

Some Recursion Formulae for the Number of Derangements and Bell Numbers

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Abstract In this paper, a further investigation for the number of Derangements and Bell numbers is performed, and some new recursion formulae for the number of Derangements and Bell numbers are established by applying the generating function methods and Padé approximation techniques. Illustrative special cases of the main results are also presented.

Keywords the number of Derangements; Bell numbers; Padé approximants; recursion formulae

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1. Introduction

Let $\varphi(n)$ denote the set of permutations $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ and let $\phi(n)$ denote the set of permutations of $\varphi(n)$ with no fixed points, the so-called derangements of $\varphi(n)$. A well-known example of this is the case of the absent-minded secretary who has to place n letters into n addressed envelopes and puts each letter into a wrong envelope. The problem of counting derangements was first considered and solved by Pierre Raymond de Montmort [1]. The number of derangements of a set of n elements is denoted by $D(n) = |\phi(n)|$. It is easy to see that the first few values for the number of derangements are $D(0) = 1, D(1) = 0, D(2) = 1, D(3) = 2, D(4) = 9$ with $D(0) = 1$ being defined by convention.

It is not difficult to find a closed formula for the number of derangements. The familiar inclusion-exclusion principle gives the explicit expression for the number of derangements, as follows [2]

$$D(n) = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \dots + \frac{(-1)^n}{n!} \right). \quad (1.1)$$

From the following interesting properties of $D(n)$ can be obtained easily,

$$\frac{D(n)}{n!} \text{ converges quickly to } e^{-1}, \quad (1.2)$$

$$D(n) = nD(n-1) + (-1)^n, \quad (1.3)$$

$$D(n) = (n-1)(D(n-1) + D(n-2)). \quad (1.4)$$

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We refer the readers to Hassani [3] for a further exposition for (1.2) and Remmel [4] for combinatorial proofs of (1.3) and (1.4). If we recognize that $\binom{n}{k}D(k)$ counts all permutations in which exactly k elements of the set $\{1, 2, \dots, n\}$ are displaced, one can get the recurrence relation for the number of derangements:

$$\sum_{k=0}^n \binom{n}{k} D(k) = n!, \quad (1.5)$$

which can be also obtained by applying the generating function of the number of derangements [5,6]:

$$\frac{e^{-t}}{1-t} = \sum_{n=0}^{\infty} D(n) \frac{t^n}{n!}. \quad (1.6)$$

In fact, there exists a more generalization of (1.5). For example, in probabilistic terms, Clarke and Sved [2] showed an interesting connection between the number of derangements and the Bell numbers, as follows,

$$\sum_{k=0}^n \binom{n}{k} k^s D(k) = n! \sum_{k=0}^s \binom{s}{k} (-1)^k n^{s-k} B(k), \quad n \geq s \geq 0, \quad (1.7)$$

where $B(n)$ is the familiar Bell numbers satisfying the generating function:

$$e^{e^t-1} = \sum_{n=0}^{\infty} B(n) \frac{t^n}{n!}, \quad (1.8)$$

and obeying the recurrence relation:

$$B(0) = 1, \quad \sum_{k=0}^n \binom{n}{k} B(k) = B(n+1), \quad n \geq 0. \quad (1.9)$$

For some interesting properties of the number of derangements and Bell numbers, one is referred to [7–14].

Inspired by the work of Clarke and Sved, we perform a further investigation for the number of Derangements and Bell numbers, and establish some new recursion formulae for the number of Derangements and Bell numbers by applying the generating function methods and Padé approximation techniques. Accordingly we consider special cases including (1.1) and (1.5) as well as immediate consequences of the main results.

This paper is organized as follows. In the second section, we recall the Padé approximation to the exponential function. The third section is contributed to the statement of some new recursion formulae for the number of Derangements and Bell numbers by applying the generating function methods and Padé approximation techniques.

2. Padé approximants

As is well known, the properties of Padé approximants have played important roles in number theory and combinatorics, for example, Hermite's proof of the transcendency of e , Lindemann's proof of the transcendency of π , continued fractions, Orthogonal polynomials and so on, see [15–17] for details. As preliminaries, we begin by recalling the definition of Padé approximation

to general series and their expression in the case of the exponential function. Let m, n be non-negative integers and let \mathcal{P}_k be the set of all polynomials of degree $\leq k$. Given a function f with a Taylor expansion

$$f(u) = \sum_{k=0}^{\infty} c_k t^k \quad (2.1)$$

in a neighborhood of the origin, a Padé form of type (m, n) is a pair (P, Q) satisfying that

$$P = \sum_{k=0}^m p_k t^k \in \mathcal{P}_m, \quad Q = \sum_{k=0}^n q_k t^k \in \mathcal{P}_n, \quad Q \not\equiv 0, \quad (2.2)$$

and

$$Qf - P = \mathcal{O}(t^{m+n+1}) \quad \text{as } t \rightarrow 0. \quad (2.3)$$

Clearly, every Padé form of type (m, n) for $f(t)$ always exists and obeys the same rational function. The uniquely determined rational function P/Q is called the Padé approximant of type (m, n) for $f(t)$, and is denoted by $[m/n]_f(t)$ or $r_{m,n}[f; t]$; see for example, [18, 19].

The study of Padé approximants to the exponential function was initiated by Hermite [20] and then continued by Padé [21]. Considering a pair (m, n) of nonnegative integers, the Padé approximant of type (m, n) for e^t is the unique rational function

$$R_{m,n}(t) = \frac{P_{m,n}(t)}{Q_{m,n}(t)}, \quad P_{m,n} \in \mathcal{P}_m, Q_{m,n} \in \mathcal{P}_n, Q_{m,n}(0) = 1, \quad (2.4)$$

with the property that

$$e^t - R_{m,n}(t) = \mathcal{O}(t^{m+n+1}) \quad \text{as } t \rightarrow 0. \quad (2.5)$$

Unlike Padé approximants to other functions, it is possible to determine explicit formulae for $P_{m,n}$ and $Q_{m,n}$ (see [22, p. 245] or [23]):

$$P_{m,n}(t) = \sum_{k=0}^m \frac{m! \cdot (m+n-k)!}{(m+n)! \cdot (m-k)!} \cdot \frac{t^k}{k!}, \quad (2.6)$$

$$Q_{m,n}(t) = \sum_{k=0}^n \frac{n! \cdot (m+n-k)!}{(m+n)! \cdot (n-k)!} \cdot \frac{(-t)^k}{k!}, \quad (2.7)$$

and

$$Q_{m,n}(t)e^t - P_{m,n}(t) = (-1)^n \frac{t^{m+n+1}}{(m+n)!} \int_0^1 x^n (1-x)^m e^{xt} dx. \quad (2.8)$$

We refer to $P_{m,n}(t)$ and $Q_{m,n}(t)$ as the Padé numerator and denominator of type (m, n) for e^t , respectively. In next section, we shall make use of the above Padé approximation to the exponential function to establish some new recursion formulae for the number of Derangements.

3. The statement of results

In this section, we shall establish some new recursion formulae for the number of Derangements and Bell numbers by applying the generating function methods and Padé approximation techniques.

3.1. Recursion formulae for Bell numbers

We first show some connections between the number of derangements, the Bell numbers and the r -Bell polynomials, which are analogous to the Clarke and Sved's formula (1.7). Mezö [24,25] studied in great detail the so-called r -Bell polynomials $B_{n,r}(x)$ satisfying the following exponential generating function:

$$e^{x(e^t-1)+rt} = \sum_{n=0}^{\infty} B_{n,r}(x) \frac{t^n}{n!}. \quad (3.1)$$

Obviously, the case $r = 0$ and $x = 1$ gives the Bell numbers $B(n) = B_{n,0}(1)$. Making m -times derivative for (1.8) with respect to t , we get

$$\frac{\partial^m}{\partial t^m}(e^{e^t-1}) = \sum_{s=0}^{\infty} B(m+s) \frac{t^s}{s!}. \quad (3.2)$$

By multiplying the exponential series $e^{xt} = \sum_{s=0}^{\infty} x^s t^s / s!$ in both sides of (3.2), with the help of the Cauchy product, we have

$$e^{-nt} \frac{\partial^m}{\partial t^m}(e^{e^t-1}) = \sum_{s=0}^{\infty} \left[\sum_{k=0}^s \binom{s}{k} (-n)^{s-k} B(m+k) \right] \frac{t^s}{s!}. \quad (3.3)$$

Assume that $\{f(n)\}, \{g(n)\}, \{h(n)\}, \{\bar{h}(n)\}$ are four sequences given by

$$F(t) = \sum_{n=0}^{\infty} f(n) \frac{t^n}{n!}, \quad G(t) = \sum_{n=0}^{\infty} g(n) \frac{t^n}{n!}, \quad (3.4)$$

and

$$H(t) = \sum_{n=0}^{\infty} h(n) \frac{t^n}{n!}, \quad \bar{H}(t) = \sum_{n=0}^{\infty} \bar{h}(n) \frac{t^n}{n!}. \quad (3.5)$$

If $H(t)\bar{H}(t) = 1$, then one can easily obtain the inverse relation: for non-negative integer n ,

$$f(n) = \sum_{k=0}^n \binom{n}{k} h(k) g(n-k) \iff g(n) = \sum_{k=0}^n \binom{n}{k} \bar{h}(k) f(n-k). \quad (3.6)$$

Notice that the familiar Leibniz rule means that

$$\frac{\partial^m}{\partial t^m}(e^{-nt} \cdot e^{e^t-1}) = \sum_{k=0}^m \binom{m}{k} (-n)^{m-k} \left\{ e^{-nt} \left(\frac{\partial^k}{\partial t^k} e^{e^t-1} \right) \right\}, \quad (3.7)$$

which together with (3.6) yields

$$e^{-nt} \left(\frac{\partial^m}{\partial t^m} e^{e^t-1} \right) = \sum_{k=0}^m \binom{m}{k} n^{m-k} \left(\frac{\partial^k}{\partial t^k} e^{-nt} \cdot e^{e^t-1} \right). \quad (3.8)$$

It follows from (3.1) and (3.8) that

$$e^{-nt} \left(\frac{\partial^m}{\partial t^m} e^{e^t-1} \right) = \sum_{s=0}^{\infty} \left(\sum_{k=0}^m \binom{m}{k} n^{m-k} B_{s+k,-n}(1) \right) \frac{t^s}{s!}. \quad (3.9)$$

Thus, equating (3.3) and (3.9) and comparing the coefficients of $t^s/s!$ gives the following result.

Theorem 3.1 *Let m, n, s be non-negative integers. Then*

$$\sum_{k=0}^m \binom{m}{k} n^{m-k} B_{s+k, -n}(1) = \sum_{k=0}^s \binom{s}{k} (-n)^{s-k} B(m+k). \quad (3.10)$$

It becomes obvious that setting $m = 0$ in Theorem 3.1 gives the connection between the Bell numbers and the r -Bell polynomials: for non-negative integers n, s ,

$$B_{s, -n}(1) = \sum_{k=0}^s \binom{s}{k} (-n)^{s-k} B(k). \quad (3.11)$$

It follows from (1.7) and (3.11) that we state the relation between the number of Derangements and the r -Bell polynomials:

$$\sum_{k=0}^n \binom{n}{k} (-k)^s D(k) = n! \cdot B_{s, -n}(1), \quad (3.12)$$

where n, s are non-negative integers with $n \geq s$.

3.2. Recursion formulae for the number of Derangements

We next present some generalizations of (1.1) and (1.5), which are different from the Clarke and Sved's formula (1.7). We rewrite (1.6) as follows,

$$e^t \left(\sum_{j=0}^{\infty} D(j) \frac{t^j}{j!} \right) = \frac{1}{1-t}. \quad (3.13)$$

If we denote the right hand side of (2.8) by $S_{m,n}(t)$, we get the following expression of Padé approximant for the exponential function e^t ,

$$e^t = \frac{P_{m,n}(t) + S_{m,n}(t)}{Q_{m,n}(t)}. \quad (3.14)$$

By applying (3.14) to (3.13), we obtain

$$(P_{m,n}(t) + S_{m,n}(t)) \sum_{j=0}^{\infty} D(j) \frac{t^j}{j!} = \frac{1}{1-t} Q_{m,n}(t). \quad (3.15)$$

If we apply the exponential series $e^{xt} = \sum_{k=0}^{\infty} x^k t^k / k!$ in the right hand side of (2.8), with the help of the familiar Beta function, we get

$$\begin{aligned} S_{m,n}(t) &= (-1)^n \frac{t^{m+n+1}}{(m+n)!} \sum_{k=0}^{\infty} \frac{t^k}{k!} \int_0^1 x^{n+k} (1-x)^m dx \\ &= \sum_{k=0}^{\infty} \frac{(-1)^n m! \cdot (n+k)!}{(m+n)! \cdot (m+n+k+1)!} \cdot \frac{t^{m+n+k+1}}{k!}. \end{aligned} \quad (3.16)$$

For convenience, we consider the coefficients $p_{m,n;k}$, $q_{m,n;k}$ and $s_{m,n;k}$ of the polynomials $P_{m,n}(t)$, $Q_{m,n}(t)$ and $S_{m,n}(t)$ such that

$$P_{m,n}(t) = \sum_{k=0}^m p_{m,n;k} t^k, \quad Q_{m,n}(t) = \sum_{k=0}^n q_{m,n;k} t^k, \quad (3.17)$$

and

$$S_{m,n}(t) = \sum_{k=0}^{\infty} s_{m,n;k} t^{m+n+k+1}. \quad (3.18)$$

Clearly, the coefficients $p_{m,n;k}$, $q_{m,n;k}$ and $s_{m,n;k}$ satisfy

$$p_{m,n;k} = \frac{m! \cdot (m+n-k)!}{(m+n)! \cdot k! \cdot (m-k)!}, \quad q_{m,n;k} = \frac{(-1)^k n! \cdot (m+n-k)!}{(m+n)! \cdot k! \cdot (n-k)!}, \quad (3.19)$$

and

$$s_{m,n;k} = \frac{(-1)^n m! \cdot (n+k)!}{(m+n)! \cdot k! \cdot (m+n+k+1)!}, \quad (3.20)$$

respectively. By applying (3.17) and (3.18) to (3.15), we discover

$$\left(\sum_{k=0}^m p_{m,n;k} t^k \right) \sum_{j=0}^{\infty} D(j) \frac{t^j}{j!} + \left(\sum_{k=0}^{\infty} s_{m,n;k} t^{m+n+k+1} \right) \sum_{j=0}^{\infty} D(j) \frac{t^j}{j!} = \frac{1}{1-t} \left(\sum_{k=0}^n q_{m,n;k} t^k \right), \quad (3.21)$$

which means

$$\sum_{l=0}^{\infty} t^l \sum_{\substack{k+j=l \\ k \geq 0, j \geq 0}} p_{m,n;k} \frac{D(j)}{j!} + \sum_{l=0}^{\infty} t^l \sum_{\substack{k+j=l-m-n-1 \\ k \geq 0, j \geq 0}} s_{m,n;k} \frac{D(j)}{j!} = \sum_{l=0}^{\infty} t^l \sum_{\substack{k+j=l \\ k \geq 0, j \geq 0}} q_{m,n;k}. \quad (3.22)$$

By comparing the coefficients of t^l in (3.22), we get that for $0 \leq l \leq m+n$,

$$\sum_{\substack{k+j=l \\ k \geq 0, j \geq 0}} p_{m,n;k} \frac{D(j)}{j!} = \sum_{\substack{k+j=l \\ k \geq 0, j \geq 0}} q_{m,n;k}. \quad (3.23)$$

It follows from (3.19) and (3.23) that we state the following recursion formula for the number of Derangements.

Theorem 3.2 *Let l, m, n be non-negative integers. Then, for positive integer l with $0 \leq l \leq m+n$,*

$$\sum_{k=0}^l \binom{m}{k} (m+n-k)! \frac{D(l-k)}{(l-k)!} = \sum_{k=0}^l \binom{n}{k} (-1)^k (m+n-k)!. \quad (3.24)$$

We now discuss some special cases of Theorem 3.2. Setting $l = m+n$ in Theorem 3.2, we obtain that for non-negative integers m, n ,

$$\sum_{k=0}^m \binom{m}{k} D(n+k) = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} (m+k)!. \quad (3.25)$$

It is trivial to see that the case $m = 0$ and $n = 0$ in (3.25) gives the formula (1.1) and (1.5), respectively.

If we compare the coefficients of t^l in (3.22) for $l \geq m+n+1$, then

$$\sum_{\substack{k+j=l \\ k \geq 0, j \geq 0}} p_{m,n;k} \frac{D(j)}{j!} + \sum_{\substack{k+j=l-m-n-1 \\ k \geq 0, j \geq 0}} s_{m,n;k} \frac{D(j)}{j!} = \sum_{\substack{k+j=l \\ k \geq 0, j \geq 0}} q_{m,n;k}. \quad (3.26)$$

Hence, by applying (3.19) and (3.20) to (3.26), we get

$$\sum_{k=0}^m \binom{m}{k} (m+n-k)! \frac{D(l-k)}{(l-k)!} + (-1)^n \frac{m! \cdot n!}{l!} \sum_{k=0}^{l-m-n-1} \binom{l-m-k-1}{n} \binom{l}{k} D(k)$$

$$= \sum_{k=0}^n \binom{n}{k} (-1)^k (m+n-k)!. \quad (3.27)$$

If we set $l = m + n + r$ in (3.27), we obtain that for non-negative integers m, n and positive integer r ,

$$\begin{aligned} & \sum_{k=0}^m \binom{m}{k} (n+k)! \frac{D(n+k+r)}{(n+k+r)!} + (-1)^n \frac{m! \cdot n!}{(m+n+r)!} \sum_{k=0}^{r-1} \binom{n+r-k-1}{n} \binom{m+n+r}{k} D(k) \\ &= \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} (m+k)!. \end{aligned} \quad (3.28)$$

Thus, multiplying both sides of (3.28) by $r!$ gives the following result.

Theorem 3.3 *Let m, n be non-negative integers. Then, for positive integer r ,*

$$\begin{aligned} & \sum_{k=0}^m \frac{\binom{m}{k}}{\binom{n+k+r}{r}} D(n+k+r) + (-1)^n r \sum_{k=0}^{r-1} \frac{\binom{r-1}{k}}{\binom{m+n+r-k-1}{m}} \cdot \frac{D(k)}{m+n+r-k} \\ &= r! \cdot \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} (m+k)!. \end{aligned} \quad (3.29)$$

It follows that we show some special cases of Theorem 3.3. By taking $r = 1$ in Theorem 3.3, in view of $D(0) = 1$, we obtain that for non-negative integers m, n ,

$$\sum_{k=0}^m \binom{m}{k} \frac{D(n+k+1)}{n+k+1} = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} (m+k)! - \frac{(-1)^n m! \cdot n!}{(m+n+1)!}. \quad (3.30)$$

If we set $n = 0$ and substitute n for m in (3.30), we get that for non-negative integer n ,

$$\sum_{k=0}^n \binom{n}{k} \frac{D(k+1)}{k+1} = n! - \frac{1}{n+1}. \quad (3.31)$$

On the other hand, by setting $m = 0$ in Theorem 3.3, we get that for non-negative integer n and positive integer r ,

$$\frac{n!}{(n+r)!} D(n+r) + \frac{(-1)^n}{(r-1)!} \sum_{k=0}^{r-1} \binom{r-1}{k} \frac{D(k)}{n+r-k} = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} k!. \quad (3.32)$$

It is obvious that the case $r = 1$ and $n = 0$ in (3.32) gives the formula (1.1) and (1.5), respectively.

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