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Weighted Sensitivity Minimization with a Stable Controller

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Abstract In this paper, we mainly consider weighted sensitivity minimization by a stable controller for linear time-invariant and time-varying systems. The problem is reduced to a distance problem from an operator to the nest algebra. And we give the existence and its computation of an optimal stable controller for the distance problem.

Keywords nest algebra; weighted sensitivity minimization; strong stabilization.

MR(2010) Subject Classification 93D99

1. Introduction

The sensitivity minimization problem was formulated by Zames [1] and solved in the scalar case by Zames and Francis [2,3]. The general sensitivity problem was solved by Doyle [4]. Vidyasagar studied the matrix case in [5]. The problem was introduced in the framework of nest algebra by Feintuch in [6].

In this paper, we mainly study the weighted sensitivity minimization problem in the framework of nest algebra. Different from the sensitivity minimization problem in [6], we consider the problem with a stable controller. Namely, the problem was studied in the condition of strong stabilization. It is based on the work in [7]. We reduce it to a distance problem from an operator to the nest algebra. For the distance problem, we give an optimal solution in the cases of linear time-invariant and time-varying systems.

The paper is organized as follows. In Section 2, we recall some notations, definitions and auxiliary properties. The weighted sensitivity minimization problem for linear time-varying and time-invariant systems are respectively studied in Section 3. Section 4 concludes the paper.

2. Preliminaries

In the following, some basic concepts will be introduced. Let $\mathcal{H}$ be a complex infinite-dimensional sequence space

$$\mathcal{H}=\{(x_0,x_1,x_2,\ldots):x_i\in\mathbb{C},\sum_{i=0}^{\infty}|x_i|^2<\infty\},$$

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where $|\cdot|$ denotes the standard Euclidean norm on $\mathbb{C}$. Obviously, $\mathcal{H}$ is a separable Hilbert space with the standard inner product $(x,y)=\sum_{i=0}^{\infty}x_i\bar{y}_i$.

Let $\mathcal{B}(\mathcal{H})$ denote the set of all bounded linear operators on $\mathcal{H}$. For $T\in\mathcal{B}(\mathcal{H})$, $\operatorname{Ran}T$ denotes the range $\{Tx : x\in\mathcal{H}\}$ of T and $\operatorname{Ker}T$ denotes the kernel $\{x\in\mathcal{H} : Tx=0\}$ of T.

Let $\mathcal{H}_e$ denote the extended space of $\mathcal{H}$

$$\mathcal{H}_e=\{(x_0,x_1,x_2,\ldots) : x_i\in\mathbb{C}\}.$$

For each $n\geq 0$, the standard truncation projection P_n on $\mathcal{H}$ and $\mathcal{H}_e$ is defined as

$$P_n(x_0,x_1,\ldots,x_n,x_{n+1},\ldots)=(x_0,x_1,\ldots,x_n,0,0,\ldots)$$

with $P_{-1}=0, P_\infty=I$. P_n sets all outputs after time n to zero, so the projection sequence $\{P_n\}_{n=-1}^{\infty}$ is crucial to the physical notion of causality for linear systems.

A linear transformation L on $\mathcal{H}_e$ is causal if $P_nL=P_nLP_n$ for all $-1\leq n\leq\infty$. A linear system on $\mathcal{H}_e$ is a causal linear transformation on $\mathcal{H}_e$, which is continuous with respect to the resolution topology. A linear system L is stable if its restriction to $\mathcal{H}$ is a bounded operator [6].

Obviously, the set of linear systems on $\mathcal{H}_e$ is an algebra with respect to the standard addition and multiplication. We denote this algebra by $\mathcal{L}$. It is easy to check that any element in $\mathcal{L}$ is a lower triangular matrix with respect to the standard basis. The set of stable linear systems on $\mathcal{H}_e$, denoted by $\mathcal{S}$, is a weakly closed algebra containing the identity, referred to in the operator theory literature as a nest algebra.

A nest is a chain $\mathcal{N}$ of closed subspaces of a Hilbert space $\mathcal{H}$ containing $\{0\}$ and $\mathcal{H}$ which is closed under intersection and closed span. The following is the nest algebra determined by $\mathcal{N}$

$$\mathcal{T}(\mathcal{N})=\{T\in\mathcal{B}(H) : TN\subseteq N, \forall N\in\mathcal{N}\}.$$

By the fact that $\mathcal{S}$ is a nest algebra determined by the nest $\{I-P_n : -1\leq n\leq\infty\}$, and Corollary 4.2.6 in [6], the following theorem holds.

Theorem 2.1([6]) *Every operator $T\in\mathcal{S}$ has a factorization $T=UA$ where U is inner and A is outer.*

From above we know that all operators in $\mathcal{S}$ have inner-outer factorizations in the discrete time case. In continuous time this is not always so but we will assume such factorizations without getting into conditions required for their existence. Also, it is standard to assume that the outer operator A from $\mathcal{S}$ satisfies that $A\mathcal{S}=\{AQ : Q\in\mathcal{S}\}$ is dense in $\mathcal{S}$, this holds in particular if A is invertible.

Let $L,C\in\mathcal{L}$. We consider the standard feedback configuration in Figure 1 with plant L and compensator C.

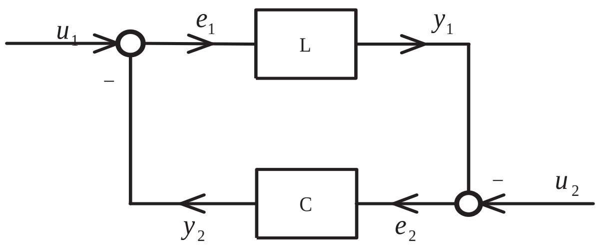

Figure 1 Standard feedback configuration

The closed loop system equation is

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} I & C \\ L & -I \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

The system $\{L, C\}$ is well posed if $\left[\begin{smallmatrix} I & C \\ L & -I \end{smallmatrix}\right]$ is invertible. This inverse is given by the transfer matrix

$$H(L,C) = \begin{bmatrix} (I+CL)^{-1} & C(I+LC)^{-1} \\ L(I+CL)^{-1} & -(I+LC)^{-1} \end{bmatrix}.$$

Let $\mathcal{D}(L) = \{u \in \mathcal{H} : Lu \in \mathcal{H}\}$ and $\mathcal{D}(C) = \{u \in \mathcal{H} : Cu \in \mathcal{H}\}$. Then $\left[\begin{smallmatrix} I & C \\ L & -I \end{smallmatrix}\right]$ can be seen as a linear transformation from $\mathcal{D}(L) \bigoplus \mathcal{D}(C)$ into $\mathcal{H} \bigoplus \mathcal{H}$.

Definition 2.5 ([6]) *The closed loop system $\{L, C\}$ is stable if all the entries of $H(L,C)$ are stable systems on $\mathcal{H}$. The plant L is stabilizable if there exists $C \in \mathcal{L}$ such that the closed loop system $\{L, C\}$ is stable. The plant L is strongly stabilizable if $C \in \mathcal{S}$.*

In order to characterize the stability, we need the notions of strong representations for linear systems. The graph of a linear transformation L with domain $\mathcal{D}(L)$ in $\mathcal{H}$ is $G(L) = \left\{\left[\begin{smallmatrix} x \\ Lx \end{smallmatrix}\right] : x \in \mathcal{D}(L)\right\}$.

Definition 2.6 ([6]) $\left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right]$ *is a strong right representation of $L \in \mathcal{L}$ if*

(1) $M, N \in \mathcal{S}$ and there exist $X, Y \in \mathcal{S}$ such that $[Y, X]\left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right] = I$,

(2) $G(L) = \mathrm{Ran}\left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right]$.

$[-\hat{N}, \hat{M}]$ is a strong left representation of $L \in \mathcal{L}$ if

(1) $\hat{M}, \hat{N} \in \mathcal{S}$ and there exist $\hat{X}, \hat{Y} \in \mathcal{S}$ such that $[-\hat{N}, \hat{M}]\left[\begin{smallmatrix} -\hat{X} \\ \hat{Y} \end{smallmatrix}\right] = I$,

(2) $G(L) = \mathrm{Ker}[-\hat{N}, \hat{M}]$.

The following result on strong representation is also important in the analysis of systems.

Theorem 2.2 ([6]) *Suppose $M, N \in \mathcal{S}$. Then $\left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right]$ is a strong right representation of $L \in \mathcal{L}$ if and only if*

(1) there exist $X, Y \in \mathcal{S}$ such that $[Y, X]\left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right]$,

(2) M is invertible in $\mathcal{L}$.

From the proof of above theorem, we know that $\begin{bmatrix} M \\ N \end{bmatrix}$ is a strong right representation of L if and only if NM^{-1} is a right coprime factorization for L. Similarly, $[-\hat{N}, \hat{M}]$ is a strong left representation of L if and only if $\hat{M}^{-1}\hat{N}$ is a left coprime factorization for L.

3. Sensitivity minimization

In the section, we introduce the sensitivity minimization problem [6, Chapter 7]. Suppose $\{L, C\}$ is stable. The sensitivity function of the system is $S(z) = [I+L(z)C(z)]^{-1}$. The sensitivity minimization problem is:

$$r_0 = \inf\{\|S(z)\|_\infty : C \in \mathcal{S}(L)\}$$

We want to design a stable C so that the closed loop system is stable and the effect of d on y is minimized. The operator $S_W = (I + LC)^{-1}W$ on H^2 is called the weighted sensitivity operator for the system described in Figure 2.

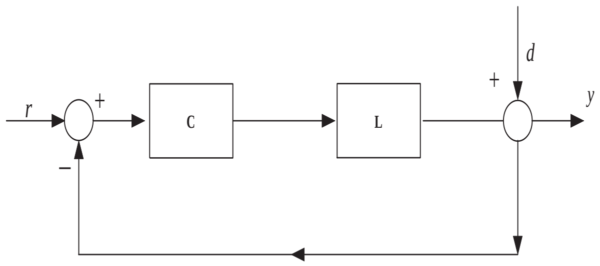

Figure 2 Standard feedback configuration with outside disturbance d.

Then the weighted sensitivity minimization problem is:

$$\inf\{\|S_W\| : C \in \mathcal{S}(L)\}.$$

3.1. Cases of the time-varying strong stabilization

Now we consider the weighted sensitivity minimization problem in the case of the time-varying strong stabilization. Firstly, we give the inner/outer factorization of L and W. Let $L = L_i L_o$, $W^* = W_i^* W_o^*$. Suppose L_o and W_o are invertible, then for the weighted sensitivity minimization problem, it has the optimal solution in the condition of time-varying strong stabilization.

Theorem 3.1.1 *Let $L \in \mathcal{L}$ admit a left coprime factorization $B^{-1}A$, where $A,\ B \in \mathcal{S}$, and $\begin{bmatrix} X \\ Y \end{bmatrix}$ is a right inverse of $[A, B]$ over $\mathcal{S}$, i.e., $AX + BY = I$. If A is a compact operator, then for the weighted sensitivity minimization problem, there exists a $\widehat{Q} \in \mathcal{S}$, such that the following equation holds:*

$$\gamma = \inf_{Q\in\mathcal{S}} \|(YB + LQ)W\| = \|YBW + L_i\widehat{Q}W_i\|.$$

Proof Since the outer operator L_o and co-outer operator W_o are invertible in $\mathcal{S}$, the mapping

$Q \to L_oQW_o$ is bijective on $\mathcal{S}$ and therefore

$$\gamma = \inf_{Q\in\mathcal{S}} \|(YB + LQ)W\| \to \inf_{Q\in\mathcal{S}} \|YBW + L_iQW_i\|.$$

By the fact that $\mathcal{S}$ is weakly closed [8] and L_i is an isometry, W_i is a co-isometry, we show that $L_i\mathcal{S}W_i$ is a weakly closed subspace in $\mathcal{S}$. In fact, if $\{X_k\}$ is a net in $\mathcal{S}$ such that $\{L_iX_kW_i\}$ converges weakly to Y in $\mathcal{S}$, then $L_i^*L_iX_kW_iW_i^* = X_k$ converges weakly to $L_i^*YW_i^*$. Since $\mathcal{S}$ is weakly closed, $X = L_i^*YW_i^* \in \mathcal{S}$. For $\forall\ x,\ y \in \mathcal{H}$, let $u = W_ix$, $v = L_i^*y$. We have $(X_ku, v) \longrightarrow (L_i^*YW_i^*u, v) = (Xu, v)$. Then $(L_iX_kW_ix, y) = (X_kW_ix, L_i^*y) \longrightarrow (XW_ix, L_i^*y) = (L_iXW_ix, y)$. Thus $\{L_iX_kW_i\}$ converges weakly to $L_iXW_i = Y$ which belongs to $L_i\mathcal{S}N_i$.

Now $\gamma = d(YBW, L_i\mathcal{S}W_i)$, hence there exists a net $\{Z_k\}$ in $L_i\mathcal{S}W_i$ such that $\|YBW - Z_k\| \to \gamma$. It follows that $\{Z_k\}$ is bounded and therefore weakly compact. Hence there exists a subnet converging weakly to some $Z \in L_i\mathcal{S}W_i$. Thus $Z = -L_i\widehat{Q}W_i$ for some $\widehat{Q} \in \mathcal{S}$, and

$$\gamma = \|YBW - Z\| = \|YBW + L_i\widehat{Q}W_i\|.$$

The proof is completed. $\Box$

In the following, we will give another solution to the sensitivity minimization problem. Note that the operator $\begin{bmatrix} L_i^* \\ I-L_iL_i^* \end{bmatrix}$ and $[\,W_i^*,\quad I - W_i^*W_i\,]^*$ are isometries, then we have

$$\begin{aligned}
\gamma &= \inf_{Q\in\mathcal{S}} \|YBW + LQW\| = \inf_{Q\in\mathcal{S}} \|YBW + L_iL_oQW_oW_i\| \\
&= \inf_{Q\in\mathcal{S}} \left\| \begin{bmatrix} L_i^* \\ I - L_iL_i^* \end{bmatrix} (YBW + L_iL_oQW_oW_i) \begin{bmatrix} W_i^*, & I - W_i^*W_i \end{bmatrix} \right\| \\
&= \inf_{Q\in\mathcal{S}} \left\| \begin{bmatrix} L_i^*YBW_o + L_oQW_o & 0 \\ (I - L_iL_i^*)YBW_o & 0 \end{bmatrix} \right\| = \inf_{Q\in\mathcal{S}} \left\| \begin{bmatrix} L_i^*YBW_o + L_oQW_o \\ (I - L_iL_i^*)YBW_o \end{bmatrix} \right\|.
\end{aligned}$$

Since L_o and W_o are invertible under the standard assumption, we have

$$\gamma = \inf_{Z\in\mathcal{S}} \left\| \begin{bmatrix} L_i^*YBW_o + Z \\ (I - L_iL_i^*)YBW_o \end{bmatrix} \right\|,$$

where $Z = L_oQW_o$.

By Corollary 7.4.3 ([6, Chapter 7]), the following theorem is given.

Theorem 3.1.2 *For weighted sensitivity minimization problem, the identity holds in the above assumption*

$$\gamma = \sup_n \left\| \begin{bmatrix} P_nL_i^*YBW_o(I - P_n) \\ (I - L_iL_i^*)YBW_o(I - P_n) \end{bmatrix} \right\|.$$

Note that the norm in Theorem 3.1.2 is the induced Hilbert space operator norm, we obtain

$$\gamma^2 \le \sup_n\{\|P_nL_i^*YBW_o(I - P_n)\|^2\} + \sup_n\{\|(I - L_iL_i^*)YBW_o(I - P_n)\|^2\}.$$

3.2. Cases of the time-invariant strong stabilization

In the next, we mainly consider weighted sensitivity minimization problem for linear time-invariant systems. The weighted sensitivity minimization problem considered here is: Find

$$\gamma = \inf\{\|S_{WC}\| : \ C \in \mathcal{S}\}.$$

Applying the formulas for C obtained in [7] to S_{WC}, we have the following result.

Theorem 3.2.1 *Suppose $L \in \mathcal{T}$ has a left coprime factorization $B^{-1}A$, where $A,\ B \in \mathcal{S}$, and $\begin{bmatrix} X \\ Y \end{bmatrix}$ is a right inverse of $[A, B]$ over $\mathcal{S}$, i.e., $AX + BY = I$. If A is a compact operator, then the weighted sensitivity minimization problem can be converted to the following distance problem:*

$$\begin{aligned} \gamma &= \inf\{\|T - UQ\| : Q \in H^\infty\} = \inf\{\|\bar{U}T - Q\| : Q \in H^\infty\} \\ &= d(\bar{U}T, H^\infty) = \|H_{\bar{U}T}\|. \end{aligned}$$

Proof As A is a compact operator, by Theorem 3.1 in [7], we know that L can be stabilized by a stable controller C, i.e., $C \in \mathcal{S}$, and C admits the following right coprime factorization $C = (X - Q)(Y + LQ)^{-1}$, where $Q \in \mathcal{S}$. Thus

$$\begin{aligned} I + LC &= I + B^{-1}A(X - Q)(Y + LQ)^{-1} \\ &= B^{-1}[B(Y + LQ) + A(X - Q)](Y + LQ)^{-1} \\ &= B^{-1}[AX + BY + (BL - A)Q](Y + LQ)^{-1} \\ &= B^{-1}[I + (BB^{-1}A - A)Q](Y + LQ)^{-1} \\ &= B^{-1}(Y + LQ)^{-1}, \end{aligned}$$

which implies

$$(I + LC)^{-1} = (Y + LQ)B.$$

The weighted sensitivity minimization problem then becomes: Find

$$\inf\{\|YBW + B^{-1}AQW\| : Q \in H^\infty\}.$$

Because H^∞ is commutative, we have $B^{-1}AQW = B^{-1}AWQ$. Let U be the inner function of $-B^{-1}AW$, and V its outer function. Since $\{VQ : Q \in H^\infty\}$ is dense in H^∞, we can rewrite the problem as: Find

$$\inf\{\|T - UQ\| : Q \in H^\infty\},$$

where $T = YBW$.

Since U is inner, $|T - UQ| = |\bar{U}||T - UQ| = |\bar{U}T - Q|$, where $T, Q \in H^\infty$. Thus, $\|T - UQ\|_\infty = \|\bar{U}T - Q\|_\infty$, and

$$\inf\{\|T - UQ\| : Q \in H^\infty\} = \inf\{\|\bar{U}T - Q\| : Q \in H^\infty\} = d(\bar{U}T, H^\infty).$$

By Theorem 3.2.10 in [6], this is just $\|H_{\bar{U}T}\|$, the norm of the Hankel operator induced by $\bar{U}T \in L^\infty$. $\square$

Using the scalar commutant lifting Theorem, we compute the sensitivity problem in the strong stabilization condition. The following theorem is given.

Theorem 3.2.2 *There exists a $\hat{Q} \in H^\infty$, such that*

$$\gamma = \inf\{\|T - UQ\| : \ Q \in H^\infty\} = \|T - U\hat{Q}\|_\infty.$$

4. Conclusion

In this paper, we mainly consider the weighted sensitivity problem with stable controllers. We reduce it to a distance problem. For the distance problem, we give an optimal solution in the cases of linear time-invariant and time-varying systems.

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