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Weighted Representation Asymptotic Basis of Integers

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Abstract Let k_1, k_2 be nonzero integers with $(k_1, k_2) = 1$ and $k_1k_2 \neq -1$. Let $R_{k_1,k_2}(A, n)$ be the number of solutions of $n = k_1a_1 + k_2a_2$, where $a_1, a_2 \in A$. Recently, Xiong proved that there is a set $A \subseteq \mathbb{Z}$ such that $R_{k_1,k_2}(A, n) = 1$ for all $n \in \mathbb{Z}$. Let $f : \mathbb{Z} \longrightarrow \mathbb{N}_0 \cup \{\infty\}$ be a function such that $f^{-1}(0)$ is finite. In this paper, we generalize Xiong's result and prove that there exist uncountably many sets $A \subseteq \mathbb{Z}$ such that $R_{k_1,k_2}(A, n) = f(n)$ for all $n \in \mathbb{Z}$.

Keywords additive basis; representation function

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1. Introduction

For sets A and B of integers and integers k_1, k_2, let

$$k_1A + k_2B = \{k_1a + k_2b : a \in A, b \in B\}.$$

The counting function for the set A is

$$A(y, x) = \mathrm{card}\{a \in A : y \leqslant a \leqslant x\}.$$

For $A \subseteq \mathbb{Z}$ and $n \in \mathbb{Z}$, let $R_{k_1,k_2}(A, n)$ be the number of solutions of $n = k_1a_1 + k_2a_2$, where $a_1, a_2 \in A$. We call A a weighted representation asymptotic basis if $R_{k_1,k_2}(A, n) \geqslant 1$ for all $n \in \mathbb{Z}$ with at most finite exceptions. In 2003, Nathanson [2] constructed a family of arbitrarily sparse bases $A \subseteq \mathbb{Z}$ satisfying $R_{1,1}(A, n) = 1$ for all $n \in \mathbb{Z}$. Let $f : \mathbb{Z} \longrightarrow \mathbb{N}_0 \cup \{\infty\}$ be any function such that $f^{-1}(0)$ is finite. In 2004, Nathanson [3] constructed a family of arbitrarily sparse bases $A \subseteq \mathbb{Z}$ satisfying $R_{1,1}(A, n) = f(n)$ for all $n \in \mathbb{Z}$. In 2005, Nathanson [4] proved that there exists a family of arbitrarily sparse bases of $A \subset \mathbb{Z}$ such that $R_{A,h}(n) = f(n)$ for all $n \in \mathbb{Z}$, where $R_{A,h}(n) = \sharp\{(a_1, \dots, a_h) \in A^h : n = a_1 + \cdots + a_h, a_1 \leqslant a_2 \leqslant \cdots \leqslant a_h\}$. In 2011, Tang et al. [5] proved that there exists a family of bases of $A \subseteq \mathbb{Z}$ satisfying $R_{1,-1}(A, n) = 1$ for all $n \neq 0$. In 2014, Xiong [7] proved that there exists a family of bases of $A \subseteq \mathbb{Z}$ satisfying $R_{l_1,l_2}(A, n) = 1$ for all $n \in \mathbb{Z}$, where l_1, l_2 are nonzero integers with $(l_1, l_2) = 1$ and $l_1l_2 \neq -1$. We refer to [1,6,8,9] for related problems.

In this paper, we obtain the following result.

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Theorem 1.1 *Let k_1, k_2 be nonzero integers with $(k_1, k_2) = 1$, $k_1k_2 \neq -1$ and $f : \mathbb{Z} \longrightarrow \mathbb{N}_0 \cup \{\infty\}$ such that*

$$\triangle = \operatorname{card}(f^{-1}(0)) < \infty.$$

Then there exist uncountably many weighted representation asymptotic bases $A \subset \mathbb{Z}$ such that

$$R_{k_1,k_2}(A, n) = f(n) \quad \text{for all} \quad n \in \mathbb{Z},$$

and

$$A(-x, x) \geqslant (\frac{x}{c})^{1/3},$$

where

$$c = M\{16 + [\frac{\triangle + 1}{2}]\}$$

and M is a constant depending on integers k_1 and k_2.

2. Proof of Theorem 1.1

To prove Theorem 1.1, we need the following Lemma:

Lemma 2.1 ([3, Lemma 1]) *Let $f : \mathbb{Z} \longrightarrow \mathbb{N}_0 \cup \{\infty\}$ be a function such that $f^{-1}(0)$ is finite. Let $\triangle$ denote the cardinality of the set $f^{-1}(0)$. Then there exists a sequence $U = \{\mu_l\}_{l=1}^{\infty}$ of integers such that, for every $n \in \mathbb{Z}$ and $l \in \mathbb{N}$, $f(n) = \operatorname{card}\{l \geqslant 1 : \mu_l = n\}$, and $|\mu_l| \leqslant [\frac{l+\triangle}{2}]$.*

Proof of Theorem 1.1 By Lemma 2.1, we know there exists a sequence $U = \{\mu_l\}_{l=1}^{\infty}$ of integers such that

$$f(n) = \operatorname{card}\{i \in \mathbb{N} : \mu_i = n\} \quad \text{for all integers} \quad n \tag{1}$$

and

$$|\mu_l| \leqslant \frac{l + \triangle}{2} \quad \text{for all} \quad l \geqslant 1. \tag{2}$$

We shall construct a strictly increasing sequence $\{i_l\}_{l=1}^{\infty}$ of positive integers and an sequence $\{A_l\}_{l=1}^{\infty}$ of finite sets of integers such that

(i) $|A_l| = 2l$;

(ii) there exists a positive number c such that $A_l \subseteq [-cl^3, cl^3]$;

(iii) $R_{k_1,k_2}(A_l, n) \leqslant f(n)$ for all $n \in \mathbb{Z}$;

(iv) $R_{k_1,k_2}(A_l, \mu_j) \geqslant \operatorname{card}\{i \leqslant i_l : \mu_i = \mu_j\}$ for $j = 1, \ldots, l$.

We shall show that the infinite set

$$A = \bigcup_{l=1}^{\infty} A_l$$

is a (k_1, k_2)-weighted representation asymptotic basis of $\mathbb{Z}$ satisfying Theorem 1.1.

We construct A_l by induction. Since $(k_1, k_2) = 1$, there exist integers x_1, x_2 such that $k_1x_1 + k_2x_2 = 1$. Let $i_1 = 1$. Let $A_1 = \{k_2a_1 + x_1\mu_{i_1}, -k_1a_1 + x_2\mu_{i_1}\}$, where integer a_1 is chosen to satisfy the following conditions

(a) $(k_1A_1 + k_2A_1) \cap f^{-1}(0) = \varnothing$,

(b) $k_2a_1 + x_1\mu_{i_1} \neq -k_1a_1 + x_2\mu_{i_1}$,

(c) μ_{i_1}, $(k_2^2 - k_1^2)a_1 + (k_2x_1 + k_1x_2)\mu_{i_1}$, $(k_1 + k_2)(k_2a_1 + x_1\mu_{i_1})$, $(k_1 + k_2)(-k_1a_1 + x_2\mu_{i_1})$ are pairwise distinct.

The conditions (a)–(c) exclude at most $7+3\triangle$ integers, so there exist more than one choice for the number a_1 such that $|a_1| \leqslant 2\triangle + 3$, and a_1 satisfies (a)–(c).

Since $|\mu_{i_1}| = |\mu_1| \leqslant (1 + \triangle)/2$ and

$$|k_2a_1 + x_1\mu_{i_1}| \leqslant |k_2||a_1| + |x_1||\mu_{i_1}| \leqslant M_1 \cdot \frac{5\triangle + 7}{2},$$
$$|-k_1a_1 + x_2\mu_{i_1}| \leqslant |k_1||a_1| + |x_2||\mu_{i_1}| \leqslant M_2 \cdot \frac{5\triangle + 7}{2},$$

where $M_1 = \max\{|k_2|, |x_1|\}$, $M_2 = \max\{|k_1|, |x_2|\}$.

It follows that $A_1 \subseteq [-c, c]$ for any $c \geqslant \max\{M_1(5\triangle+7)/2, M_2(5\triangle+7)/2\}$, and A_1 satisfies conditions (i)–(iv).

Assume that for some l, we have constructed $A_1 \subseteq \cdots \subseteq A_{l-1}$ satisfying (i)–(iv). Now we construct A_l. Let $i_l > i_{l-1}$ be the least integer such that

$$R_{k_1,k_2}(A_{l-1}, \mu_{i_l}) < f(\mu_{i_l}).$$

Then if $n = \mu_{i_{l-1}+1}, \ldots, \mu_{i_l-1}$, by (iii) and (1) we have

$$R_{k_1,k_2}(A_{l-1}, n) = f(n) \geqslant 1. \tag{3}$$

Thus by the fact that $A_1 \subseteq \cdots \subseteq A_{l-1}$ and (3), we have

$$\begin{aligned}
i_l - 1 &\leqslant R_{k_1,k_2}(A_1, \mu_{i_1}) + \sum_{j=2}^{l} \sum_{n\in\{\mu_{i_{j-1}+1},\ldots,\mu_{i_j-1}\}} R_{k_1,k_2}(A_{j-1}, n) \\
&\leqslant \sum_{n\in\{\mu_1,\ldots,\mu_{i_l-1}\}} R_{k_1,k_2}(A_{l-1}, n) \leqslant \sum_{n\in\mathbb{Z}} R_{k_1,k_2}(A_{l-1}, n) \\
&= \binom{2l-1}{2} < 2l^2.
\end{aligned}$$

Therefore $i_l \leqslant 2l^2$, and $\mu_{i_l} \leqslant l^2 + \frac{\triangle}{2}$. Let

$$A_l = A_{l-1} \cup \{k_2a_l + x_1\mu_{i_l}, -k_1a_l + x_2\mu_{i_l}\}.$$

So

$$k_1A_l + k_2A_l = \bigcup_{i=1}^{6} T_i,$$

where

$$\begin{aligned}
T_1 =& k_1A_{l-1} + k_2A_{l-1}, \quad T_2 = k_1A_{l-1} + k_2(k_2a_l + x_1\mu_{i_l}), \\
T_3 =& k_1A_{l-1} + k_2(-k_1a_l + x_2\mu_{i_l}), \quad T_4 = k_2A_{l-1} + k_1(k_2a_l + x_1\mu_{i_l}), \\
T_5 =& k_2A_{l-1} + k_1(-k_1a_l + x_2\mu_{i_l}), \\
T_6 =& \{\mu_{i_l}, (k_2^2 - k_1^2)a_l + (k_2x_1 + k_1x_2)\mu_{i_l}, \\
& (k_1 + k_2)(k_2a_l + x_1\mu_{i_l}), (k_1 + k_2)(-k_1a_l + x_2\mu_{i_l})\}.
\end{aligned}$$

The set A_l satisfies (i) if $k_2a_l + x_1\mu_{i_l} \notin A_{l-1}$, $-k_1a_l + x_2\mu_{i_l} \notin A_{l-1}$ and $k_2a_l + x_1\mu_{i_l} \neq -k_1a_l + x_2\mu_{i_l}$, and we exclude at most $4l-3$ integers as possible choices a_l.

The set A_l satisfies (iii), (iv) if

$$(k_1A_l + k_2A_l) \cap f^{-1}(0) = \emptyset$$

and

$$R_{k_1,k_2}(A_l, n) = \begin{cases} R_{k_1,k_2}(A_{l-1}, n), & \text{if } n \in (k_1A_{l-1} + k_2A_{l-1}) \backslash \{\mu_{i_l}\}, \\ R_{k_1,k_2}(A_{l-1}, n) + 1, & \textit{if } n = \mu_{i_l}, \\ 1, & \textit{if } n \in (k_1A_l + k_2A_l) \backslash ((k_1A_{l-1} + k_2A_{l-1}) \cup \{\mu_{i_l}\}). \end{cases}$$

Since $k_1A_l + k_2A_l = \bigcup_{i=1}^{6} T_i$, it suffices to require that

(d) $(k_1A_l + k_2A_l) \cap f^{-1}(0) = \varnothing$,

(e) $T_i \cap T_j = \varnothing, 1 \leqslant i, j \leqslant 5, i \neq j$,

(f) $T_i \cap (T_6 \backslash \{\mu_{i_l}\}) = \varnothing, 1 \leqslant i \leqslant 5$,

(g) μ_{i_l}, $(k_2^2 - k_1^2)a_l + (k_2x_1 + k_1x_2)\mu_{i_l}$, $(k_1 + k_2)(k_2a_l + x_1\mu_{i_l})$, $(k_1 + k_2)(-k_1a_l + x_2\mu_{i_l})$ are pairwise distinct.

Noting that $k_1k_2 \neq -1$, we know that the numbers of integers excluded as possible choices for a_l satisfying conditions (d), (e), (f), and (g) are at most $8(l-1)\triangle + 3\triangle$, $32(l-1)^3 + 24(l-1)^2$, $12(l-1)^2 + 24(l-1)$, 6, respectively.

Case 1 $l = 2$. Then it excludes at most $103 + 11\triangle$ integers, so there exist more than one choice for the number $|a_2| \leqslant 6\triangle + 51$ to satisfy conditions (d)–(g). So there exist integers c (depending on integers k_1 and k_2) such that $A_2 \subseteq [-cl^3, cl^3]$.

Case 2 $l \geqslant 3$. Then

$$\begin{aligned} &32(l-1)^3 + 36(l-1)^2 + 24(l-1) + 8(l-1)\triangle + 3\triangle + 6 + 4l - 3 \\ &\quad = 32l^3 - 60l^2 + (52 + 8\triangle)l - 5\triangle - 17 \\ &\quad \leqslant (32 + \triangle)l^3 - 8l^2 - 52l(l-1) - 5\triangle - 17. \end{aligned}$$

Write $M = \max\{|k_1|, |k_2|, |x_1|, |x_2|\}$ and let

$$c = M\{16 + [\frac{\triangle + 1}{2}]\}.$$

Then the number of integers a with $|a| \leqslant (16 + [\frac{\triangle+1}{2}])l^3 - l^2 - [\frac{\triangle+1}{2}]$ is

$$2(16 + [\frac{\triangle + 1}{2}])l^3 - 2l^2 - 2[\frac{\triangle + 1}{2}] + 1 \geqslant (32 + \triangle)l^3 - 2l^2 - \triangle.$$

So there exists an integer a such that

$$|k_2a_l + x_1\mu_{i_l}| \leqslant |k_2||a_l| + |x_1||\mu_{i_l}| \leqslant M(|a_l| + |\mu_{i_l}|) \leqslant cl^3,$$

$$|-k_1a_l + x_2\mu_{i_l}| \leqslant |k_1||a_l| + |x_2||\mu_{i_l}| \leqslant M(|a_l| + |\mu_{i_l}|) \leqslant cl^3,$$

and it follows that there exists an integer a_l such that the set A_l satisfies conditions (i)–(iv). Since this is true at each step of the induction, there are uncountably many sequences $\{A_l\}_{l=1}^{\infty}$ that satisfy conditions (i)–(iv).

Let $x \geqslant 8c$, and let l be the unique positive integer such that $cl^3 \leqslant c < c(l+1)^3$. Conditions (i) and (ii) imply that

$$A(-x,x) \geqslant |A_l| = 2l > 2(\frac{x}{c})^{1/3} - 2 \geqslant (\frac{x}{c})^{1/3}.$$

By (iv), we have

$$R_{k_1,k_2}(A_l,\mu_j) \geqslant \lim_{l\to\infty} \text{card}\{i \leqslant i_l : \mu_i = \mu_j\}, \quad j = 1,\ldots,l. \tag{4}$$

Since $U = \{\mu_l\}_{l=1}^{\infty}$ is a sequence of integers such that $f(n) = \text{card}\{i \in \mathbb{N} : \mu_i = n\}$ for all integers n, it follows that $n \in U = \{\mu_l\}_{l=1}^{\infty}$. By (4) we have

$$\lim_{l\to\infty} R_{k_1,k_2}(A_l,n) \geqslant \lim_{l\to\infty} \text{card}\{i \leqslant i_l : \mu_i = n\}. \tag{5}$$

Since

$$f(n) = \lim_{l\to\infty} \text{card}\{i \leqslant i_l : \mu_i = n\},$$

by (iii) and (5), we have

$$R_{k_1,k_2}(A,n) = \lim_{l\to\infty} R_{k_1,k_2}(A_l,n) = f(n)$$

for all $n \in \mathbb{Z}$. This completes the proof of Theorem 1.1. □

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