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Epigroups in which the Idempotent-generated Subsemigroups Are Completely Regular

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Abstract The purpose of the paper is to characterize epigroups in which the idempotent-generated subsemigroups are completely regular. Some special subclasses of the class of epigroups having the same property are also described.

Keywords epigroup; core; completely regular semigroup

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1. Introduction

A semigroup S is called an epigroup if for each element a in S there exists some power a^n of a such that a^n is a member of some maximal subgroup G of S. Let a^ω stand for the identity of the subgroup G. It is known that $aa^\omega(=a^\omega a)$ lies in G; we then denote the group inverse of aa^ω in G by $\overline{a}$ and call it the pseudo-inverse of a. Thus, an epigroup can alternatively be regarded as a unary semigroup with the unary operation of taking pseudo-inverse $x \mapsto \overline{x}$. For these and more informations on the theory of epigroups we refer to Shevrin [1,2] and his survey [3]. For the further growth of the related topics on the theory of epigroups, we refer to Shevrin and Ovsyannikov [4], Volkov [5], Wang and Jin [6], Wang and Luo [7] and Liu [8–10].

It is known that a recurring theme throughout the study of some subclasses of the class of semigroups (for examples, regular semigroups, epigroups) is the role of the idempotent. The idempotent-generated subsemigroup of a semigroup S is called the core of S. Since the 1970s, a number of works concerning the core of a regular semigroup have been given (see, for example, Fitz-Gerald [11], Eberhart et al [12] and Hall [13]). Theorem 3 in [13] says that for a regular semigroup S, the core of S is completely regular if and only if S is E-solid: i.e., for all idempotents e,f,g in S with $e\mathcal{L}g\mathcal{R}f$, there exists an idempotent h in S such that $e\mathcal{R}h\mathcal{L}f$.

For an epigroup, its core is itself an epigroup (and this satement will be repeated in Section 2). Some works have been done devotedly or partly on epigroups (including its subclasses, for example, the class of finite semigroups) whose cores belong to some known subclass of semigroups (see, for example, Moura [14], Almeida and Moura [15], Auinger and Szendrei [16] and certain contents of Almeida [17]). In this paper we will focus on epigroups whose cores are completely regular. In Section 2, we start with some basic properties of this type; in particular, we present an

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example to illustrate that even for an epigroup with a completely regular core, the subepigroup and the subsemigroup generated by idempotens in the same regular $\mathcal{D}$-class D of it, respectively, may be different. Under what conditions they are equal, in Section 4, we give one answer, where we show that, for an epigroup with a completely regular core, the subsemigroup $\langle E(D)\rangle$ generated by idempotents in a regular $\mathcal{D}$-class D is completely regular if and only if the identity of the group containing the product of any two idempotents in $\langle E(D)\rangle$ also lies in $\langle E(D)\rangle$. Moreover, in this section, some special subclasses of epigroups with completely regular cores are also considered. The principal result in the paper is formulated in Section 3, where we characterize epigroups whose cores are completely regular in terms of the behaving of the product of two idempotents, in terms of epidivisors, in terms of certain decomposition, as well as in terms of identities respectively. The readers will see that some proofs in the paper mainly use mathematical induction and Fitz-Gerald's method.

2. Preliminaries

In this paper, we adopt the notation and terminology of [18–21], and for a background of epigroups we refer to [1,2] or [3].

The set of idempotents of a subset A of a semigroup S is denoted by $E(A)$. The set of idempotents of S is denoted by $E(S)$. On $E(S)$ there is a natural partial order relation defined by the rule that

$$e \le f \Leftrightarrow ef = fe = e.$$

Let S be a semigroup, and let a be an element in S. An element x in S is an inverse of a in S if $axa = a$ and $xax = x$. The set of inverses of a is denoted by $V(a)$, and a is regular if $V(a)$ is not empty. The set of regular elements of S is denoted by $\mathrm{Reg}(S)$, and we say that S is regular if $\mathrm{Reg}(S) = S$. The element a in S is completely regular if there exists $x \in V(a)$ such that $ax = xa$, and S is called completely regular if all its elements have this property. Obviously, $a \in S$ is completely regular if and only if a is a member of some subgroup G of S, that is, a is a group element of S. Then the set of all completely regular elements of a semigroup S (i.e., the group part of S) is denoted by $\mathrm{Gr}(S)$. For a $\mathcal{D}$-class D of S, if D contains at least one idempotent, then it is called a regular $\mathcal{D}$-class of S. The collection of all the regular $\mathcal{D}$-classes of S is denoted by $\mathrm{Reg}(S/\mathcal{D})$. Then for any $D \in \mathrm{Reg}(S/\mathcal{D})$, $E(D) \neq \emptyset$ and every element of D is regular [21, Proposition 2.3.1]. For the element $a \in S$, write the principal ideal S^1aS^1 generated by a as $J(a)$ and the $\mathcal{D}$-class containing a by D_a. It should be emphasized that Green's relation $\mathcal{D}$ coincides with Green's relation $\mathcal{J}$ in an epigroup S, that is, $b \in D_a$ if and only if $J(b) = J(a)$.

In the remainder of this paper, S always stands for an epigroup unless it is stated.

For any $a \in S$, the least positive integer k such that $a^k \in \mathrm{Gr}(S)$ is called the index of a and will be denoted by $\mathrm{ind}(a)$. Of course $a \in \mathrm{Gr}(S)$ (in this case $\mathrm{ind}(a) = 1$) if and only if $aa^\omega = a$. Formally, we sometimes write $a^{k+\omega}$ instead of a^ka^ω (note that this is well-defined: since $aa^\omega = a^\omega a$). Recall that for any $a \in S$ and any $n \ge \mathrm{ind}(a)$, $a^n \in \mathrm{Gr}(S)$ (see [1, Lemma 1]); then $a^{n+\omega}$ can be written in the abbreviation a^n if $n \ge \mathrm{ind}(a)$.

A subepigroup of an epigroup is a subsemigroup closed under taking pseudo-inverse as well. A homomorphic image of a subsemigroup of a given semigroup is called a divisor of the semigroup. A divisor obtained from a subepigroup of an epigroup is called an epidivisor.

For a subset $X \subseteq S$, $\langle X\rangle$ is the subsemigroup of S generated by X, while $\langle\!\langle X\rangle\!\rangle$ is the subepigroup of S generated by X. Any element of $\langle\!\langle X\rangle\!\rangle$ can be represented as a unary semigroup term over X, where operations are multiplication and taking pseudo-inverse. As stated at the beginning of the 3rd paragraph in Section 1, the core $\langle E(S)\rangle$ of S is in fact a subepigroup of S, that is, $\langle E(S)\rangle = \langle\!\langle E(S)\rangle\!\rangle$; for this conclusion we refer to Lemma 2.6 in [3] (followed by further details on its origin there). We recall that if E^n denotes the set of all elements of S which can be written as the product of n idempotents of S, then $\langle E(S)\rangle = \bigcup_{n=1}^{\infty} E^n$.

We will denote the class of all epigroups by $\mathcal{E}$ and the variety of all epigroups of index at most n (treated as unary semigroups) by $\mathcal{E}_n$. The variety $\mathcal{E}_1$ is the variety of all completely regular semigroups, and in this paper we use the conventional notation $\mathcal{CR}$ to denote this variety. For a subclass $\mathcal{V}$ of $\mathcal{E}$, we define

$$\mathbf{E}(\mathcal{V}) = \{S \in \mathcal{E} \mid \langle E(S)\rangle \in \mathcal{V}\}.$$

In this paper we mainly pay attention to the subclass $\mathbf{E}(\mathcal{CR})$ within the class of epigroups.

The next lemma can be drawn from Lemma II.2.3 in [22].

Lemma 2.1 *Let $S \in \mathbf{E}(\mathcal{CR})$ and $e_1, e_2, \ldots, e_n \in E(S)$. Then*

$$\begin{aligned}\overline{e_1e_2\ldots e_n} =&(e_1e_2\ldots e_n)^{\omega}(e_ne_1\ldots e_{n-1})^{\omega}(e_{n-1}e_n\ldots e_{n-2})^{\omega}\ldots\\ &(e_2e_3\ldots e_ne_1)^{\omega}(e_1e_2\ldots e_n)^{\omega}.\end{aligned}$$

From Lemma 2.1, the pseudo-inverse of the product of idempotents coming from the ordered n-tuple $(e_1, e_2, \ldots, e_n)$ successively is a factorization into the product of terms

$$(e_ie_{i+1}\ldots e_ne_1e_2\ldots e_{i-1})^{\omega}$$

beginning with e_i in the turn $(e_1, e_n, \ldots, e_2, e_1)$ as a clockwise rotation, as follows:

e_3

e_2 e_n

e_1

We recall that from Observation 2.3 in [10] S is a completely regular if and only if for any $\in S$, $aa^{\omega} = a$.

emma 2.2 *$S \in \mathbf{E}(\mathcal{CR})$ if and only if $\langle E(S)\rangle \subseteq \mathrm{Gr}(S)$.*

roof Necessity is clear. Conversely, assume that $\langle E(S)\rangle \subseteq \mathrm{Gr}(S)$. Then for any $a \in \langle E(S)\rangle$,

$aa^\omega = a$, and, since as mentioned $\langle E(S)\rangle$ is a subepigroup of S, we get that $\langle E(S)\rangle$ is a completely regular subsemigroup of S. Therefore, $S \in \mathbf{E}(\mathcal{CR})$. $\Box$

We remark that if $S \in \mathbf{E}(\mathcal{CR})$, then, from Proposition 2.4.2 in [21], for $\mathcal{K} \in \{\mathcal{L}, \mathcal{R}, \mathcal{H}\}$, we have $\mathcal{K}^{\langle E(S)\rangle} = \mathcal{K}^S \cap (\langle E(S)\rangle \times \langle E(S)\rangle)$, since $\langle E(S)\rangle$ is a (completely) regular subsemigroup of S (according to this comment, if $S \in \mathbf{E}(\mathcal{CR})$, an element in $\langle E(S)\rangle$ is a completely regular element of S must be a completely regular element of $\langle E(S)\rangle$). But the corresponding assertion for $\mathcal{D}$ is not true if we consider, for example, the semigroup B_2, where

$$B_2 = \langle c, d | c^2 = d^2 = 0, cdc = c, dcd = d\rangle.$$

As mentioned in [21, Section 2.4], $cd, dc \in \langle E(S)\rangle = E(S)$ and $cd\mathcal{D}^S dc$, while $(cd, dc) \notin \mathcal{D}^{\langle E(S)\rangle}$; therefore $\mathcal{D}^{\langle E(S)\rangle} \neq \mathcal{D}^S \cap (\langle E(S)\rangle \times \langle E(S)\rangle)$.

Naturally, it is worthwhile to study both $\langle E(D)\rangle$ and $\langle\langle E(D)\rangle\rangle$ of S generated by all idempotents in some regular $\mathcal{D}$-class D of S. We emphasize that even for $S \in \mathbf{E}(\mathcal{CR})$, in general, $\langle E(D)\rangle$ is not equal to $\langle\langle E(D)\rangle\rangle$. This is demonstrated by the following example.

Example 2.3 Let F_2 be defined in the variety $\mathcal{E}_2$ by the presentation

$$F_2 = \langle\langle c, d \,|\, cdc = c, dcd = d, c^2 = c^{2+\omega}, d^2 = d^{2+\omega}, c^{k+\omega}d = cd^{k+\omega}, d^{l+\omega}c = dc^{l+\omega}, k, l \in \mathbb{Z}\rangle\rangle.$$

The semigroup F_2 is a B_2-extension of a completely simple semigroup $M_2(\infty)$ (that is, an ideal extension of $M_2(\infty)$ by B_2), where $M_2(\infty) \simeq \mathcal{M}[2, C_{1,\infty}, 2; \left(\begin{smallmatrix}\varepsilon & \varepsilon \\ \varepsilon & \gamma^2\end{smallmatrix}\right)]$ (ε is the identity element and γ is a generator of infinite cyclic group $C_{1,\infty}$), and its egg-box picture can be presented in Figure 1. Clearly, $F_2 \in \mathbf{E}(\mathcal{CR})$, and for the $\mathcal{D}$-class D containing the idempotent cd,

$$\langle E(D)\rangle = \{cd, dc, c^{2n}, d^{2n}, c^{2n+1}d, d^{2n+1}c, n \in \mathbb{Z}^+\}.$$

It is true that $\langle E(D)\rangle \neq \langle\langle E(D)\rangle\rangle$, since $\overline{c^2} \notin \langle E(D)\rangle$, while $\overline{c^2} \in \langle\langle E(D)\rangle\rangle$.

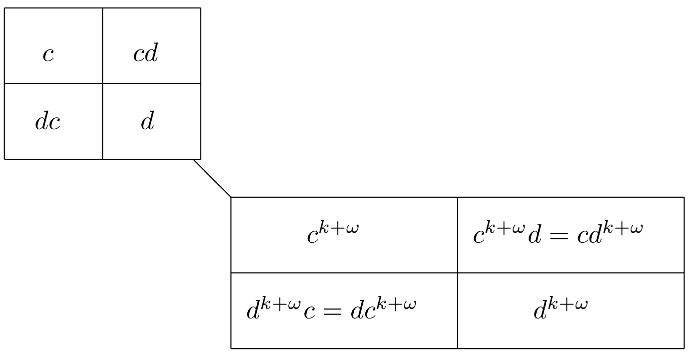

Figure 1 The egg-box picture of F_2

Recall that for any $a, b \in S$, $(ab)^\omega\mathcal{D}(ba)^\omega$ (see [1, Lemma 5]). As a corollary, we have

Lemma 2.4 *Let $a_1, a_2, \ldots, a_n \in S$ and $a = a_1a_2\ldots a_n$. Then for $i = 1, 2, \ldots, n$,*

$$(a_ia_{i+1}\ldots a_na_1a_2\ldots a_{i-1})^\omega\mathcal{D}a^\omega.$$

Lemma 2.5 *If $S \in \mathbf{E}(\mathcal{CR})$, then*

$$\langle E(S)\rangle = \bigcup_{D\in \mathrm{Reg}(S/\mathcal{D})} \langle E(D)\rangle = \bigcup_{D\in \mathrm{Reg}(S/\mathcal{D})} \langle\langle E(D)\rangle\rangle.$$

Proof Obviously, $\bigcup_{D\in \mathrm{Reg}(S/\mathcal{D})}\langle E(D)\rangle \subseteq \langle E(S)\rangle$, since $\langle E(D)\rangle \subseteq \langle E(S)\rangle$. For the reverse inclusion, take $x = e_1e_2\dots e_n$, $e_i \in E(S), i = 1,2,\dots,n$. Analogous to the proof of Lemma II.6.2 in [22] (for the more general case concerning regular semigroups, see [20, Theorem 1.4.18], and this technique appeared firstly in Fitz-Gerald [11]), set

$$f_i = e_ie_{i+1}\dots e_n\overline{x}e_1\dots e_i, \quad i = 1,2,\dots,n.$$

Since by hypothesis $x \in \mathrm{Gr}(S)$, that is, $xx^\omega = x$, it is routine to check that $x = f_1f_2\dots f_n$ and $f_i \in E(S)$. Also from Lemma 2.4, for $i = 1,2,\dots,n$, $f_i\mathcal{D}f_1$; hence, $x \in \langle E(D)\rangle$, where D is a regular $\mathcal{D}$-class containing elements f_i. Therefore $\langle E(S)\rangle \subseteq \bigcup_{D\in \mathrm{Reg}(S/\mathcal{D})}\langle E(D)\rangle$, as required.

For the second equality in this lemma, since $\langle E(S)\rangle = \bigcup_{D\in \mathrm{Reg}(S/\mathcal{D})}\langle E(D)\rangle$ as we have just proved in the preceding paragraph, and $\langle E(D)\rangle \subseteq \langle\langle E(D)\rangle\rangle$ for any $D \in \mathrm{Reg}(S/\mathcal{D})$, we have $\langle E(S)\rangle \subseteq \bigcup_{D\in \mathrm{Reg}(S/\mathcal{D})}\langle\langle E(D)\rangle\rangle$. Thus

$$\langle E(S)\rangle \subseteq \bigcup_{D\in \mathrm{Reg}(S/\mathcal{D})} \langle\langle E(D)\rangle\rangle \subseteq \langle\langle E(S)\rangle\rangle.$$

But $\langle E(S)\rangle = \langle\langle E(S)\rangle\rangle$ and so $\langle E(S)\rangle = \bigcup_{D\in \mathrm{Reg}(S/\mathcal{D})}\langle\langle E(D)\rangle\rangle$. $\square$

Before stating and proving the next proposition using induction on the number of elementary operations, let us first present a related definition. For any subset $X \subseteq S$, the depth of a term over X is defined recursively as follows (due to Shevrin, for example, see [4, Subsection 1.2]): the depth of any element of X is equal to 0; if $u, v \in \langle\langle X\rangle\rangle$ are terms of depth m, n, respectively, then the depth of uv is $m + n + 1$, and the depth of $\overline{u}$ is $m + 1$.

Proposition 2.6 *If $S \in \mathbf{E}(\mathcal{CR})$, then for an element $e \in E(S)$,*

$$\langle\langle E(D_e)\rangle\rangle \subseteq \bigcup_{\substack{f\in E(S)\\ f\le e}} \langle E(D_f)\rangle,$$

and $\bigcup_{\substack{f\in E(S)\\ f\le e}} \langle E(D_f)\rangle \in \mathcal{CR}$.

Proof Let $u \in \langle\langle E(D_e)\rangle\rangle$. Then u can be represented as a unary semigroup term over $E(D_e)$, where operations are multiplication and taking pseudo-inverse. The proof is proceeded by induction on the depth $n \ge 0$ of u. The base step $n = 0$ is obviously true, and so we proceed to the inductive step. If $n \ge 1$, then the term u can be written possibly into the following cases: either $u = u_1u_2$ for some "shorter" terms u_1, u_2, or $u = \overline{v}$ where v is of depth $n - 1$.

In the first case, by the inductive hypothesis, the terms u_i $(i = 1, 2)$ may be written in the forms

$$u_1 = f_1f_2\dots f_k,\ u_2 = g_{k+1}g_{k+2}\cdots g_m,$$

where $f_1, f_2,\dots, f_k \in E(D_f)$, $g_{k+1}, g_{k+2},\dots, g_m \in E(D_g)$, $f, g \le e$, whence

$$u = f_1f_2\dots f_k \cdot g_{k+1}g_{k+2}\cdots g_m.$$

Using Fitz-Gerald's method, as we have already done for an element in the core in the proof of Lemma 2.5, u can also be viewed as the product of the idempotents in the same $\mathcal{D}$-class, that is, $u = h_1h_2\dots h_m$, where $h_1, h_2, \dots, h_m \in E(D_{h_1})$ and

$$\begin{aligned} h_1 &= f_1f_2\dots f_k g_{k+1}g_{k+2}\dots g_m\overline{u}f_1, \\ h_2 &= f_2f_3\dots f_k g_{k+1}g_{k+2}\dots \overline{u}f_1f_2, \\ &\dots \\ h_k &= f_k g_{k+1}\dots g_m\overline{u}f_1f_2\dots f_k, \\ h_{k+1} &= g_{k+1}g_{k+2}\dots g_m\overline{u}f_1f_2\dots f_k g_{k+1}, \\ &\dots \\ h_m &= g_m\overline{u}f_1f_2\dots f_k g_{k+1}g_{k+2}\dots g_m. \end{aligned}$$

Notice that $f_1\mathcal{D}f$ implies that there exist s, t in S^1 such that $f_1 = sft$ and $f \le e$ if and only if $ef = fe = f$. Then

$$\begin{aligned} h_1 = h_1^\omega &= (f_1f_2\dots f_k g_{k+1}g_{k+2}\dots g_m\overline{u}f_1)^\omega = (sftf_2\dots f_k g_{k+1}g_{k+2}\dots g_m\overline{u}f_1)^\omega \\ &= (s\cdot eftf_2\dots f_k g_{k+1}g_{k+2}\dots g_m\overline{u}f_1)^\omega\mathcal{D}(eftf_2\dots f_k g_{k+1}g_{k+2}\dots g_m\overline{u}f_1s)^\omega \\ &= (e\cdot eftf_2\dots f_k g_{k+1}g_{k+2}\dots g_m\overline{u}f_1s)^\omega\mathcal{D}(eftf_2\dots f_k g_{k+1}g_{k+2}\dots g_m\overline{u}f_1se)^\omega. \end{aligned}$$

If we set $(eftf_2\dots f_k g_{k+1}g_{k+2}\dots g_m\overline{u}f_1se)^\omega = h$, then obviously $h \in E(S), h \le e, h_1, h_2, \dots, h_m \in E(D_h)$. Therefore, $u \in \langle E(D_h)\rangle$ for some idempotent $h \le e$.

For the second case, by the inductive hypothesis, there exists some $f \in E(S)$ with $f \le e$ such that $v \in \langle E(D_f)\rangle$, namely, $v = f_1f_2\dots f_k$, $f_i \in E(D_f), i = 1, 2, \dots, k$. Then from Lemma 2.1,

$$u = \overline{v} = (f_1f_2\dots f_k)^\omega(f_kf_1\dots f_{k-1})^\omega(f_{k-1}f_k\dots f_{k-2})^\omega\dots(f_1f_2\dots f_k)^\omega,$$

and also from Lemma 2.4

$$(f_1f_2\dots f_k)^\omega\mathcal{D}(f_kf_1\dots f_{k-1})^\omega\mathcal{D}(f_{k-1}f_k\dots f_{k-2})^\omega\mathcal{D}\dots\mathcal{D}(f_2f_3\dots f_kf_1)^\omega.$$

Notice that $f_1\mathcal{D}f$ implies that there exist s, t in S^1 such that $f_1 = sft$, and $f \le e$ if and only if $ef = fe = f$. Then

$$\begin{aligned} (f_1f_2\dots f_k)^\omega &= (sftf_2\dots f_k)^\omega = (s\cdot eftf_2\dots f_k)^\omega\mathcal{D}(eftf_2\dots f_ks)^\omega \\ &= (e\cdot eftf_2\dots f_ks)^\omega\mathcal{D}(eftf_2\dots f_kse)^\omega. \end{aligned}$$

If we set $(eftf_2\dots f_kse)^\omega = g$, then obviously $g \in E(S), g \le e$, $u \in \langle E(D_g)\rangle$.

Similar manipulation yields $\bigcup_{\substack{f\in E(S)\\ f\le e}}\langle E(D_f)\rangle$ is a subepigroup of $\langle E(S)\rangle$. Then by hypothesis we have $\bigcup_{\substack{f\in E(S)\\ f\le e}}\langle E(D_f)\rangle \in \mathcal{CR}$. $\square$

To formulate the main result in this paper, we need a semigroup given by the following

$$V = \langle e, f \mid e^2 = e, f^2 = f, fe = 0\rangle;$$

and we end this section by giving a lemma which will be used in the next section to prove the main result, and of course it is independently interesting.

Lemma 2.7 ([5, Lemma 1.3]) *Every semigroup S in which the product of any two idempotents is completely regular is E-solid.*

3. Epigroups whose cores are completely regular

Now we give the principal result in this paper presented in the following theorem which gives some characterizations of epigroups whose cores are completely regular.

Theorem 3.1 *The following conditions on an epigroup S are equivalent:*

(i) $S \in \mathbf{E}(\mathcal{CR})$;

(ii) *There is no semigroup V among the epidivisors of S;*

(iii) $\langle E(S)\rangle = \bigcup_{e\in E(S)}\langle E(D_e)\rangle$ *and* $\bigcup_{\substack{f\in E(S)\\ f\le e}}\langle E(D_f)\rangle \in \mathcal{CR}$ *for any* $e\in E(S)$;

(iv) $\langle E(S)\rangle = \bigcup_{D\in \mathrm{Reg}(S/\mathcal{D})}\langle\langle E(D)\rangle\rangle$ *and* $\langle\langle E(D)\rangle\rangle \in \mathcal{CR}$ *for any regular $\mathcal{D}$-class D of S;*

(v) $\langle E(S)\rangle = \bigcup_{e\in E(S)}\langle E(J(e))\rangle$ *and* $\langle E(J(e))\rangle \in \mathcal{CR}$ *for any* $e\in E(S)$;

(vi) *S satisfies the identity* $(x^\omega y^\omega)^{\omega+1} = x^\omega y^\omega$.

Proof The equivalence of (i) and (ii) is a corollary of Proposition 1.2 in [5].

(i)⇒(iii). This follows from Lemma 2.5 and Proposition 2.6.

(iii)⇒(iv). We show first that $\bigcup_{e\in E(S)}\langle E(D_e)\rangle = \bigcup_{e\in E(S)}(\bigcup_{\substack{f\in E(S)\\ f\le e}}\langle E(D_f)\rangle)$. Clearly

$$\bigcup_{e\in E(S)}\langle E(D_e)\rangle \subseteq \bigcup_{e\in E(S)}\Big(\bigcup_{\substack{f\in E(S)\\ f\le e}}\langle E(D_f)\rangle\Big),$$

since $\langle E(D_e)\rangle \subseteq \bigcup_{\substack{f\in E(S)\\ f\le e}}\langle E(D_f)\rangle$. Also

$$\bigcup_{e\in E(S)}\Big(\bigcup_{\substack{f\in E(S)\\ f\le e}}\langle E(D_f)\rangle\Big) \subseteq \bigcup_{e\in E(S)}\langle E(D_e)\rangle,$$

since $\bigcup_{\substack{f\in E(S)\\ f\le e}}\langle E(D_f)\rangle \subseteq \bigcup_{e\in E(S)}\langle E(D_e)\rangle$. Thus

$$\langle E(S)\rangle = \bigcup_{e\in E(S)}\langle E(D_e)\rangle = \bigcup_{e\in E(S)}\Big(\bigcup_{\substack{f\in E(S)\\ f\le e}}\langle E(D_f)\rangle\Big).$$

Then by hypothesis every element in $\langle E(S)\rangle$ is completely regular, which, by Lemma 2.2, implies $S\in\mathbf{E}(\mathcal{CR})$. Furthermore, from Lemma 2.5, $\langle E(S)\rangle = \bigcup_{D\in\mathrm{Reg}(S/\mathcal{D})}\langle\langle E(D)\rangle\rangle$; and $\langle\langle E(D)\rangle\rangle \in \mathcal{CR}$ follows immediately, since a subepigroup of a completely regular (sub)semigroup must be completely regular.

(iv)⇒(v). It is clear that (iv) implies (i) and then $S\in\mathbf{E}(\mathcal{CR})$. Clearly, $\bigcup_{e\in E(S)}\langle E(J(e))\rangle \subseteq \langle E(S)\rangle$ and the reverse inclusion follows by Lemma 2.5, since for any $e\in E(S)$, $D_e\subseteq J(e)$. It is also easy to check that $\langle E(J(e))\rangle\in\mathcal{CR}$ by Lemma 2.2.

(v)⇒(vi). For any $x,y\in S$, $x^\omega y^\omega \in \langle E(S)\rangle$, then the assumption in (v) means that $x^\omega y^\omega \in \mathrm{Gr}(S)$, that is, $(x^\omega y^\omega)^{\omega+1} = x^\omega y^\omega$.

(vi)⇒(i). Let $a\in\langle E(S)\rangle$. Notice that $\langle E(S)\rangle = \bigcup_{n=1}^{\infty}E^n$. Then $a\in E^n$ for some $n\in\mathbb{N}$. Once we prove that $a\in\mathrm{Gr}(S)$, then, by Lemma 2.2, the proof will be completed. We prove it by induction on $n \geq 1$. The base step $n = 1$ is trivially true and by the assumption in (vi) the step $n = 2$ is also true.

For the step $n = 3$, write $a = efg$, where $e, f, g \in E(S)$. By the hypothesis that the product of two arbitrary idempotents is a group element, we have

$$
\begin{aligned}
efg &= e(fg)^{\omega} \cdot fg = ef \cdot (ef)^{\omega} g, \\
e(fg)^{\omega} &= efg \cdot \overline{fg}, \quad (ef)^{\omega} g = \overline{ef} \cdot efg, \\
(ef)^{\omega}(fg)^{\omega} &= \overline{ef} \cdot ef(fg)^{\omega} = \overline{ef} \cdot e(fg)^{\omega} = (ef)^{\omega} g \cdot \overline{fg}, \\
e(fg)^{\omega} &= ef(fg)^{\omega} = ef \cdot (ef)^{\omega}(fg)^{\omega}, (ef)^{\omega} g = (ef)^{\omega}(fg)^{\omega} \cdot fg,
\end{aligned}
$$

so that

$$efg \mathcal{R} e(fg)^{\omega} \mathcal{L} (ef)^{\omega}(fg)^{\omega} \mathcal{R} (ef)^{\omega} g \mathcal{L} efg.$$

Thus $efg, e(fg)^{\omega}, (ef)^{\omega} g$ and $(ef)^{\omega}(fg)^{\omega}$ are located in the egg-box picture of D_{efg} as indicated in Figure 2.

efg		$e(fg)^{\omega} = ef(fg)^{\omega}$
$(ef)^{\omega} g = (ef)^{\omega} fg$		$(ef)^{\omega}(fg)^{\omega}$

Figure 2 The locations of $efg, e(fg)^{\omega}, (ef)^{\omega} g$ and $(ef)^{\omega}(fg)^{\omega}$ in D_{efg}

Since, by the hypothesis in (vi), $(ef)^{\omega} g, e(fg)^{\omega}, (ef)^{\omega}(fg)^{\omega}$ are all completely regular, say,

$$(ef)^{\omega} g \mathcal{H} ((ef)^{\omega} g)^{\omega}, e(fg)^{\omega} \mathcal{H} (e(fg)^{\omega})^{\omega}, (ef)^{\omega}(fg)^{\omega} \mathcal{H} ((ef)^{\omega}(fg)^{\omega})^{\omega}.$$

Thus the locations of efg, $((ef)^{\omega} g)^{\omega}$, $(e(fg)^{\omega})^{\omega}$ and $((ef)^{\omega}(fg)^{\omega})^{\omega}$ in the egg-box picture of D_{efg} are as indicated in Figure 3.

Now from Lemma 2.7 there exists an idempotent in H_{efg} and then H_{efg} is a subgroup of S (see [21, Corollary 2.2.6]), so that $efg \in \mathrm{Gr}(S)$, as required.

efg		$(e(fg)^{\omega})^{\omega}$
$((ef)^{\omega} g)^{\omega}$		$((ef)^{\omega}(fg)^{\omega})^{\omega}$

Figure 3 The locations of $efg, (e(fg)^{\omega})^{\omega}, ((ef)^{\omega} g)^{\omega}$ and $((ef)^{\omega}(fg)^{\omega})^{\omega}$ in D_{efg}

For the inductive step, suppose that $n \geq 4$. Then $a = e_1 e_2 \dots e_n$ for some $e_i \in E(S)$. Since $e_3 e_4 \dots e_n \in E^{n-2}$, by the inductive hypothesis, we get $e_3 e_4 \dots e_n = (e_3 e_4 \dots e_n)^{\omega} e_3 e_4 \dots e_n$. Then $a = e_1 e_2 \cdot e_3 \dots e_n = e_1 e_2 (e_3 e_4 \dots e_n)^{\omega} \cdot e_3 e_4 \dots e_n$. While

$$e_1 e_2 (e_3 e_4 \dots e_n)^{\omega} = e_1 e_2 \cdot e_3 e_4 \dots e_n \cdot \overline{e_3 e_4 \dots e_n} = a \cdot \overline{e_3 e_4 \dots e_n};$$

therefore $a\,\mathcal{R}\,e_1e_2(e_3e_4\dots e_n)^\omega$. On the other hand,

$$e_1e_2(e_3e_4\dots e_n)^\omega = e_1e_2\cdot(e_1e_2)^\omega(e_3e_4\dots e_n)^\omega,$$
$$(e_1e_2)^\omega(e_3e_4\dots e_n)^\omega = \overline{e_1e_2}\cdot e_1e_2(e_3e_4\dots e_n)^\omega,$$

so that $e_1e_2(e_3e_4\dots e_n)^\omega\,\mathcal{L}\,(e_1e_2)^\omega(e_3e_4\dots e_n)^\omega$.

As above, considering the element $(e_1e_2)^\omega e_3e_4\dots e_n$ in a similar way, we have

$$a = e_1e_2\cdot e_3e_4\dots e_n = e_1e_2\cdot(e_1e_2)^\omega e_3e_4\dots e_n,$$
$$(e_1e_2)^\omega e_3e_4\dots e_n = \overline{e_1e_2}\cdot e_1e_2\dots e_n = \overline{e_1e_2}\cdot a,$$
$$(e_1e_2)^\omega e_3e_4\dots e_n = (e_1e_2)^\omega(e_3e_4\dots e_n)^\omega\cdot e_3e_4\dots e_n,$$
$$(e_1e_2)^\omega(e_3e_4\dots e_n)^\omega = (e_1e_2)^\omega e_3e_4\dots e_n\cdot\overline{e_3e_4\dots e_n},$$

so that

$$a\,\mathcal{L}\,(e_1e_2)^\omega e_3e_4\dots e_n\,\mathcal{R}\,(e_1e_2)^\omega(e_3e_4\dots e_n)^\omega.$$

Thus $a, (e_1e_2)^\omega e_3e_4\dots e_n, e_1e_2(e_3e_4\dots e_n)^\omega$ and $(e_1e_2)^\omega(e_3e_4\dots e_n)^\omega$ are located in the egg-box picture of D_a as indicated in Figure 4.

a		$e_1e_2(e_3e_4\dots e_n)^\omega$
$(e_1e_2)^\omega e_3e_4\dots e_n$		$(e_1e_2)^\omega(e_3e_4\dots e_n)^\omega$

Figure 4 The locations of $a, (e_1e_2)^\omega e_3e_4\dots e_n, e_1e_2(e_3e_4\dots e_n)^\omega$ and $(e_1e_2)^\omega(e_3e_4\dots e_n)^\omega$ in D_a

Since $(e_1e_2)^\omega e_3e_4\dots e_n \in E^{n-1}, e_1e_2(e_3e_4\dots e_n)^\omega \in E^3, (e_1e_2)^\omega(e_3e_4\dots e_n)^\omega \in E^2$, from the facts we have proved and the inductive hypothesis, these elements are all completely regular. Similar to the proof for the step $n=3$, the locations of a, $((e_1e_2)^\omega e_3e_4\dots e_n)^\omega$, $(e_1e_2(e_3e_4\dots e_n)^\omega)^\omega$ and $((e_1e_2)^\omega(e_3e_4\dots e_n)^\omega)^\omega$ in the egg-box picture of D_a are as indicated in Figure 5. Then again from Lemma 2.7, there exists an idempotent in H_a and then H_a is a subgroup of S, and so $a\in \mathrm{Gr}(S)$, as required. $\square$

a		$(e_1e_2(e_3e_4\dots e_n)^\omega)^\omega$
$((e_1e_2)^\omega e_3e_4\dots e_n)^\omega$		$((e_1e_2)^\omega(e_3e_4\dots e_n)^\omega)^\omega$

Figure 5 The locations of $a, ((e_1e_2)^\omega e_3e_4\dots e_n)^\omega, (e_1e_2(e_3e_4\dots e_n)^\omega)^\omega$ and $((e_1e_2)^\omega(e_3e_4\dots e_n)^\omega)^\omega$ in D_{a^ω}

Remark 3.2 The equivalence of (i) and (vi) in Theorem 3.1 for a finite semigroup is known (for example, see [17, Exercise 5.2.17]). For an epigroup S, besides the direct proof of "(vi)⇒(i)" in our proof, we can check it from the point of view of properties of regularity. From Result 2 in [23], for an arbitrary semigroup U, $\langle E(U)\rangle$ is the regular subsemigroup of U if and only if the product of two arbitrary idempotents in U is regular. Thus for the given S, the identity in (vi) in Theorem 3.1 implies that $\langle E(S)\rangle$ is the regular subepigrop of S (recall that $\langle E(S)\rangle = \langle\langle E(S)\rangle\rangle$ mentioned early) and is inherited by $\langle E(S)\rangle$ (that is, $\langle E(S)\rangle$ satisfies the identity $(x^\omega y^\omega)^{\omega+1} = x^\omega y^\omega$). Now we may consider solely the regular subepigroup $T = \langle E(S)\rangle$ instead of S, since we only investigate the behaving of $\langle E(S)\rangle$ and nothing else. Trivially T is a semiband (recall that a semiband is a regular semigroup generated by its idempotent). From Lemma 2.7, T is E-solid and thus T is completely regular, as required, in view of Theorem 3 in [13] (which says that for a regular semigroup U, $\langle E(U)\rangle$ is completely regular if and only if U is E-solid). We remark that here the check showing that any element a in $\langle E(S)\rangle$ is completely regular relys on the regularity (for example, the proof of [13, Theorem 3] utilizes the inverse of a, while our proof mainly takes advantage of the unary operation of taking pseudo-inverse since we deal with epigroups viewed as unary semigroups). In fact, by using the technique of biordered sets, as in the proof of [20, Theorem 3.4.11], it was also shown that a semiband is completely regular if and only if the semiband is E-solid.

4. Some restrictions

In this section we consider some special subclasses of the class of epigroups whose cores are completely regular and begin by determining whether a completely 0-simple semigroup has this property.

In general, the core of a completely 0-simple semgiroup is not completely regular. For example, the completely 0-simple semgiroup A_2 given by a presentation

$$A_2 = \langle c, d \mid c^2 = 0, d^2 = d, cdc = c, dcd = d\rangle$$

is idempotent-generated, so the core of A_2 is A_2 itself and is not completely regular. We remark that A_2 has an epidivisor consisting precisely of elements $0, c, cd$ and dc, which is isomorphic to V (here $e = cd, f = dc$).

We recall that every completely 0-simple semigroup S is isomorphic to some Rees matrix semigroup $\mathcal{M}^0(I, G, \Lambda; P)$ over the 0-group G^0 with the regular sandwich matrix $P = (p_{\lambda i})$ (see [21, Theorem 3.2.3]). It is easy to identify that for $S = \mathcal{M}^0(I, G, \Lambda; P)$,

$$E(S) = \{(i, \overline{p_{\lambda i}}, \lambda) \in S \mid i \in I, \lambda \in \Lambda, p_{\lambda i} \neq 0\} \cup \{0\},$$
$$\mathrm{Gr}(S) = \{(i, g, \lambda) \in S \mid i \in I, \lambda \in \Lambda, g \in G, p_{\lambda i} \neq 0\} \cup \{0\}.$$

We refer to Howie [24] for an investigation of the core of a completely 0-simple semgiroup.

The following proposition indicates that for a completely 0-simple semgiroup $S = \mathcal{M}^0(I, G, \Lambda; P)$, the condition $S \in \mathbf{E}(\mathcal{CR})$ is involved solely with an arrangement of non-zero entries of P.

Proposition 4.1 *Let $S = \mathcal{M}^0(I, G, \Lambda; P)$ be a Rees matrix semigroup. Then $S \in \mathbf{E}(\mathcal{CR})$ if and only if for all $i, j \in I$, $\lambda, \mu \in \Lambda$,*

$$p_{\lambda i} \neq 0, p_{\lambda j} \neq 0, p_{\mu j} \neq 0 \Rightarrow p_{\mu i} \neq 0.$$

Proof Suppose that $p_{\lambda i} \neq 0, p_{\lambda j} \neq 0, p_{\mu j} \neq 0$. Then $(i, \overline{p_{\lambda i}}, \lambda), (j, \overline{p_{\mu j}}, \mu) \in E(S)$, so that by Theorem 3.1

$$(i, \overline{p_{\lambda i}}, \lambda)(j, \overline{p_{\mu j}}, \mu) = (i, \overline{p_{\lambda i}} p_{\lambda j} \overline{p_{\mu j}}, \mu) \in \mathrm{Gr}(S),$$

which gives $p_{\mu i} \neq 0$, since by hypothesis $(i, \overline{p_{\lambda i}} p_{\lambda j} \overline{p_{\mu j}}, \mu) \neq 0$.

For the converse, by Theorem 3.1, it suffices to prove that for any $e, f \in E(S)$, $ef \in \mathrm{Gr}S$. If $e = 0$ or $f = 0$, obviously $ef \in \mathrm{Gr}S$. It remains to show the case $e \neq 0, f \neq 0$. In this case, let $e = (i, \overline{p_{\lambda i}}, \lambda), f = (j, \overline{p_{\mu j}}, \mu)$ with $i, j \in I$, $\lambda, \mu \in \Lambda$ such that $p_{\lambda i} \neq 0, p_{\mu j} \neq 0$. Now

$$ef = (i, \overline{p_{\lambda i}}, \lambda)(j, \overline{p_{\mu j}}, \mu) = (i, \overline{p_{\lambda i}} p_{\lambda j} \overline{p_{\mu j}}, \mu).$$

If $p_{\lambda j} = 0$, clearly $ef = 0 \in \mathrm{Gr}(S)$; if $p_{\lambda j} \neq 0$, then by hypothesis $p_{\mu i} \neq 0$, which also gives $ef \in \mathrm{Gr}(S)$. $\square$

As we have mentioned earlier (see Example 2.3), if $S \in \mathbf{E}(\mathcal{CR})$, we cannot obtain $\langle E(D) \rangle = \langle\langle E(D) \rangle\rangle$ for every regular $\mathcal{D}$-class D of S. We now ask what else is necessary to conclude that $\langle E(D) \rangle = \langle\langle E(D) \rangle\rangle$ for some regular $\mathcal{D}$-class D of S.

Proposition 4.2 *Let $S \in \mathbf{E}(\mathcal{CR})$ and D be a regular $\mathcal{D}$-class of S. Then for any two idempotents $e, f \in \langle E(D) \rangle$, $(ef)^\omega \in \langle E(D) \rangle$ if and only if $\langle\langle E(D) \rangle\rangle = \langle E(D) \rangle$. In this case, $\langle E(D) \rangle$ is a completely regular subsemigroup of S.*

Proof If $\langle\langle E(D) \rangle\rangle = \langle E(D) \rangle$, then for any $e, f \in \langle E(D) \rangle$, $ef, \overline{ef} \in \langle E(D) \rangle$, so that $(ef)^\omega \in \langle E(D) \rangle$. The fact that $\langle E(D) \rangle$ is a completely regular subsemigroup of S is clear.

Conversely, to show that $\langle\langle E(D) \rangle\rangle = \langle E(D) \rangle$, we need to show that for any $a \in \langle E(D) \rangle$, $\overline{a} \in \langle E(D) \rangle$. Write $a = e_1 e_2 \ldots e_n$, $e_i \in D$, $i = 1, 2, \ldots, n$. Then from Lemma 2.1

$$\overline{a} = (e_1 e_2 \ldots e_n)^\omega (e_n e_1 \ldots e_{n-1})^\omega (e_{n-1} e_n \ldots e_{n-2})^\omega \ldots (e_1 e_2 \ldots e_n)^\omega.$$

If we can show that elements $(e_i e_{i+1} \ldots e_n e_1 e_2 \ldots e_{i-1})^\omega$ $(i = 1, 2, \ldots, n)$ are all in $\langle E(D) \rangle$, then $\overline{a} \in \langle E(D) \rangle$, and this will complete the proof. The fact that these elements are all in $\langle E(D) \rangle$ is guaranteed by the following Lemma. $\square$

Lemma 4.3 *Let $S \in \mathbf{E}(\mathcal{CR})$ and D be a regular $\mathcal{D}$-class of S. If for any two idempotents $e, f \in \langle E(D) \rangle$, $(ef)^\omega \in \langle E(D) \rangle$, then for any $a \in \langle E(D) \rangle$, $a^\omega \in \langle E(D) \rangle$.*

Proof Let $a = e_1 e_2 \ldots e_n$, $e_i \in D$, $i = 1, 2, \ldots, n$. We show that $a^\omega \in \langle E(D) \rangle$. The proof is by induction on $n \geq 1$, The base step $n = 1$ is trivially true. By hypothesis the step $n = 2$ is also true.

For the step $n = 3$, analogous to the proof of "(vi)$\Rightarrow$(i)" in Theorem 3.1, the locations of $a^\omega, ((e_1 e_2)^\omega e_3)^\omega, (e_1 (e_2 e_3)^\omega)^\omega$ and $((e_1 e_2)^\omega (e_2 e_3)^\omega)^\omega$ in the egg-box picture of D_{a^ω} are as indicated in Figure 6; then $a^\omega = \big((e_1 (e_2 e_3)^\omega)^\omega \cdot ((e_1 e_2)^\omega e_3)^\omega\big)^\omega$. By hypothesis it is easy to check that $a^\omega \in \langle E(D) \rangle$.

For the inductive step, suppose that $n \geq 4$. Analogous to the proof of "(vi)$\Rightarrow$(i)" in Theorem 3.1, the locations of $a^\omega, ((e_1 e_2)^\omega e_3 e_4 \ldots e_n)^\omega, (e_1 e_2 (e_3 e_4 \ldots e_n)^\omega)^\omega$ and $((e_1 e_2)^\omega (e_3 e_4 \ldots e_n)^\omega)^\omega$ in D_{a^ω} are as indicated in Figure 7. Since by the hypothesis that $\langle E(S) \rangle$ is completely regular, it is easy to obtain that $a^\omega = ((e_1e_2(e_3e_4\ldots e_n)^\omega)^\omega \cdot ((e_1e_2)^\omega e_3e_4\ldots e_n)^\omega)^\omega$. By the (inductive) hypothesis, it is now routine to check $a^\omega \in \langle E(D)\rangle$. $\square$

a^ω		$(e_1(e_2e_3)^\omega)^\omega$
$((e_1e_2)^\omega e_3)^\omega$		$((e_1e_2)^\omega(e_2e_3)^\omega)^\omega$

Figure 6 The locations of $a^\omega, ((e_1e_2)^\omega e_3)^\omega, (e_1(e_2e_3)^\omega)^\omega$ and $((e_1e_2)^\omega(e_2e_3)^\omega)^\omega$ in D_{a^ω}

Someone might take for granted that in Proposition 4.2 if $S \in \mathbf{E}(\mathcal{CR})$, then $\langle\langle E(D)\rangle\rangle$ is a completely simple subsemigroup of S. But this is not true in general; for example, the subepigroup $\langle\langle E(D_{cd})\rangle\rangle$ of B_2 generated by $E(D_{cd})$ is not completely simple.

a^ω		$(e_1e_2(e_3e_4\ldots e_n)^\omega)^\omega$
$((e_1e_2)^\omega e_3e_4\ldots e_n)^\omega$		$((e_1e_2)^\omega(e_3e_4\ldots e_n)^\omega)^\omega$

Figure 7 The locations of $a^\omega, ((e_1e_2)^\omega e_3e_4\ldots e_n)^\omega, (e_1e_2(e_3e_4\ldots e_n)^\omega)^\omega$ and $((e_1e_2)^\omega(e_3e_4\ldots e_n)^\omega)^\omega$ in D_{a^ω}

Recall that a semigroup is called periodic if the monogenic subsemigroup generated by its each element is finite. Since each element in a periodic semigroup has a power which is idempotent, by Proposition 4.2, we get

Corollary 4.4 *Let S be a periodic semigroup. If $S \in \mathbf{E}(\mathcal{CR})$, then for any regular $\mathcal{D}$-class D of S, $\langle\langle E(D)\rangle\rangle = \langle E(D)\rangle$. In this case, $\langle E(D)\rangle$ is a completely regular subsemigroup of S.*

We now turn to epigroups decomposable into semilattices of archimedean epigroups. For the characterizations of these epigroups from different points of view, we refer to Theorem 3 in [1] (or Theorem 3.16 in [3]), and especially it says that an epigroup S decomposes into a semilattice of archimedean epigroups if and only if each regular $\mathcal{D}$-class in S forms a (completely simple) subsemigroup. Now let $S \in \mathbf{E}(\mathcal{CR})$. Comparing with periodic semigroups, if S is a semilattice of archimedean epigroups, then for any regular $\mathcal{D}$-class of S, we also get $\langle\langle E(D)\rangle\rangle = \langle E(D)\rangle$; while here $\langle E(D)\rangle$ is completely simple subsemigroup of S and this is not the same with a periodic semigroup. These facts will be reflected in the following proposition.

Proposition 4.5 *The following conditions on an epigroup S are equivalent:*

(i) S is a semilattice of archimedean epigroups, and $\langle E(S)\rangle$ is a completely regular subsemigroup of S;

(ii) $\langle E(S)\rangle = \bigcup_{D\in \mathrm{Reg}(S/\mathcal{D})}\langle E(D)\rangle$, and for any regular $\mathcal{D}$-class D of S, $\langle E(D)\rangle$ is a completely simple subsemigroup of S;

(iii) There are no semigroups B_2, V among the epidivisors of S;

(iv) S satisfies the identities $((xy)^\omega(yx)^\omega(xy)^\omega)^\omega = (xy)^\omega$, $(x^\omega y^\omega)^{\omega+1} = x^\omega y^\omega$.

Proof The equivalence of (iii) and (iv) is clear by virtue of Theorem 3 in [1] and Theorem 3.1 in the paper.

(i) $\Leftrightarrow$ (iii). From Theorem 3 in [1], S is a semilattice of archimedean epigroups if and only if there are no semigroups A_2, B_2 among the epidivisors of S. This together with Theorem 3.1 implies that the condition in (i) is equivalent to saying that there are no semigroups A_2, B_2, V among the epidivisors of S. As mentioned in the 2nd paragraph in this section, V is among the epidivisors of A_2, so that the equivalence of (i) and (iii) holds.

(i) $\Rightarrow$ (ii). Suppose that (i) holds. On the one hand, by Theorem 3.1,

$$\langle E(S)\rangle = \bigcup_{D\in \mathrm{Reg}(S/\mathcal{D})} \langle\langle E(D)\rangle\rangle,$$

where $\langle\langle E(D)\rangle\rangle$ is a completely regular subsemigroup of S for any regular $\mathcal{D}$-class D of S; on the other hand, by Theorem 3 in [1], each regular $\mathcal{D}$-class D is a completely simple subsemigroup, and then $\langle\langle E(D)\rangle\rangle$ is a completely simple subsemigroup of S (since its class forms a variety). Also from Proposition 4.2 $\langle\langle E(D)\rangle\rangle = \langle E(D)\rangle$, and this completes the proof of this implication.

(ii) $\Rightarrow$ (iv). Clearly the conditions in (ii) imply that $S \in \mathbf{E}(\mathcal{CR})$, then from Theorem 3.1 the second identity holds. For the first identity, let $x, y \in S$. Then, from Lemma 5 in [1], there exists some regular $\mathcal{D}$-class D of S such that $(xy)^\omega, (yx)^\omega \in \langle E(D)\rangle$ and so by the hypothesis that $\langle E(D)\rangle$ is completely simple, we obtain $((xy)^\omega(yx)^\omega(xy)^\omega)^\omega = (xy)^\omega$. $\Box$

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