On Jeśmanowicz’ Conjecture Concerning Pythagorean Triples

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Abstract Let \((a, b, c)\) be a primitive Pythagorean triple. Jeśmanowicz conjectured in 1956 that for any positive integer \(n\), the Diophantine equation \((an)^x + (bn)^y = (cn)^z\) has only the positive integer solution \((x, y, z) = (2, 2, 2)\). Let \(p \equiv 3 \pmod{4}\) be a prime and \(s\) be some positive integer. In the paper, we show that the conjecture is true when \((a, b, c) = (4p^2s - 1, 4p^s, 4p^2s + 1)\) and certain divisibility conditions are satisfied.

Keywords Jeśmanowicz’ conjecture; Diophantine equation

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1. Introduction

Let \(a, b, c\) be a primitive Pythagorean triple, that is, \(a, b, c\) are relatively prime positive integers such that \(a^2 + b^2 = c^2\). In 1956, Jeśmanowicz [2] conjectured that for any positive integer \(n\), the Diophantine equation

\[(an)^x + (bn)^y = (cn)^z\]  \hspace{1cm} (1)

has no positive integer solutions other than \((x, y, z) = (2, 2, 2)\).

Whether there are other solutions has been investigated by many authors. Sierpiński [7] showed that Eq. (1) has no other positive integer solutions when \(n = 1\) and \((a, b, c) = (3, 4, 5)\). Jeśmanowicz [2] further proved the same conclusion for \(n = 1\) and \((a, b, c) = (5, 12, 13), (7, 24, 25), (9, 40, 41), (11, 60, 61)\). For any positive integer \(k\), Lu [4] showed that Eq. (1) has only the positive integer solution \((x, y, z) = (2, 2, 2)\) if \(n = 1\) and \((a, b, c) = (4k^2 - 1, 4k, 4k^2 + 1)\). Recently, Tang and Weng [9] proved that for any positive integer \(m\), if \(c = 2^{2m} + 1\) is a Fermat number, then Eq. (1) has only the positive integer solution \((x, y, z) = (2, 2, 2)\). Deng [1] showed that if \(k = 2^s\) for some positive integer \(s\) and certain divisibility conditions are satisfied, then Jeśmanowicz’ conjecture is true. For related problems, we refer to [3,5,6,8,10].

For any positive integer \(N\) with \(N > 1\), let \(P(N)\) denote the product of distinct prime factors of \(N\) and \((\frac{x}{y})\) denote the Legendre symbol. If \(q^e\) is a prime power, we write \(p^e \parallel N\) to mean that \(p^e | N\) while \(p^{e+1} \nmid N\). Let \(p \equiv 3 \pmod{4}\) be a prime and \(s\) be some positive integer.
In this paper, we consider the case $k = p^s$ and the following results will be proved.

**Theorem 1.1** Let $p \equiv 3 \pmod{4}$ be a prime and $s$ be some positive integer. Let $(a, b, c) = (4p^{2s} - 1, 4p^s, 4p^{2s} + 1)$ be a primitive Pythagorean triple. Suppose that the positive integer $n$ is such that either $P(a) | n$ or $P(n) \nmid a$. Then Eq. (1) has only the positive integer solution $(x, y, z) = (2, 2, 2)$.

**Corollary 1.2** Let $s$ be some positive integer and $(a, b, c) = (4 \cdot 3^{2s} - 1, 4 \cdot 3^s, 4 \cdot 3^{2s} + 1)$ be a primitive Pythagorean triple. If $a$ has only two distinct prime divisors, then for any positive integer $n$, Eq.(1) has only the positive integer solution $(x, y, z) = (2, 2, 2)$.

### 2. Proofs

We shall begin with the following two Lemmas.

**Lemma 2.1** ([4, Theorem]) Let $(a, b, c) = (4k^2 - 1, 4k, 4k^2 + 1)$ and $n = 1$. Then Eq. (1) has only the positive integer solution $(x, y, z) = (2, 2, 2)$.

**Lemma 2.2** ([1, Corollary]) Let $(a, b, c)$ be any primitive Pythagorean triple such that the Diophantine equation $a^x + b^y = c^z$ has only the positive integer solution $(x, y, z) = (2, 2, 2)$. If $(x, y, z)$ is a solution of Eq.(1) with $(x, y, z) \neq (2, 2, 2)$, then one of the following conditions is satisfied:

(i) $x > z > y$ and $P(n) | b$;
(ii) $y > z > x$ and $P(n) | a$.

**Proof of Theorem 1.1** By Lemma 2.1, we may suppose that $n > 2$. We also suppose that $(x, y, z)$ is a solution of Eq.(1) with $(x, y, z) \neq (2, 2, 2)$.

**Case 1** $P(n) \nmid a$. By Lemma 2.2, we have $x > z > y$ and $P(n) | b$. Then

$$a^x b^{z-y} + b^y = c^z$$

Since $b = 2^2p^s$, we can write $n = 2^a p^\beta$ with $\alpha + \beta \geq 1$. Thus

$$2^{\alpha(x-y)} p^{\beta(x-y)} a^x + 2^{2y} p^{\beta(y)} = 2^{\alpha(z-y)} p^{\beta(z-y)} c^z.$$  \hspace{1cm} (2)

**Subcase 1.1** $\alpha = 0, \beta \geq 1$. Then

$$p^{\beta(x-y)} a^x + 2^{2y} p^{\beta(y)} = p^{\beta(z-y)} c^z.$$  \hspace{1cm} (3)

Since $\beta(x-y) > \beta(z-y)$, we have $sy = \beta(z-y)$. Thus

$$p^{\beta(x-z)} a^x = c^z - 2^{2y}.$$  \hspace{1cm} (3)

If $p = 3$, then $(-1)^{x+\beta(z-y)} \equiv 1 \pmod{4}$, and then $x + \beta(x-z) \equiv 0 \pmod{2}$. Thus taking modulo 8 in (3), we have

$$5^z \equiv 2^{2y} + 1 \pmod{8}$$  \hspace{1cm} (4)

If $y = 1$, then $z \equiv 1 \pmod{2}$ and $3^{\beta(x-z)} a^x = c^z - 4$. Since $3 | c^z - 4$ and $3 \nmid a$, we have
\(\beta(x-z) = 1\), thus \(x\) is even, \(x + \beta(x-z) \equiv 1 \pmod{2}\), a contradiction. Thus \(y \geq 2\). By (4) we have \(5^z \equiv 1 \pmod{8}\) and \(z \equiv 0 \pmod{2}\).

If \(p > 3\), then \(a = 4p^{2z} - 1 \equiv 0 \pmod{3}, c = 4p^{2z} + 1 \equiv 2 \pmod{3}\). Thus \(2^z \equiv 1 \pmod{3}\) and \(z \equiv 0 \pmod{2}\).

Now we can write \(z = 2z_1\). By (3),

\[
p^{\beta(x-z)}(2p^s + 1)^x(2p^s - 1)^x = p^{\beta(x-z)}a^x = (c^{z_1} - 2^y)(c^{z_1} + 2^y).
\]

Noting that \(\gcd(c^{z_1} - 2^y, c^{z_1} + 2^y) = 1\), we can write \(a = a_1a_2\) with \(\gcd(a_1, a_2) = 1, a_1^2|c^{z_1} + 2^y\) and \(a_2^2|c^{z_1} - 2^y\). Then either \(a_1 \geq 2p^s + 1\) or \(a_2 \geq 2p^s + 1\). Otherwise, if \(a_1 < 2p^s + 1\) and \(a_2 < 2p^s + 1\), then \(a_1 \leq 2p^s - 1\) and \(a_2 \leq 2p^s - 1\) by \(a_1a_2 = 4p^{2s} - 1\). Thus

\[
a = a_1a_2 \leq (2p^s - 1)^2 < (2p^s - 1)(2p^s + 1) = a,
\]

which is impossible. If \(a_1 \geq 2p^s + 1\), then

\[
a_1^2 \geq (2p^s + 1)^2 = c + 4p^s > c + 4,
\]

thus

\[
a_1^2 > (a_1^2)^{z_1} > (c + 4)^{z_1} > c^{z_1} + 2^y,
\]

but \(a_1^2|c^{z_1} + 2^y\), a contradiction. If \(a_2 \geq 2p^s + 1\), we similarly get \(a_2^2 > c^{z_1} + 2^y > c^{z_1} - 2^y\), but \(a_2^2|c^{z_1} - 2^y\), a contradiction.

**Subcase 1.2** \(\alpha \geq 1, \beta = 0\). Then by (2) we have

\[
2^{\alpha(x-y)}a^x + 2^{2y}p^y = 2^{\alpha(x-y)}c^x.
\]

Since \(\alpha(x-y) > \alpha(z-y)\), we have \(2y = \alpha(z-y)\). Thus

\[
2^{\alpha(x-z)}a^x = c^z - p^{y}\tag{5}
\]

If \(p = 3\), then \((-1)^{x+z}(x-z) \equiv 1 \pmod{3}\), thus \(x + \alpha(x-z) \equiv 0 \pmod{2}\). Moreover, \((-1)^{x}2^{\alpha(x-z)} \equiv 1 - 3^y \pmod{4}\) and \((-1)^{x}2^{\alpha(x-z)} \equiv 1 - 3^y \pmod{9}\). If \(sy \equiv 1 \pmod{2}\), then \(1 - 3^y \equiv 2 \pmod{4}\), thus \(\alpha(x-z) = 1, x \equiv 1 \pmod{2}\). It follows that \(\alpha = 1, z \equiv 0 \pmod{2}\), \(y \equiv 1 \pmod{2}\), \(2y = \alpha(z-y)\), a contradiction. Then \(sy \equiv 0 \pmod{2}\) and \((-1)^{x}2^{\alpha(x-z)} \equiv 1 \pmod{9}\). Hence \(\alpha(x-z) \geq 3\). Further, \(5^z \equiv 1 \pmod{8}\), thus \(z \equiv 0 \pmod{2}\).

If \(p > 3\), then \((-1)^{x}2^{\alpha(x-z)} \equiv 1 \pmod{p}\). Since \(p \equiv 3 \pmod{4}\) and \(p > 3\), we have \(\alpha(x-z) \geq 3\). Moreover, \((-1)^{y} \equiv 1 \pmod{4}\) and \(5^z \equiv 1 \pmod{8}\), thus \(sy \equiv 0 \pmod{2}\), \(z \equiv 0 \pmod{2}\).

Now we can write \(z = 2z_1, sy = 2yn\). By (5),

\[
2^{\alpha(x-z)}a^x = (c^{z_1} - p^y)(c^{z_1} + p^y).
\]

Let \(a = a_1a_2\) with \(\gcd(a_1, a_2) = 1, a_1^2|c^{z_1} + p^y\) and \(a_2^2|c^{z_1} - p^y\). Then either \(a_1 \geq 2p^s + 1\) or \(a_2 \geq 2p^s + 1\). Otherwise, if \(a_1 \leq 2p^s - 1\) and \(a_2 \leq 2p^s - 1\), then

\[
a = a_1a_2 \leq (2p^s - 1)^2 < (2p^s - 1)(2p^s + 1) = a,
\]
which is impossible. If \( a_1 \geq 2p^s + 1 \), then

\[
a_1^2 \geq (2p^s + 1)^2 = c + 4p^s > c + p^s.
\]

Further, we have

\[
a_1^2 > (a_1^2)^{z_1} > (c + p^s)^{z_1} \geq c^{z_1} + p^{s z_1} > c^{z_1} + p^{h_1},
\]

but \( a_1^2 | c^{z_1} + p^{h_1} \), a contradiction. If \( a_2 \geq 2p^s + 1 \), we similarly get \( a_2^2 > c^{z_1} + p^{h_1} > c^{z_1} - p^{h_1} \), but \( a_2^2 | c^{z_1} - p^{h_1} \), a contradiction.

**Subcase 1.3** \( \alpha \geq 1, \beta \geq 1 \). Then \( \alpha(x - y) > \alpha(z - y), \beta(x - y) > \beta(z - y) \), so \( 2y = \alpha(z - y), sy = \beta(z - y) \), thus

\[
2^{\alpha(x-z)}p^{\beta(x-z)}a^x = c^z - 1. \tag{6}
\]

Since \( 4 | c^z - 1 \) and \( p^{2s} | c^z - 1 \), we have \( \alpha(x - z) \geq 2 \) and \( \beta(x - z) \geq 2s \). Then \( \alpha(x - z) \geq 3 \).

Otherwise, if \( \alpha(x - z) = 2 \), then \( \alpha = 1, x - z = 2 \) or \( \alpha = 2, x - z = 1 \). By \( 2y = \alpha(z - y) \), we have \( z = 3y \) or \( z = 2y \). By \( sy = \beta(z - y) \), we have \( s = 2 \beta \) or \( s = \beta \), but \( \beta(x - z) \geq 2s \), it is impossible.

Taking modulo 8 in (6), we have \( 5^z \equiv 1 \pmod{8} \), thus \( z \equiv 0 \pmod{2} \). Write \( z = 2z_1 \), we have

\[
2^{\alpha(x-z)}p^{\beta(x-z)}a^x = (c^{z_1} - 1)(c^{z_1} + 1).
\]

Let \( a = a_1a_2 \) with \( \gcd(a_1, a_2) = 1, a_1^2 | c^{z_1} + 1 \) and \( a_2^2 | c^{z_1} - 1 \). Then either \( a_1 \geq 2p^s + 1 \) or \( a_2 \geq 2p^s + 1 \). Otherwise, if \( a_1 \leq 2p^s - 1 \) and \( a_2 \leq 2p^s - 1 \), then

\[
a = a_1a_2 \leq (2p^s - 1)^2 < (2p^s - 1)(2p^s + 1) = a,
\]

which is impossible. If \( a_1 \geq 2p^s + 1 \), then

\[
a_1^2 \geq (2p^s + 1)^2 = c + 4p^s > c + 1.
\]

Further, we have

\[
a_1^2 > (a_1^2)^{z_1} > (c + 1)^{z_1} \geq c^{z_1} + 1,
\]

but \( a_1^2 | c^{z_1} + 1 \), a contradiction. If \( a_2 \geq 2p^s + 1 \), we similarly get \( a_2^2 > c^{z_1} + 1 > c^{z_1} - 1 \), but \( a_2^2 | c^{z_1} - 1 \), a contradiction.

**Case 2** \( P(a)|n \). By Lemma 2.2, we have \( y > z > x \). Then

\[
a^x + b^yn^{y-x} = c^{z-x}.
\]

Since \( y - x > z - x > 0 \), we have \( P(n)|a \) and \( n^{z-x} | a^x \), which means that \( P(n) = P(a) \) and \( n^{z-x} = a^x \). Thus

\[
b^yn^{y-x} = c^z - 1. \tag{7}
\]

It follows that \( 5^z \equiv 1 \pmod{8} \), so \( z \equiv 0 \pmod{2} \). Write \( z = 2z_1 \). Since \( c \equiv 1 \pmod{b} \), \( c^{z_1} + 1 \equiv 2 \pmod{b} \), we have \( \gcd(c^{z_1} + 1, b) = 2 \). Then by (7), we have \( b^y | c^{z_1} - 1 \). But

\[
\frac{b^y}{2} > \frac{b^{2z_1}}{2} = \left(\frac{c - a}{2}\right)^{z_1}\left(\frac{c + a}{2}\right)^{z_1} \geq c^{z_1} + a^{z_1} > c^{z_1} - 1,
\]

which is impossible.
This completes the proof of Theorem 1.1 □

Proof of Corollary 1.2 By Lemma 2.1, we may suppose that \( n \geq 2 \) and suppose that \( (x, y, z) \) is a solution of (1) with \( (x, y, z) \neq (2, 2, 2) \). By Case 2 of Theorem 1.1 and Lemma 2.2, we may suppose that \( y > z > x, P(n)|a \) and \( P(n) < P(a) \). Since \( a = 4 \cdot 3^{2k} - 1 \) has only two distinct prime divisors and gcd\( (2 \cdot 3^s - 1, 2 \cdot 3^s + 1) = 1 \), we can write \( 2 \cdot 3^s - 1 = q_1^{a_1}, 2 \cdot 3^s + 1 = q_2^{a_2} \), where \( q_1 \) and \( q_2 \) are distinct odd primes and \( a_1, a_2 \geq 1 \). Then we have either \( n = q_1^{a_1} \) or \( n = q_2^{a_2} \) with \( a_1, a_2 \geq 1 \).

If \( n = q_1^{a_1} \), then we have

\[
q_1^{a_1} x^{a_2} + (4 \cdot 3^s)^y q_1^{a(y-x)} = (4 \cdot 3^{2s} + 1)^z q_1^{a(z-x)}.
\]

Since \( a(y-x) > a(z-x) \), we have \( a_1 x = a(z-x) \) and

\[
(4 \cdot 3^s)^y q_1^{a(y-x)} = (4 \cdot 3^{2s} + 1)^z - q_2^{a_2 x} = (4 \cdot 3^{2s} + 1)^z - (2 \cdot 3^s + 1)^z.
\]

(8)

If \( s \) is even, then \( 3^s \equiv 1 \) (mod 4) and \( 5^s \equiv 3^s \) (mod 8), thus \( x \) and \( z \) are both even. If \( s \) is odd, then \( 5^s \equiv 7^s \) (mod 8), thus \( x \) and \( z \) are both even. Now write \( z = 2z_1, x = 2x_1 \). By (8),

\[
(4 \cdot 3^s)^y q_1^{a(y-z)} = ((4 \cdot 3^{2s} + 1)^z_1 - (2 \cdot 3^s + 1)^z_1) ((4 \cdot 3^{2s} + 1)^z_2 + (2 \cdot 3^s + 1)^z_2).
\]

(9)

Noting that gcd\((4 \cdot 3^{2s} + 1)^z_1 - (2 \cdot 3^s + 1)^z_1, (4 \cdot 3^{2s} + 1)^z_2 + (2 \cdot 3^s + 1)^z_2\) = 2 and

\[
\frac{(4 \cdot 3^s)^y}{2} = \frac{b^y}{2} > \frac{b_1^{2z_1}}{2} = \frac{2z_1(c + a)^z_1}{2} \geq (c + a)^z_1 \geq c_1^{z_1} + a_1^{z_1},
\]

we deduce that (9) cannot hold.

If \( n = q_2^{a_2} \), then we have

\[
q_1^{a_1} x^{a_2} + (4 \cdot 3^s)^y q_2^{a(y-x)} = (4 \cdot 3^{2s} + 1)^z q_2^{a(z-x)}.
\]

Since \( \beta(y-x) > \beta(z-x) \), we have \( a_2 x = \beta(z-x) \) and

\[
(4 \cdot 3^s)^y q_2^{a(y-x)} = (4 \cdot 3^{2s} + 1)^z - q_1^{a_1 x} = (4 \cdot 3^{2s} + 1)^z - (2 \cdot 3^s - 1)^z.
\]

(10)

It follows that \( x \) is even since \( (-1)^x \equiv 1 \) (mod 3) and \( z \) is even since \( 5^z \equiv 1 \) (mod 8). Now write \( z = 2z_1, x = 2x_1 \). By (10),

\[
(4 \cdot 3^s)^y q_2^{a(y-z)} = ((4 \cdot 3^{2s} + 1)^z_1 - (2 \cdot 3^s - 1)^z_1) ((4 \cdot 3^{2s} + 1)^z_2 + (2 \cdot 3^s - 1)^z_2).
\]

(11)

Noting that gcd\((4 \cdot 3^{2s} + 1)^z_1 - (2 \cdot 3^s - 1)^z_1, (4 \cdot 3^{2s} + 1)^z_2 + (2 \cdot 3^s - 1)^z_2\) = 2 and

\[
\frac{b^y}{2} > \frac{b_2^{2z_1}}{2} \geq (c + a)^z_1 \geq c_1^{z_1} + a_1^{z_1} > (4 \cdot 3^{2s} + 1)^z_1 + (2 \cdot 3^s - 1)^z_1 > (4 \cdot 3^{2s} + 1)^z_1 - (2 \cdot 3^s - 1)^z_1,
\]

we deduce that (11) cannot hold.

This completes the proof of Corollary 1.2 □

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References


