\to\infty\\ r\in E_3}} \frac{\log M(r,A_{l0})}{r^{\mu(A_{l0})}} \le \tau_2+\varepsilon < \underline{\tau}(A_{l0})-\varepsilon,$$

a contradiction.

Therefore, $\mu(f)\ge\mu(A_{l0})+1$ holds. □

Proof of Theorem 1.5 Without loss of generality, we assume $f(z)$ to be a finite order transcendental meromorphic solution of (1.3).

Denote $H_1=\{r=|z|: z\in H\}$. Since $\overline{\log \mathrm{dens}} H_1>0$, H_1 is a set of r of infinite logarithmic measure. Clearly, (3.3) and (3.4) hold for all z satisfying $|z|=r\notin[0,1]\cup E_1\cup E_2$, where E_1 and E_2 are defined similarly as in the proof of Theorem 1.3. Substituting (1.9), (1.10), (3.3), (3.4) into (3.6) yields that

$$\exp\{r^{\alpha_1-\varepsilon}\}\le O\big(\exp\{r^{\alpha_2}\}\big)\cdot\exp\{r^{\sigma(f)-1+\varepsilon}\}\cdot r^{m(\sigma(f)-1)+\varepsilon},$$

where $z\in H$ and $|z|=r\in H_1\backslash([0,1]\cup E_1\cup E_2)$. Consequently, $\sigma(f)\ge\alpha_1+1$ holds by the assumption that $\alpha_1>\alpha_2$. □

Proof of Theorem 1.6 Without loss of generality, we assume $f(z)$ to be a finite order meromorphic solution to (1.3).

It follows by Lemma 2.6 that for sufficiently large r and any given $\varepsilon(>0)$,

$$m\big(r,\frac{f(z+c_i)}{f(z+c_l)}\big)=O(r^{\sigma(f)-1+\varepsilon}),\quad i=0,1,\ldots,n,\ i\ne l. \tag{3.14}$$

The logarithmic derivative lemma and Remark 2.2 result in

$$m\big(r,\frac{f^{(j)}(z+c_i)}{f(z+c_i)}\big)=O(\log r). \tag{3.15}$$

Substituting (3.14) and (3.15) into (3.6) yields that

$$m(r,A_{l0})\le \sum_{(i,j)\ne(l,0)} m(r,A_{ij})+O(r^{\sigma(f)-1+\varepsilon})+O(\log r). \tag{3.16}$$

Then (3.16) and the assumption (1.11) result in $\sigma(f)\ge\sigma(A_{l0})+1$. □

Proof of Theorem 1.7 Clearly, (1.3) has no nonzero rational solution. Now suppose that $f(z)$ is a transcendental meromorphic solution of (1.3) with $\sigma(f)<\infty$. Set

$$s=\max\{\sigma(A_{ij}):(i,j)\ne(l,0)\}<\sigma(A_{l0})=\sigma \quad\text{and}\quad \delta(\infty,A_{l0})=\delta>0. \tag{3.17}$$

Thus, we have $m(r,A_{l0})>\frac{1}{2}\delta T(r,A_{l0})$. By Lemma 2.6 and the logarithmic derivative lemma, we see that for any given $\varepsilon > 0$,

$$m\big(r, \frac{f(z+c_i)}{f(z+c_l)}\big) = O(r^{\sigma(f)-1+\varepsilon}), i \neq l, \quad m\big(r, \frac{f^{(j)}(z+c_i)}{f(z+c_i)}\big) = O(\log r), \ j \neq 0.$$

Thus, we have

$$\begin{aligned}\frac{1}{2}\delta T(r, A_{l0}) \leq & m(r, A_{l0}) \\ & \leq \sum_{j=0}^{m}\sum_{\substack{i=0\\ i\neq l}}^{n} m(r, A_{ij}) + \sum_{j=1}^{m} m(r, A_{lj}) + \sum_{j=1}^{m}\sum_{i=0}^{n} m\big(r, \frac{f^{(j)}(z+c_i)}{f(z+c_i)}\big) + \\ & \quad \sum_{\substack{i=0\\ i\neq l}}^{n} m\big(r, \frac{f(z+c_i)}{f(z+c_l)}\big) + O(1) \\ & \leq \sum_{j=0}^{m}\sum_{\substack{i=0\\ i\neq l}}^{n} T(r, A_{ij}) + \sum_{j=1}^{m} T(r, A_{lj}) + O(r^{\sigma(f)-1+\varepsilon}) + O(\log r). \end{aligned} \tag{3.18}$$

By (3.18), we can obtain that

$$\sigma(A_{l0}) \leq \max\{\sigma(A_{ij}), (i,j) \neq (l,0), \sigma(f) - 1 + \varepsilon\}. \tag{3.19}$$

Then (3.17) and (3.19) result in $\sigma(f) \geq \sigma(A_{l0}) + 1$. □

Proof of Theorem 1.8 Without loss of generality, we assume $f(z)$ to be a finite order transcendental entire solution of (1.4). We divide (1.4) by $f(z+c_l)$ to get

$$\begin{aligned} -A_{l0}(z) = & \sum_{\substack{i=0\\ i\neq l}}^{n}\sum_{j=0}^{m} A_{ij}(z)\frac{f^{(j)}(z+c_i)}{f(z+c_i)}\frac{f(z+c_i)}{f(z+c_l)} + \sum_{j=1}^{m} A_{lj}(z)\frac{f^{(j)}(z+c_l)}{f(z+c_l)} - \\ & \frac{F(z)}{f(z)} \cdot \frac{f(z)}{f(z+c_l)}. \end{aligned} \tag{3.20}$$

It follows by (1.12) that for all sufficiently large $|z| = r$,

$$|A_{ij}(z)| \leq \exp\{r^{b+\varepsilon}\}, \quad (i,j) \neq (l,0), \tag{3.21}$$

$$|F(z)| \leq \exp\{r^{b+\varepsilon}\}. \tag{3.22}$$

Since $M(r,f) > 1$, for all sufficiently large $r = |z|$, we have by (3.22) that

$$\big|\frac{F(z)}{M(r,f)}\big| \leq |F(z)| \leq \exp\{r^{b+\varepsilon}\}. \tag{3.23}$$

Moreover, (3.3), (3.4) hold for all $|z| = r \notin [0,1] \cup E_1 \cup E_2$.

(i) If $\sigma(A_{l0}) < \frac{1}{2}$, then by Lemmas 2.7 or 2.8, there exists a subset $E_4 \subset (1, +\infty)$ having infinite logarithmic measure such that for all $|z| = r \in E_4$, we have

$$|A_{l0}(z)| \geq \exp\{r^{\sigma(A_{l0})-\varepsilon}\}. \tag{3.24}$$

Substituting (3.3), (3.4), (3.21)–(3.24) into (3.20) yields that for all z satisfying $|z| = r \in E_4 \backslash ([0,1] \cup E_1 \cup E_2)$,

$$\exp\{r^{\sigma(A_{l0})-\varepsilon}\} \leq |A_{l0}(z)| \leq O(\exp\{r^{b+\varepsilon}\}) \cdot \exp\{r^{\sigma(f)-1+\varepsilon}\} \cdot r^{m(\sigma(f)-1)+\varepsilon}. \tag{3.25}$$

Then (3.25) results in $\sigma(f) \geq \sigma(A_{l0}) + 1$.

(ii) By using Lemma 2.9 instead of Lemmas 2.7 or 2.8 in the proof of (i), we can prove (ii) similarly.

(iii) Since $m(r, A_{l0}) \sim \log M(r, A_{l0})$ as $r \to \infty, r \notin E$, by the definition of $m(r, f)$, there exists a set $H \subset [0, 2\pi)$ having linear measure zero such that for all z satisfying $\arg z = \theta \in [0, 2\pi)\backslash H$, we have

$$|A_{l0}(re^{i\theta})| > M(r, A_{l0})^{1-\varepsilon}, \quad r \to \infty, \ r \notin E. \tag{3.26}$$

By Lemma 2.10, for any given $\varepsilon > 0$, there exists a set $E_5 \subset (1, +\infty)$ with positive upper logarithmic density, such that

$$M(r, A_{l0}) > \exp\{r^{\sigma(A_{l0})-(\varepsilon/2)}\}. \tag{3.27}$$

By (3.26) and (3.27), for any given $\varepsilon > 0$ and for all z satisfying $|z| = r \in E_5\backslash E$, and $\arg z = \theta \in [0, 2\pi)\backslash H$, we have

$$|A_{l0}(re^{i\theta})| > M(r, A_{l0})^{1-\varepsilon} > (\exp\{r^{\sigma(A_{l0})-(\varepsilon/2)}\})^{1-\varepsilon} > \exp\{r^{\sigma(A_{l0})-\varepsilon}\}. \tag{3.28}$$

Substituting (3.3), (3.4), (3.21)-(3.23) and (3.28) into (3.20) yields that for all $|z| = r \in E_5\backslash(E \cup E_1 \cup E_2)$, and $\arg z = \theta \in [0, 2\pi)\backslash H$, we have

$$\exp\{r^{\sigma(A_{l0})-\varepsilon}\} \leq |A_{l0}(re^{i\theta})| \leq O(\exp\{r^{b+\varepsilon}\}) \cdot \exp\{r^{\sigma(f)-1+\varepsilon}\} \cdot r^{m(\sigma(f)-1)+\varepsilon}. \tag{3.29}$$

Then (3.29) results in $\sigma(f) \geq \sigma(A_{l0}) + 1$. $\square$

Proof of Theorem 1.9 Without loss of generality, we suppose that $f(z)$ is a finite order transcendental meromorphic solution of (1.15). We divide this proof into the following two cases.

Case 1 Suppose that $f(z)$ has only finitely many poles. Now we suppose that $\sigma(f) < 1$, then $\sigma(f^{(k)}) < 1, k \in \mathbb{N}$. By Lemma 2.12, there exist ε-sets $G_j, j = 0, \ldots, m-1$ such that

$$f^{(j+1)}(z+i) = o(1)f^{(j)}(z+i), \quad i = 0, \ldots, n, \text{ as } z \to \infty \text{ in } \mathbb{C}\backslash G_j, \tag{3.30}$$

and

$$f^{(j)}(z+i) = f^{(j)}(z)(1+o(1)), \quad i = 1, \ldots, n, \text{ as } z \to \infty \text{ in } \mathbb{C}\backslash G_j. \tag{3.31}$$

Set $H_j = \{|z| = r > 1 : z \in G_j\}, j = 0, \ldots, m-1$, and $H = \bigcup_{j=0}^{m-1} H_j$. By Remark 2.13, $H_0, H_1, \ldots, H_{m-1}$ are of finite logarithmic measure, then so is H. By (1.15), we obtain that, as $|z| = r \notin H, z \to \infty$,

$$A_{00}(z)f(z) + \sum_{i=1}^{n} A_{i0}(z)(1+o(1))f(z) + \sum_{j=1}^{m}\sum_{i=0}^{n} A_{ij}(z)o(1)f(z) = F(z). \tag{3.32}$$

Then we have

$$f(z) = \frac{F(z)}{A_{00}(z) + \sum\limits_{i=1}^{n} A_{i0}(z)(1+o(1)) + \sum\limits_{j=1}^{m}\sum\limits_{i=0}^{n} A_{ij}(z)o(1)}. \tag{3.33}$$

Thus, noting that $f(z)$ has only finitely many poles, we deduce from (3.33) that when $|z| = r \notin [0,1] \cup H$,

$$\begin{aligned}T(r,f) =& m(r,f) + N(r,f) = m(r,f) + O(\log r)\\ =& m(r, \frac{F}{A_{00} + \sum\limits_{i=1}^{n} A_{i0}(1+o(1)) + \sum\limits_{j=1}^{m}\sum\limits_{i=0}^{n} A_{ij}o(1)}) + O(\log r)\\ \leq& T(r, \frac{F}{A_{00} + \sum\limits_{i=1}^{n} A_{i0}(1+o(1)) + \sum\limits_{j=1}^{m}\sum\limits_{i=0}^{n} A_{ij}o(1)}) + O(\log r)\\ \leq& T(r,F) + \sum_{j=0}^{m}\sum_{i=0}^{n} T(r, A_{ij}) + O(\log r) = O(\log r),\end{aligned}$$

which contradicts the fact that $f(z)$ is transcendental.

Hence, $\sigma(f) \geq 1$ holds for Case 1.

Case 2 Suppose that $f(z)$ is a meromorphic function with infinitely many poles. Since $F(z), A_{ij}(z), i = 0,\ldots,n, j = 0,\ldots,m$ are polynomials, we see that there is a constant $M > 0$ such that all zeros of $F(z), A_{ij}(z), i = 0,\ldots,n, j = 0,\ldots,m$ are in $E_6 = \{z : |\mathrm{Re}\, z| < M, |\mathrm{Im}\, z| < M\}$.

Set $D_1 = \{z : \mathrm{Re}\, z > M_1\}$, $D_2 = \{z : \mathrm{Re}\, z < -M_1\}$, $D_3 = \{z : \mathrm{Im}\, z > M_1\}$, $D_4 = \{z : \mathrm{Im}\, z < -M_1\}$, where $M_1 = M + n$. Since $f(z)$ has infinitely many poles, we see that there exists at least one of $D_j, j = 1,2,3,4$, say D_1, such that $f(z)$ has infinitely many poles in D_1. Suppose that a point $z_0 \in D_1$ satisfies $f(z_0) = \infty$. Then, we claim that there exist $i_1 \in \{1,\ldots,n\}$ and $j_1 \in \{0,\cdots,m\}$ such that $z_0+i_1 \in D_1$ and $f^{(j_1)}(z_0+i_1) = \infty$, then $f(z_0+i_1) = \infty$. Indeed, if not, then by (1.15), there must be some $j \in \{1,\ldots,m\}$ such that $f^{(j)}(z_0) = \infty$. However, $f^{(i)}(z)$ and $f^{(j)}(z)$ $(j \neq i)$ have different pole multiplicities at z_0, then (1.15) is a contradiction. Similarly, there is a sequence $\{i_d : d = 1,\ldots\}$ satisfying $i_d \in \{1,\ldots,n\}(d = 1,\ldots)$, $z_0 + i_1 + \cdots + i_d \in D_1$ and $z_0 + i_1 + \cdots + i_d$ are poles of $f(z)$. Since $|i_d| \leq n$ for $d = 1,\ldots$ and n is fixed, we see that $\lambda(\frac{1}{f}) \geq 1$, consequently $\sigma(f) \geq 1$.

If $f(z)$ has infinitely many poles in D_3 (or D_4), then we may use the similar method as above.

If $f(z)$ has infinitely many poles in D_2, then we can consider the other form of (1.15), that is

$$\sum_{i=0}^{n}\sum_{j=0}^{m} A_{ij}(z-n)f^{(j)}(z+i-n) = F(z-n),$$

and get a sequence $\{l_d : d = 1,\ldots\}$ satisfying $l_d \in \{-1,\ldots,-n\}$. So, $\lambda(\frac{1}{f}) \geq 1$, and $\sigma(f) \geq 1$.

Hence, $\sigma(f) \geq 1$ holds for Case 2. $\square$

Proof of Theorem 1.10 Suppose that $f(z)$ is a transcendental meromorphic solution of (1.16) with $\sigma(f) < 1$ and $d(\neq 0)$ is a constant. Set $g(z) = f(z) - d$, then $\sigma(g) = \sigma(f)$. Substituting

$f(z)=g(z)+d$ into (1.16) results in

$$\sum_{i=0}^{n}\sum_{j=0}^{m}A_{ij}(z)g^{(j)}(z+i)=-d\Big(\sum_{i=0}^{n}A_{i0}\Big). \tag{3.34}$$

Since $\sum_{i=0}^{n}A_{i0}\not\equiv 0$, we see that the coefficients of (3.34) satisfy the conditions of Theorem 1.9. Hence, if $f(z)$ is a transcendental meromorphic solution of (1.16), then $\sigma(f)\geq 1$.

Now, it suffices to prove that (1.16) has no nonzero rational solution, when (1.17) holds. Since $\sum_{i=0}^{n}A_{i0}\not\equiv 0$, we clearly know that (1.16) has no nonzero constant solution. Now we suppose that (1.16) has a non-constant rational solution

$$f(z)=\frac{c_m z^m+c_{m-1}z^{m-1}+\cdots+c_0}{d_s z^s+d_{s-1}z^{s-1}+\cdots+d_0}=az^{m-s}(1+o(1)), \tag{3.35}$$

where $a\neq 0, m, s\in\mathbb{N}, m+s\geq 1$. Then, we have

$$f(z+i)=az^{m-s}(1+o(1)),\quad i=1,\ldots,n, \tag{3.36}$$

and

$$f^{(j)}(z+i)=O(z^{m-s-j}),\quad i=0,\ldots,n,\ j=1,\ldots,m. \tag{3.37}$$

Substituting (3.35)-(3.37) into (1.16) results in

$$\Big(\sum_{i=0}^{n}A_{i0}\Big)az^{m-s}+\sum_{i=0}^{n}(A_{i0}o(az^{m-s}))+\sum_{j=1}^{m}\sum_{i=0}^{n}(A_{ij}O(z^{m-s-1}))=0.$$

Thus, we have

$$\Big(\sum_{i=0}^{n}A_{i0}\Big)+\sum_{i=0}^{n}(A_{i0}o(1))+\sum_{j=1}^{m}\sum_{i=0}^{n}(A_{ij}o(1))=0. \tag{3.38}$$

Since (1.17) holds, we see that (3.38) is a contradiction. Thus, (1.16) has no nonzero rational solution. That is, every meromorphic solution $f(z)(\not\equiv 0)$ of (1.16) satisfies $\sigma(f)>1$ when (1.17) holds. $\square$

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