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A Class of Weighted Composition Operators on the Fock Space

Liankuo ZHAO*, **Changbao PANG**
School of Mathematics and Computer Science, Shanxi Normal University, Shanxi 041004, *P. R. China*

Abstract This paper studies a class of weighted composition operators and their spectrum on the Fock space. As an application, bounded self-adjoint, a class of complex symmetric weighted composition operators on the Fock space are characterized.

Keywords Fock space; weighted composition operator; self-adjoint; complex symmetric; spectrum

MR(2010) Subject Classification 47B32

1. Introduction

Recently weighted composition operators on various function spaces have been studied deeply [1–4]. In this paper, we study a class of weighted composition operators on the Fock space and give a complete characterization for such weighted composition operators to be bounded and their spectrum. We will see that such weighted composition operators are closely related to the self-adjoint, normal and complex symmetric weighted composition operators on the Fock space.

Recall the Fock space $\mathcal{F}^2$ is the space of entire functions f on $\mathbb{C}$ for which

$$\|f\|^2=\frac{1}{2\pi}\int_{\mathbb{C}}|f(z)|^2e^{-\frac{|z|^2}{2}}\mathrm{d}m(z),$$

where $\mathrm{d}m$ is the usual Lebesgue measure on $\mathbb{C}$.

It is well known that $\mathcal{F}^2$ is a reproducing kernel Hilbert space with inner product

$$\langle f,g\rangle=\frac{1}{2\pi}\int_{\mathbb{C}}f(z)\overline{g(z)}e^{-\frac{|z|^2}{2}}\mathrm{d}m(z),\quad f,g\in\mathcal{F}^2$$

and reproducing kernel function $K_w(z)=e^{\frac{\bar{w}z}{2}}$, $w,z\in\mathbb{C}$. Let k_w be the normalization of K_w. Then $k_w(z)=e^{\frac{\bar{w}z}{2}-\frac{|w|^2}{4}}$. The book [5] is a good reference for the Fock spaces and their operators.

For entire function φ on $\mathbb{C}$ and $\psi\in\mathcal{F}^2$, the weighted composition operator $C_{\psi,\varphi}$ on $\mathcal{F}^2$ is defined as

$$C_{\psi,\varphi}f=\psi(f\circ\varphi),\quad f\in\mathcal{F}^2.$$

When $\psi=1$, denote $C_{\psi,\varphi}$ as C_φ, the composition operator defined by φ, which has been studied in [6,7]. In some sense, bounded composition operators on the Fock space are trivial. But there

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* Corresponding author
E-mail address: liankuozhao@sina.com (Liankuo ZHAO)

are plenty of nontrivial bounded weighted composition operators on the Fock space as we will show. Note that in [8–11], the bounded and compact weighted composition operators between Fock spaces are characterized for general symbols.

In the next section, we firstly characterize bounded weighted composition operator $C_{\psi,\varphi}$ and their spectrum $\sigma(C_{\psi,\varphi})$ on $\mathcal{F}^2$ with $\varphi(z)=az+b$ and $\psi(z)=K_c(z)$ for some constants a,b,c. Then, we consider self-adjoint, a class of complex symmetric weighted composition operators on $\mathcal{F}^2$ and obtain a complete characterization for such operators.

In summary, we have the following results.

Theorem 1.1 *Let $\varphi(z)=az+b$ and $\psi(z)=K_c(z)$ for some constants a,b,c. Then $C_{\psi,\varphi}$ is bounded on $\mathcal{F}^2$ if and only if one of the following conditions holds:*

(1) $|a|<1$; (2) $|a|=1$ and $c+\bar{a}b=0$.

Furthermore,

(1) If $|a|<1$, then $\sigma(C_{\psi,\varphi})=\{0,\psi(p),a\psi(p),a^2\psi(p),a^3\psi(p),\ldots\}$, where $p=\frac{b}{1-a}$;

(2) If $a=1$, then $\sigma(C_{\psi,\varphi})=\{z\in\mathbb{C}:|z|=e^{\frac{|b|^2}{4}}\}$;

(3) If $|a|=1,a\neq 1$, then $\sigma(C_{\psi,\varphi})=\overline{\{\beta a^m\}_{m=0}^{\infty}}$, where $\beta=e^{\frac{|b|^2}{2(a-1)}}$.

Theorem 1.2 *Let φ be an entire function on $\mathbb{C}$ and ψ be a nonzero function in $\mathcal{F}^2$. Then*

(1) $C_{\psi,\varphi}$ is a bounded self-adjoint operator on $\mathcal{F}^2$ if and only if

$$\varphi(z)=az+b,\quad \psi(z)=sK_b(z)$$

for some constant b, real constants a,s with $s\neq 0$, either $|a|<1$ or $|a|=1$ and $b+ab=0$.

(2) $C_{\psi,\varphi}$ is a bounded complex symmetric operator with conjugation $\mathcal{J}_\lambda$ on $\mathcal{F}^2$ if and only if

$$\varphi(z)=az+b,\quad \psi(z)=sK_{\lambda\bar{b}}(z)$$

for some constants a,b,s with $s\neq 0$, either $|a|<1$ or $|a|=1$ and $\lambda\bar{b}+\bar{a}b=0$. Here

$$(\mathcal{J}_\lambda f)(z)=\overline{f(\lambda\bar{z})},\quad f\in\mathcal{F}^2$$

with $|\lambda|=1$.

2. Main results

2.1 Boundedness of $C_{\psi,\varphi}$ with $\varphi(z)=az+b$ and $\psi(z)=K_c(z)$

In this subsection, we characterize bounded weighted composition operator $C_{\psi,\varphi}$ and their spectrum $\sigma(C_{\psi,\varphi})$ on $\mathcal{F}^2$ with $\varphi(z)=az+b$ and $\psi(z)=K_c(z)$.

The following lemma is easy to verify and the similar result holds on any reproducing kernel function spaces.

Lemma 2.1 *Let φ be an entire function and $\psi\in\mathcal{F}^2$. If $C_{\psi,\varphi}$ is bounded on $\mathcal{F}^2$, then*

$$C^*_{\psi,\varphi}K_w=\overline{\psi(w)}K_{\varphi(w)}.$$

In [8], bounded and compact weighted composition operators with general symbols on $\mathcal{F}^2$ are characterized. As an application, we obtain the following result.

Proposition 2.2 *Let $\varphi(z) = az + b$ and $\psi(z) = K_c(z)$ for some constants a, b, c. Then $C_{\psi,\varphi}$ is bounded on $\mathcal{F}^2$ if and only if one of the following condition holds:*

(1) $|a| < 1$; (2) $|a| = 1$ and $c + \bar{a}b = 0$.

Proof By [8, Theorem 2.2], $C_{\psi,\varphi}$ is bounded on $\mathcal{F}^2$ if and only if $\varphi(z) = az + b$ with $|a| \le 1$ and

$$\sup\{|\psi(z)|^2 e^{\frac{|\varphi(z)|^2 - |z|^2}{2}} :\ z \in \mathbb{C}\} < \infty. \tag{1}$$

For $\varphi(z) = az + b$ and $\psi(z) = K_c(z)$, it is easy to verify that

$$|\psi(z)|^2 e^{\frac{|\varphi(z)|^2 - |z|^2}{2}} = e^{\frac{(|a|^2-1)|z|^2 + (a\bar{b}+\bar{c})z + (\bar{a}b+c)\bar{z} + |b|^2}{2}}.$$

So $\sup\{|\psi(z)|^2 e^{\frac{|\varphi(z)|^2 - |z|^2}{2}} : z \in \mathbb{C}\} < \infty$ if and only if

$$\sup\{(|a|^2 - 1)|z|^2 + (a\bar{b} + \bar{c})z + (\bar{a}b + c)\bar{z} + |b|^2 :\ z \in \mathbb{C}\} < \infty.$$

Obviously,

$$\sup\{(|a|^2 - 1)|z|^2 + (a\bar{b} + \bar{c})z + (\bar{a}b + c)\bar{z} + |b|^2 :\ z \in \mathbb{C}\} < \infty$$

if and only if either $|a| < 1$ or $|a| = 1$ and $c + \bar{a}b = 0$. The conclusion follows. □

Remark 2.3 The Eq. (1) is a little different from the Eq. (5) in [8, Theorem 2.2] since in our paper the reproducing kernel function $K_w(z) = e^{\frac{\bar{w}z}{2}}$, but $K_w(z) = e^{\bar{w}z}$ in [8].

The following corollary shows that the bounded operator $C_{\psi,\varphi}$ on $\mathcal{F}^2$ with $\varphi(z) = az + b$ and $\psi(z) = K_c(z)$ for some constants a, b, c is either a unitary operator or a compact operator.

Corollary 2.4 *Let $\varphi(z) = az + b$, $\psi(z) = K_c(z)$ for some constants a, b, c and $C_{\psi,\varphi}$ be bounded on $\mathcal{F}^2$.*

(1) If $|a| = 1$, then $C_{\psi,\varphi}$ is constant multiples of a unitary operator.

(2) If $|a| < 1$, then $C_{\psi,\varphi}$ is compact.

Proof (1) If $|a| = 1$, then $c + \bar{a}b = 0$ by Proposition 2.2, i.e., $c = -\bar{a}b$. It follows from [12, Corollary 1.2] that $C_{k_c,\varphi}$ is a unitary operator on $\mathcal{F}^2$. Obviously, $C_{\psi,\varphi} = sC_{k_c,\varphi}$ with $s = e^{\frac{|c|^2}{4}}$.

(2) If $|a| < 1$, then

$$\lim_{|z|\to\infty} [(|a|^2 - 1)|z|^2 + (a\bar{b} + \bar{c})z + (\bar{a}b + c)\bar{z} + |b|^2] = -\infty,$$

which implies that

$$\lim_{|z|\to\infty} |\psi(z)|^2 e^{\frac{|\varphi(z)|^2 - |z|^2}{2}} = \lim_{|z|\to\infty} e^{\frac{(|a|^2-1)|z|^2 + (a\bar{b}+\bar{c})z + (\bar{a}b+c)\bar{z} + |b|^2}{2}} = 0.$$

It follows from [8, Theorem 2.4] that $C_{\psi,\varphi}$ is compact on $\mathcal{F}^2$. □

By Corollary 2.4 and [12, Corollary 1.4], we obtain the following result.

Corollary 2.5 *Let $\varphi(z) = az + b$, $\psi(z) = K_c(z)$ for some constants a, b, c with $|a| = 1$ and $C_{\psi,\varphi}$ be bounded on $\mathcal{F}^2$.*

(1) If $a \neq 1$, then $\sigma(C_{\psi,\varphi}) = \overline{\{\beta a^m\}_{m=0}^{\infty}}$, where $\beta = e^{\frac{|b|^2}{2(a-1)}}$.

(2) If $a = 1$, then $\sigma(C_{\psi,\varphi}) = \{z \in \mathbb{C} : |z| = e^{\frac{|b|^2}{4}}\}$.

In [13], the spectra of a class of compact weighted composition operators on weighted Hardy space were characterized. Check the proof of Theorem 1 in [13] carefully, it is easy to obtain the the following result.

Proposition 2.6 *Let φ be a nonconstant entire function on $\mathbb{C}$ with $\varphi(p) = p$ for some $p \in \mathbb{C}$ and ψ be a nonzero function in $\mathcal{F}^2$. If $C_{\psi,\varphi}$ is a compact operator on $\mathcal{F}^2$, then*

$$\sigma(C_{\psi,\varphi}) = \{0, \psi(p), \psi(p)\varphi'(p), \psi(p)(\varphi'(p))^2, \psi(p)(\varphi'(p))^3, \ldots\}.$$

In fact, Proposition 2.6 gives the spectral characterization of all compact weighted composition operators on $\mathcal{F}^2$ by [8, Theorem 2.4]. Combining Corollary 2.4 and Proposition 2.6, we obtain the following result.

Corollary 2.7 *Let $\varphi(z) = az + b$ and $\psi(z) = K_c(z)$ for some constants a, b, c with $|a| < 1$. Then*

$$\sigma(C_{\psi,\varphi}) = \{0, \psi(p), a\psi(p), a^2\psi(p), a^3\psi(p), \ldots\},$$

where $p = \frac{b}{1-a}$.

Proof When $0 < |a| < 1$, by Corollary 2.4, $C_{\psi,\varphi}$ is compact on $\mathcal{F}^2$. Let $p = \frac{b}{1-a}$. Then $\varphi(p) = p$. Obviously $\varphi'(z) = a$. By Proposition 2.6, the conclusion follows.

When $a = 0$, $C_{\psi,\varphi}$ is the rank-one operator $K_c \otimes K_b$, it is easy to verify that

$$\sigma(K_c \otimes K_b) = \{0, K_c(b)\},$$

which implies that $\sigma(C_{\psi,\varphi}) = \{0, \psi(p)\}$. $\square$

By Proposition 2.2, Corollaries 2.5 and 2.7, we obtain Theorem 1.1.

2.2 Self-adjoint weighted composition operators

In this subsection, we consider bounded self-adjoint weighted composition operators on $\mathcal{F}^2$. The self-adjoint weighted composition operators on the Hardy space were characterized in [14,15]. More generally, we consider the problem when the adjoint of a weighted composition operator is another weighted composition operator.

Theorem 2.8 *Let φ_1, φ_2 be entire functions and ψ_1, ψ_2 be nonzero functions in $\mathcal{F}^2$. Then $C^*_{\psi_1,\varphi_1} = C_{\psi_2,\varphi_2}$ if and only if*

$$\varphi_1(z) = az + b,\ \psi_1(z) = dK_c(z),\ \varphi_2(z) = \bar{a}z + c,\ \psi_2(z) = \bar{d}K_b(z),$$

where a, d are constants with $d \neq 0$, either $|a| < 1$ or $|a| = 1$ and $c + \bar{a}b = 0$.

Proof Necessity. Since $C^*_{\psi_1,\varphi_1} = C_{\psi_2,\varphi_2}$, we have

$$\psi_2(z)K_w(\varphi_2(z)) = (C_{\psi_2,\varphi_2}K_w)(z) = (C^*_{\psi_1,\varphi_1}K_w)(z) = \overline{\psi_1(w)}K_{\varphi_1(w)}(z) \tag{2}$$

for all $w, z \in \mathbb{C}$.

Let $w = 0$ in Eq. (2). Then $\psi_2(z) = \overline{\psi_1(0)} K_{\varphi_1(0)}(z),\ z \in \mathbb{C}$.

Let $z = 0$ in Eq. (2). Then $\overline{\psi_1(w)} = \psi_2(0) K_w(\varphi_2(0)) = \psi_2(0)\overline{K_{\varphi_2(0)}(w)},\ w \in \mathbb{C}$.

In particular, let $w = 0,\ z = 0$ in Eq. (2). We have $\overline{\psi_1(0)} = \psi_2(0)$.

Denote $b = \varphi_1(0),\ c = \varphi_2(0),\ d = \overline{\psi_2(0)}$, then $d \neq 0$ since ψ_1 is nonzero, and

$$\psi_1(z) = dK_c(z),\ \psi_2(z) = \bar{d}K_b(z).$$

Taking the formulas above into Eq. (2), we obtain

$$\bar{d}e^{\frac{1}{2}\bar{b}z} e^{\frac{1}{2}\bar{w}\varphi_2(z)} = \bar{d}e^{\frac{1}{2}\bar{w}c} e^{\frac{1}{2}\overline{\varphi_1(w)}z} \tag{3}$$

for all $w, z \in \mathbb{C}$. So we have

$$\frac{1}{2}(\bar{b}z + \bar{w}\varphi_2(z)) = \frac{1}{2}(\bar{w}c + \overline{\varphi_1(w)}z) + 2n(z, w)\pi i,$$

where $n(z, w)$ is a continuous integer-valued function. Let $z = w = 0$ in the formula above. Then $n(0, 0) = 0$, which implies that $n(z, w) = 0$. We get

$$\frac{\varphi_2(z) - c}{z} = \overline{(\frac{\varphi_1(w) - b}{w})} \tag{4}$$

for all $w, z \in \mathbb{C}$, so

$$\frac{\varphi_2(z) - c}{z} = \overline{(\frac{\varphi_1(w) - b}{w})} = \bar{a}$$

for some constant a. Therefore $\varphi_1(z) = az + b$, $\varphi_2(z) = \bar{a}z + c$.

Since $C_{\psi_1,\varphi_1},\ C_{\psi_2,\varphi_2}$ are bounded, by Theorem 1.1, we have either $|a| < 1$ or $|a| = 1$ and $c + \bar{a}b = 0$.

Sufficiency. By Theorem 1.1, $C_{\psi_1,\varphi_1},\ C_{\psi_2,\varphi_2}$ are bounded. It is easy to verify that

$$(C^*_{\psi_1,\varphi_1} K_w)(z) = \overline{\psi_1(w)} K_{\varphi_1(w)}(z) = \bar{d}e^{\frac{1}{2}c\bar{w}} e^{\frac{1}{2}\overline{(aw+b)}z} = \bar{d}e^{\frac{1}{2}(c\bar{w} + \bar{a}\bar{w}z + \bar{b}z)},$$

$$(C_{\psi_2,\varphi_2} K_w)(z) = \psi_2(z) K_w(\varphi_2(z)) = \bar{d}e^{\frac{1}{2}\bar{b}z} e^{\frac{1}{2}\bar{w}(\bar{a}z+c)} = \bar{d}e^{\frac{1}{2}(c\bar{w} + \bar{a}\bar{w}z + \bar{b}z)}$$

for all $z, w \in \mathbb{C}$. So we have $C^*_{\psi_1,\varphi_1} = C_{\psi_2,\varphi_2}$. $\square$

As an application, we obtain the characterization of bounded self-adjoint weighted composition operator on $\mathcal{F}^2$, which is Theorem 1.2 (1).

Corollary 2.9 *Let φ be an entire function on $\mathbb{C}$ and ψ be a nonzero function in $\mathcal{F}^2$. Then $C_{\psi,\varphi}$ is a bounded self-adjoint operator on $\mathcal{F}^2$ if and only if*

$$\varphi(z) = az + b, \quad \psi(z) = sK_b(z)$$

for some constant b, real constants $a,\ s$ with $s \neq 0$, either $|a| < 1$ or $|a| = 1$ and $b + ab = 0$.

2.3 Complex symmetric weighted composition operators

Recall a conjugation on a separable complex Hilbert space $\mathcal{H}$ is an antilinear operator $\mathcal{C}$ on $\mathcal{H}$ which satisfies $\langle \mathcal{C}f, \mathcal{C}g \rangle = \langle g, f \rangle$, $f, g \in \mathcal{H}$ and $\mathcal{C}^2 = I$, where I is the identity operator on $\mathcal{H}$.

A bounded operator T on $\mathcal{H}$ is said to be complex symmetric with respect to $\mathcal{C}$ if

$$\mathcal{C}T\mathcal{C} = T^*.$$

In [16,17], complex symmetric weighted composition operators on the Hardy space were studied. Inspired by [16,17], in this subsection, we consider a class of complex symmetric weighted composition operator on $\mathcal{F}^2$.

For $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, define $(\mathcal{J}_\lambda f)(z) = \overline{f(\lambda \bar{z})}$, $f \in \mathcal{F}^2$. It is easy to verify that $\mathcal{J}_\lambda$ is a conjugation on $\mathcal{F}^2$ and $\mathcal{J}_\lambda K_w = K_{\lambda \bar{w}}$.

To characterize $\mathcal{J}_\lambda$-complex symmetric weighted composition operators on $\mathcal{F}^2$, we consider the more general problem when $C_{\psi_1,\varphi_1}\mathcal{J}_\lambda = \mathcal{J}_\lambda C^*_{\psi_2,\varphi_2}$.

Theorem 2.10 *Let φ_1, φ_2 be entire functions and ψ_1, ψ_2 be nonzero functions in $\mathcal{F}^2$. Then $C_{\psi_1,\varphi_1}\mathcal{J}_\lambda = \mathcal{J}_\lambda C^*_{\psi_2,\varphi_2}$ on $\mathcal{F}^2$ if and only if*

$$\varphi_1(z) = az + b,\ \psi_1(z) = sK_{\lambda\bar{c}}(z),\ \varphi_2(z) = az + c,\ \psi_2(z) = sK_{\lambda\bar{b}}(z)$$

for some constants a, b, s with $s \neq 0$, either $|a| < 1$ or $|a| = 1$ and $\lambda\bar{c} + \bar{a}b = 0$.

Proof Necessity. Assume $C_{\psi_1,\varphi_1}\mathcal{J}_\lambda = \mathcal{J}_\lambda C^*_{\psi_2,\varphi_2}$.

Since

$$(C_{\psi_1,\varphi_1}\mathcal{J}_\lambda K_w)(z) = \psi_1(z)K_{\lambda\bar{w}}(\varphi_1(z)) = \psi_1(z)e^{\frac{1}{2}\bar{\lambda}w\varphi_1(z)},$$
$$(\mathcal{J}_\lambda C^*_{\psi_2,\varphi_2}K_w)(z) = \psi_2(w)K_{\lambda\overline{\varphi_2(w)}}(z) = \psi_2(w)e^{\frac{1}{2}\bar{\lambda}\varphi_2(w)z},$$

we have

$$\psi_1(z)e^{\frac{1}{2}\bar{\lambda}w\varphi_1(z)} = \psi_2(w)e^{\frac{1}{2}\bar{\lambda}\varphi_2(w)z}. \tag{5}$$

Let $w = 0$ in Eq. (5). Then $\psi_1(z) = \psi_2(0)e^{\frac{1}{2}\bar{\lambda}\varphi_2(0)z}$. In particular, $\psi_1(0) = \psi_2(0)$. Denote $s = \psi_1(0) = \psi_2(0),\ c = \varphi_2(0)$, then

$$\psi_1(z) = se^{\frac{1}{2}\bar{\lambda}cz} = sK_{\lambda\bar{c}}(z). \tag{6}$$

Since ψ_1 is nonzero, $s \neq 0$.

Let $z = 0$ in Eq. (5). Then $\psi_2(w) = \psi_1(0)e^{\frac{1}{2}\bar{\lambda}w\varphi_1(0)}$. Denote $b = \varphi_1(0)$, then

$$\psi_2(w) = se^{\frac{1}{2}\bar{\lambda}bw} = sK_{\lambda\bar{b}}(w). \tag{7}$$

Taking Eqs. (6) and (7) into Eq. (5), we obtain

$$e^{\frac{\bar{\lambda}}{2}(cz+w\varphi_1(z))} = e^{\frac{\bar{\lambda}}{2}(bw+z\varphi_2(w))}, \tag{8}$$

so

$$\frac{\bar{\lambda}}{2}(cz + w\varphi_1(z)) = \frac{\bar{\lambda}}{2}(bw + z\varphi_2(w)) + 2n(z,w)\pi i,$$

where $n(z,w)$ is a continuous integer-valued function. Let $w = 0,\ z = 0$ in the formula above. Then $n(0,0) = 0$, which implies that $n(z,w) = 0$. We get

$$cz + w\varphi_1(z) = bw + z\varphi_2(w), \tag{9}$$

i.e., $\frac{\varphi_1(z)-b}{z} = \frac{\varphi_2(w)-c}{w}$. By the arbitrary of z and w, we have

$$\varphi_1(z) = az + b, \quad \varphi_2(w) = aw + c$$

for some constant a.

Since C_{ψ_1,φ_1} and C_{ψ_2,φ_2} are bounded, by Theorem 1.1, we have either $|a| < 1$ or $|a| = 1$ and $\lambda\bar{c} + \bar{a}b = 0$.

Sufficiency. By Theorem 1.1, C_{ψ_1,φ_1} and C_{ψ_2,φ_2} are bounded. It is easy to verify that

$$(C_{\psi_1,\varphi_1}\mathcal{J}_\lambda K_w)(z) = \psi_1(z)K_{\lambda\bar{w}}(\varphi_1(z)) = se^{\frac{1}{2}\bar{\lambda}cz}e^{\frac{1}{2}\bar{\lambda}w(az+b)} = se^{\frac{\bar{\lambda}}{2}(cz+awz+bw)},$$

$$(\mathcal{J}_\lambda C^*_{\psi_2,\varphi_2}K_w)(z) = \psi_2(w)K_{\lambda\overline{\varphi_2(w)}}(z) = se^{\frac{1}{2}\bar{\lambda}bw}e^{\frac{1}{2}\bar{\lambda}(aw+c)z} = se^{\frac{\bar{\lambda}}{2}(cz+awz+bw)}$$

for all $z, w \in \mathbb{C}$. So we have $C_{\psi_1,\varphi_1}\mathcal{J}_\lambda = \mathcal{J}_\lambda C^*_{\psi_2,\varphi_2}$. $\Box$

As a corollary, we obtain the characterization of bounded complex symmetric weighted composition operator with conjugation $\mathcal{J}_\lambda$ on $\mathcal{F}^2$, which is Theorem 1.2 (2).

Corollary 2.11 Let φ be an entire function and ψ be a nonzero function in $\mathcal{F}^2$. Then $C_{\psi,\varphi}$ is a bounded complex symmetric operator with conjugation $\mathcal{J}_\lambda$ on $\mathcal{F}^2$ if and only if

$$\varphi(z) = az + b, \quad \psi(z) = sK_{\lambda\bar{b}}(z)$$

for some constants a, b, s with $s \neq 0$, either $|a| < 1$ or $|a| = 1$ and $\lambda\bar{b} + \bar{a}b = 0$.

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