A Note on Almost Topological Groups

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Abstract In this paper, we mainly discuss some generalized metric properties and the cardinal invariants of almost topological groups. We give a characterization for an almost topological group to be a topological group and show that: (1) Each almost topological group that is of countable τ-character is submetrizable; (2) Each left λ-narrow almost topological group is λ-narrow; (3) Each separable almost topological group is ω-narrow. Some questions are posed.

Keywords almost topological group; submetrizable; λ-narrow; separable

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1. Introduction

All spaces are $T_2$ unless stated otherwise. We denote by $\mathbb{N}$ the set of all natural numbers and $\omega = \mathbb{N} \cup \{0\}$. The letter $e$ denotes the neutral element of a group. Readers may refer [3, 5, 7] for notations and terminology not explicitly given here.

A group $G$ endowed with a topology $\tau$ is called a semitopological group if the left and right translations of $G$ are continuous. We also say that $G$ is a paratopological group if the multiplication in $G$ is continuous as a mapping of $G \times G$ into $G$, where $G \times G$ is given product topology. A topological group is a paratopological group with continuous inversion. Obviously, each topological group is a paratopological group, and each paratopological group is a semitopological group. Paratopological groups were discussed and many results have been obtained in [1, 3, 8, 9, 11–14].

In the class of paratopological groups, it is well known that the closure of a subgroup of a paratopological group is not necessarily a subgroup. Therefore, Fernández in [6] introduced some class of paratopological groups (that is, almost topological groups) such that the closure of each subgroup of arbitrary such paratopological group must be a subgroup. In this paper, we shall discuss some generalized metric properties and the cardinal invariants of almost topological groups.

2. Preliminaries
Definition 2.1 ([3]) Let \( \lambda \) be a cardinal. A subset \( H \) of a semitopological group is left \( \lambda \)-narrow (resp., right \( \lambda \)-narrow) if for every open neighborhood \( U \) of the neutral element \( e \) in \( G \), there exists a subset \( F \) of \( H \) such that \( |F| \leq \lambda \) and \( H \subset FU \) (resp., \( H \subset UF \)). A subset \( H \) of a semitopological group is \( \lambda \)-narrow if it is left \( \lambda \)-narrow and right \( \lambda \)-narrow.

Definition 2.2 ([15]) A semitopological group is left precompact (resp., right precompact) if for each open neighborhood \( U \) of the neutral element \( e \) in \( G \), there exists a finite set \( A \subset G \) such that \( AU = G \) (resp., \( UA = G \)). A semitopological group is precompact if it is left precompact and right precompact.

Remark 2.3 Recently, the following results have been obtained:

1. Every left precompact paratopological group is right precompact [17];
2. Every left \( \omega \)-narrow Baire paratopological group is \( \omega \)-narrow [15];
3. A dense subgroup of a precompact paratopological group is precompact [18].

However, in the class of topological groups, the following results are well known:

(i) Every left \( \omega \)-narrow topological group is \( \omega \)-narrow;
(ii) The subgroup \( H \) of an \( \omega \)-narrow topological group is \( \omega \)-narrow [3].

Definition 2.4 ([6]) An almost topological group is a paratopological group \((G, \tau)\) which satisfies the following conditions:

(a) The group \( G \) admits a Hausdorff topological group topology \( \gamma \) weaker than \( \tau \), and

(b) There exists a local base \( \mathcal{B} \) at the neutral element \( e \) of the paratopological group \((G, \tau)\) such that the set \( V = U \setminus \{e\} \) is open in \((G, \gamma)\) for each \( U \in \mathcal{B} \).

We will say that \( G \) is an almost topological group with structure \((\tau, \gamma, \mathcal{B})\).

Remark 2.5 (1) It is easy to check that Sorgenfrey line is an almost topological group. However, Sorgenfrey line is not a topological group.

(2) The closure of any subgroup of the product of a family of almost topological groups is a subgroup [6].

(3) Any discrete subgroup of a product of a family of almost topological groups is closed [6].

Recall that a family \( \mathcal{U} \) of non-empty open sets of a space \( X \) is called a \( \pi \)-base at a point \( x \) if for each non-empty open neighborhood \( V \) of \( x \) in \( X \), there exists \( U \in \mathcal{U} \) such that \( U \subset V \).

The \( \pi \)-character of \( x \) in \( X \) is defined by

\[
\pi \chi(x, X) = \min\{|\mathcal{U}| : \mathcal{U} \text{ is a local } \pi \text{-base at } x \text{ in } X\}.
\]

The \( \pi \)-character of \( X \) is defined by

\[
\pi \chi(X) = \sup\{\pi \chi(x, X) : x \in X\}.
\]

The character of \( x \) in \( X \) is defined by

\[
\chi(x, X) = \min\{|\mathcal{U}| : \mathcal{U} \text{ is a neighborhood base at } x \text{ in } X\}.
\]
The character of $X$ is defined by
\[ \chi(X) = \sup \{ \chi(x, X) : x \in X \}. \]
The density of $X$ is defined by
\[ d(X) = \min \{|F| : F \subset X, F = X\}. \]
The cellularity of $X$ is defined by
\[ c(X) = \sup \{|\mathcal{U}| : \mathcal{U} \text{ is a disjoint family of open subsets of } X\}. \]

3. Generalized metric properties on almost topological groups

First, we shall give a condition under which an almost topological group is a topological group.

**Proposition 3.1** A non-discrete almost topological group $G$ is a topological group if and only if $G$ satisfies the following $(\Diamond)$:

$(\Diamond)$ For each open neighborhood $U$ of the neutral element $e$ there exist a point $y \in U \setminus \{e\}$ and an open neighborhood $V$ of $e$ such that $e \in yV \subset U$.

**Proof** Let $G$ be an almost topological group with structure $(\tau, \gamma, \mathcal{B})$.

Let $(G, \tau)$ be a topological group. For each open neighborhood $U$ of $e$ in $\tau$, there exists a symmetric open neighborhood $V$ of $e$ in $\tau$ such that $e \in V^2 \subset U$. Since $(G, \tau)$ is non-discrete, there exists a point $y \in V \setminus \{e\}$. Obviously, we have $e \in yV \subset U$.

Conversely, it suffices to show that $e \in \text{int}_\tau(U^{-1})$ for each open neighborhood $U$ of $e$ in $\tau$. By $(\Diamond)$, there exist a point $y \in U \setminus \{e\}$ and an open neighborhood $V$ of $e$ in $(G, \tau)$ such that $e \in yV \subset U$. We can assume that $V \in \mathcal{B}$, then $e \in y(V \setminus \{e\}) \subset U$. Since $V \in \mathcal{B}$, we know that $y(V \setminus \{e\})$ is an open neighborhood of $e$ in $(G, \gamma)$. Therefore, $(y(V \setminus \{e\}))^{-1}$ is also a neighborhood of $e$ in $\gamma$, thus $e \in \text{int}_\gamma((y(V \setminus \{e\}))^{-1})$. Since $\gamma \subset \tau$, the set $\text{int}_\gamma((y(V \setminus \{e\}))^{-1}) \subset U^{-1}$ is an open neighborhood of $e$ in $(G, \tau)$. Therefore, $e \in \text{int}_\tau(U^{-1})$. \[\square\]

**Example 3.2** There exists a Hausdorff paratopological group $G$ which satisfies $(\Diamond)$. However, it is not a topological group.

**Proof** Consider the additive group $(\mathbb{R}, +)$. Fix a natural number $k$ and put $U_n(k) = k(\mathbb{N} \cup \{0\}) + (-\frac{1}{k}, \frac{1}{k})$ for each $n \in \mathbb{N}$. Let $\mathcal{U} = \{U_n(k) : k, n \in \mathbb{N}\}$. Then there exists a topology $\sigma$ on $\mathbb{R}$ such that $G = (\mathbb{R}, \sigma)$ is a Hausdorff paratopological group and the family $\mathcal{U}$ is a local base at $0$ in $G$, see [10]. Obviously, $G$ is not a topological group and satisfies $(\Diamond)$. \[\square\]

The following question is still open in the class of paratopological groups.

**Question 3.3** ([2, Problem 20]) Is every regular first countable paratopological group submetrizable?

However, in the class of almost topological groups, the following theorem gives a positive answer to Question 3.3.
Theorem 3.4  Let $G$ be an almost topological group that is of countable $\pi$-character. Then $G$ is submetrizable.

Proof  Let $G$ be an almost topological group with structure $(\tau, \gamma, B)$, and let $\{U_n : n \in \omega\}$ be a countable $\pi$-base at the neutral element $e$ in $(G, \tau)$. If $G$ is discrete, then it is obvious that $G$ is submetrizable. Therefore, we may assume that $G$ is non-discrete. For each $n \in \omega$, take $x_n \in U_n$. Then we can find $B_n \in B$ such that $x_n B_n \subset U_n$ since $(G, \tau)$ is a paratopological group. Note that $G$ is a non-discrete almost topological group, hence the set $B_n \setminus \{e\}$ is a non-empty open set in $(G, \gamma)$ for each $n \in \omega$. So $x_n (B_n \setminus \{x_n\})$ is also an open set in $(G, \gamma)$. Then the family $V = \{x_n B_n \setminus \{x_n\} : n \in \omega\}$ is countable. We claim that $V$ is a $\pi$-base at the neutral element $e$ of $(G, \gamma)$. Indeed, let $W$ be an arbitrary open neighbourhood of the neutral element $e$ in $(G, \gamma)$. Clearly, $W$ is also an open neighbourhood of the neutral element $e$ in $(G, \tau)$, hence there exists $n \in \omega$ such that $U_n \subset W$. Therefore, we have $x_n B_n \setminus \{x_n\} \subset x_n B_n \subset U_n \subset W$. Thus $V$ is a $\pi$-base at the neutral element $e$ of $(G, \gamma)$. It is well known that a Hausdorff topological group with a countable $\pi$-character is metrizable, so $G$ is submetrizable. □

By the proof of Theorem 3.4, we have the following.

Corollary 3.5  If $G$ is an almost topological group with structure $(\tau, \gamma, B)$, then $\pi\chi(G, \gamma) \leq \pi\chi(G, \tau)$.

However, the following question is still open.

Question 3.6  Let $G$ be an almost topological group with structure $(\tau, \gamma, B)$. Does the equation $\chi(G, \tau) = \chi(G, \gamma)$ hold?

The following question is posed by Liu and Lin.

Question 3.7  (14, Question 2.2) Let $G$ be a first-countable paratopological group. If $G$ is a $p$-space, is $G$ developable?

A space $X$ is a w\(\Delta\)-space [4] if there exists a sequence $\{\mathcal{H}_n\}$ of open covers of $X$ such that if $x_n \in \text{st}(x, \mathcal{H}_n)$ for each $n \in \mathbb{N}$, then the set $\{x_n : n \in \mathbb{N}\}$ has a cluster point in $X$.

Definition 3.8  ([19]) Let $X$ be a space and $\{\mathcal{P}_n\}_n$ a sequence of collections of open subsets of $X$.

(1) $\{\mathcal{P}_n\}_n$ is called a development for $X$ if $\{\text{st}(x, \mathcal{P}_n)\}_n$ is a neighborhood base at $x$ in $X$ for each point $x \in X$.

(2) $X$ is called developable, if $X$ has a development.

(3) $X$ is called Moore, if $X$ is regular and developable.

Clearly, each developable space is a w\(\Delta\)-space.

The following corollary gives a partial answer to Question 3.7.

Corollary 3.9  Let $G$ be an almost topological group that is of countable $\pi$-character. If $G$ is a w\(\Delta\)-space, then $G$ is developable.

Proof  It follows from Theorem 3.4 that $G$ is submetrizable. Then $(G, \tau)$ is developable since $G$
is a \( \kappa \Delta \)-space [7]. □

It is well known that a topological group with countable pseudocharacter is submetrizable. Moreover, the authors in [11] have given a paratopological group which is of countable pseudocharacter and non-submetrizable. Therefore, we have the following question.

**Question 3.10** Let \( G \) be an almost topological group that is of countable pseudocharacter. Is \( G \) submetrizable?

The authors in [11] showed that a Moore paratopological group needs not be metrizable. Indeed, that paratopological group is an almost topological group. However, the following questions are still open in the class of paratopological groups.

**Question 3.11** Let \( G \) be a regular paratopological group or an almost topological group. If \( G \) is regular and has a uniform base, is \( G \) metrizable?

**Question 3.12** Let \( G \) be a paratopological group or an almost topological group. If \( G \) is regular and has a point-countable base, is \( G \) metrizable?

4. Some cardinal invariants of almost topological groups

The following question was posed by Guran.

**Question 4.1** ([15, Question 2.2]) Is every left \( \omega \)-narrow paratopological group right \( \omega \)-narrow?

The following theorem gives a partial answer to Question 4.1.

**Theorem 4.2** Let \( G \) be an almost topological group with structure \((\tau, \gamma, \mathcal{B})\). If \((G, \tau)\) is left \( \lambda \)-narrow, then \((G, \gamma)\) is \( \lambda \)-narrow.

**Proof** It suffices to show that \((G, \tau)\) is right \( \lambda \)-narrow. If \((G, \tau)\) is discrete, then it is obvious that \((G, \tau)\) is right \( \lambda \)-narrow. Therefore, we may assume that \((G, \tau)\) is non-discrete. Let \( U \) be an arbitrary open neighborhood of the identity \( e \) in \((G, \tau)\). Since \((G, \tau)\) is left \( \lambda \)-narrow, we can find a subset \( F_0 \) of \((G, \tau)\) such that \( |F_0| \leq \lambda \) and \( F_0U = G \). Since \( e \in U \), there exists \( B_0 \in \mathcal{B} \) such that \( B_0 \subset U \) and \( B_0 \setminus \{e\} \) is open in \((G, \gamma)\). Since \( G \) is non-discrete, we have \( B_0 \setminus \{e\} \neq \emptyset \).

Let \( y \in B_0 \setminus \{e\} \). Then \((B_0 \setminus \{e\})y^{-1} \in \gamma \) and \( e \in (B_0 \setminus \{e\})y^{-1} \). Since \((G, \tau)\) is left \( \lambda \)-narrow, topological group \((G, \gamma)\) is \( \lambda \)-narrow. So there exists a subset \( F_1 \) of \( G \) such that \( |F_1| \leq \lambda \) and \((B_0 \setminus \{e\})y^{-1}F_1 = G \). Thus \( G = U(y^{-1}F_1) \) and \( |y^{-1}F_1| \leq \lambda \). □

**Theorem 4.3** Let \( G \) be an almost topological group with structure \((\tau, \gamma, \mathcal{B})\). Then \((G, \gamma)\) is \( \lambda \)-narrow if and only if \((G, \tau)\) is \( \lambda \)-narrow.

**Proof** Obviously, if \((G, \tau)\) is discrete, then theorem holds. Therefore, we may assume that \((G, \tau)\) is non-discrete.

**Sufficiency.** Since \((G, \gamma)\) is weaker than \((G, \tau)\), it is obvious that \((G, \gamma)\) is \( \lambda \)-narrow if \((G, \tau)\) is \( \lambda \)-narrow.

**Necessity.** By Theorem 4.2, it suffices to show that \((G, \tau)\) is left \( \lambda \)-narrow. Let \( U \) be an
arbitrary non-empty open neighborhood of $e$ in $(G, \tau)$. Then there exists $B \in \mathcal{B}$ such that $B \subseteq U$ and $B \setminus \{e\}$ is a non-empty open set in $(G, \gamma)$. Take $x \in B \setminus \{e\}$. Since $(G, \gamma)$ is topological group, there exists an open neighborhood $V$ of $e$ in $(G, \gamma)$ such that $xV \subseteq B \setminus \{e\}$. Since $(G, \gamma)$ is $\lambda$-narrow, there exists a subset $F$ of $G$ such that $|F| \leq \lambda$ and $FV = G$. Let $H = Fx^{-1}$. Clearly, $|H| \leq \lambda$. Then

$$G = FV = Fx^{-1}xV \subset H(B \setminus \{e\}) \subset HU.$$ 

Therefore, $(G, \tau)$ is left $\lambda$-narrow. □

**Theorem 4.4** If $G$ is an almost topological group with structure $(\tau, \gamma, \mathcal{B})$, then $d(G, \gamma) = d(G, \tau)$.

**Proof** Obviously, we have $d(G, \tau) \geq d(G, \gamma)$. Next, we shall show that $d(G, \gamma) \geq d(G, \tau)$. Clearly, if $(G, \tau)$ is discrete, then theorem holds. Therefore, we may assume that $(G, \tau)$ is non-discrete. Let $D$ be a dense subset of $(G, \gamma)$. Next we shall show that $D$ is a dense subset in $(G, \tau)$. Indeed, for an arbitrary non-empty open set $U$ in $(G, \tau)$, take $x \in U$. There exists $B \in \mathcal{B}$ such that $xB \subseteq U$ and $xB \setminus \{x\}$ is open in $(G, \gamma)$. Since $(G, \gamma)$ is non-discrete, the interior of $xB \setminus \{x\}$ in $(G, \gamma)$ is non-empty. So we have $(xB \setminus \{x\}) \cap D \neq \emptyset$, and therefore $(xB \setminus \{x\}) \cap D \subseteq xB \cap D \subseteq U \cap D \neq \emptyset$. Therefore, $D$ is dense in $(G, \tau)$. □

**Theorem 4.5** Every separable almost topological group is $\omega$-narrow.

**Proof** Let $G$ be an almost topological group with structure $(\tau, \gamma, \mathcal{B})$. Since $(G, \tau)$ is separable, $(G, \gamma)$ is also separable since $(G, \gamma)$ is weaker than $(G, \tau)$. So $(G, \gamma)$ is $\omega$-narrow [3]. Therefore, Theorem 4.3 implies that $(G, \tau)$ is $\omega$-narrow. □

**Theorem 4.6** If $G$ is an almost topological group with structure $(\tau, \gamma, \mathcal{B})$, then $c(G, \tau) = c(G, \gamma)$.

**Proof** Obviously, $c(G, \gamma) \leq c(G, \tau)$. Next, it suffices to show that $c(G, \tau) \leq c(G, \gamma)$. Let $c(G, \tau) = \kappa$. If $(G, \tau)$ is discrete, then the theorem holds. Therefore, we may assume that $(G, \tau)$ is non-discrete. Let $\mathcal{W}$ be the maximum family of pairwise disjoint open sets in $(G, \tau)$. Since $c(G, \tau) = \kappa$, we may assume that $\mathcal{W} = \{U_\alpha : \alpha \in \kappa\}$. For each $\alpha \leq \kappa$, take $x_\alpha \in U_\alpha$, then there exists $B_\alpha \in \mathcal{B}$ such that $x_\alpha B_\alpha \subseteq U_\alpha$. By Definition 2.4, we know that $x_\alpha B_\alpha \setminus \{x_\alpha\}$ is a non-empty open set in $(G, \gamma)$. Let

$$\mathcal{V} = \{x_\alpha B_\alpha \setminus \{x_\alpha\} : \alpha \leq \kappa\}.$$ 

Clearly, $\mathcal{V}$ is the family of pairwise disjoint open sets in $(G, \gamma)$, hence $|\mathcal{V}| \leq c(G, \gamma)$. Moreover, it is obvious that $|\mathcal{V}| = |\mathcal{W}|$. Thus $\kappa \leq c(G, \gamma)$. Therefore, we have $c(G, \tau) \leq c(G, \gamma)$. □

**Theorem 4.7** Suppose that $G$ is an almost topological group with structure $(\tau, \gamma, \mathcal{B})$. If $c(G, \tau) \leq \omega$, then $(G, \tau)$ is $\omega$-narrow.

**Proof** Suppose $c(G, \tau) \leq \omega$. Then it follows from Theorem 4.6 that $c(G, \gamma) \leq \omega$. Since $(G, \gamma)$ is topological group, $(G, \gamma)$ is $\omega$-narrow [3]. By Theorem 4.3, $(G, \tau)$ is $\omega$-narrow. □
In [16], the author showed that a paratopological group $G$ is $\lambda$-narrow if it contains a dense $\lambda$-narrow subgroup. The following theorem is complementary to I. Sánchez’s result.

**Theorem 4.8** Every subgroup $H$ of an $\omega$-narrow almost topological group $G$ is $\omega$-narrow.

**Proof** Let $G$ be an almost topological group with structure $(\tau, \gamma, \mathcal{B})$. If $(G, \tau)$ is discrete, then the theorem holds. Therefore, we may assume that $(G, \tau)$ is non-discrete. Suppose $H$ is an arbitrary subgroup of $G$. By Theorem 4.3, $(G, \gamma)$ is $\omega$-narrow, so $H$ is $\omega$-narrow in $(G, \gamma)$. For an arbitrary open neighborhood $U$ of the neutral element $e$ in $(G, \tau)$, take $x \in U$. Then there exists $B \in \mathcal{B}$ such that $xB \subset U$ and $xB \setminus \{x\}$ is a non-empty open subset of $(G, \gamma)$. So there exists a countable subset $A$ of $H$ such that $(xB \setminus \{x\}) \cdot A \supset H$. Then

$$H \subset (xB \setminus \{x\}) \cdot A \subset xB \cdot A \subset U \cdot A.$$ 

Therefore, $H$ is $w$-narrow in $(G, \tau)$. □

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**References**