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The Application of Rough Set Theory in Pseudo-BCK-Algebra

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Abstract In this paper, we apply the rough set theory to pseudo-BCK-algebras. As a generalization of pseudo-BCK-algebras, the notions of rough pseudo-BCK-algebras, rough subalgebras and rough pseudo-filters are introduced and some of their properties are discussed in an algebra-like approximation space. Furthermore, we investigate rough subalgebras and rough pseudo-filters in a pseudo-BCK-algebra approximation space. Finally, we give several verification programs of pseudo-BCK-algebras, pseudo-filters and subalgebra.

Keywords lower and upper approximation; pseudo-BCK-algebra; pseudo-filter; rough set; subalgebra

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1. Introduction

The theory of rough set was firstly proposed by Pawlak in 1982 (see [1–3]). It is an extension of set theory for the study of intelligent systems characterized by insufficient and incomplete information. Presently, with the rapid development of the rough set theory, such theory has been demonstrated to be useful in the fields such as pattern recognition, decision support system, medical analysis, data mining and so on [4–9].

In 1987, an algebraic approach to rough sets has been given by Iwinski [10]. Since then, the possible connection between rough sets and algebraic systems was studied by some authors. They applied rough set theory to many algebraic systems, such as groups [11,12], semigroups [13], BCI-algebras [14,15], BCK-algebras [16], rings [17], BCC-algebras [18], MV-algebras [19], BL-algebras [20], and so on.

Pseudo-BCK-algebras is a kind of important logical algebraic system introduced by Georgescu and Iorgulescu in 2001, as a non-commutative generalization of BCK-algebras [21]. More properties of pseudo-BCK-algebras were estabilished by Iorgulescu in [22,23]. In order to further characterize pseudo-BCK-algebras, Jun etc. introduced the notion of (positive implicative) pseudo-ideals and displayed characterizations of pseudo-ideals in [24]. Liu, Liu and Xu extended

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the ideal and congruence theory to pseudo-BCK-algebras, and characterized the ideals generated by a set [25].

In this paper, we apply the rough set theory to pseudo-BCK-algebra, and introduce the notions of rough pseudo-BCK-algebras, rough subalgebras and rough pseudo-filters. Then, we discuss some of their properties and give several verification programs.

2. Preliminaries

This part describes the required prior knowledge, including basic definitions and theorems of rough sets and pseudo-BCK-algebras.

Definition 2.1 ([1]) *Let U be a certain set called the universe, and let R be an equivalence relation on U. The pair $A{=}(U;R)$ is called an approximation space. We call R an indiscernibility relation. If $x,y{\in}U$ and $(x,y){\in}R$, then we say that x and y are indistinguishable in A.*

Definition 2.2 ([2,26]) *Let U be a finite, nonempty set called the universe, and let R be a equivalence relation on U. By $R(x)$ we mean the set of all y such that xRy, then $R(x){=}[x]$, i.e., $R(x)$ is an equivalence class of the relation R containing element x. We define two basic operations on sets in the rough set theory, called the R-lower and the R-upper approximation, and defined respectively by*

$$\underline{R(X)}{=}\{x{\in}U\colon R(x){\subseteq}X\},\quad \overline{R(X)}{=}\{x{\in}U\colon R(x){\cap}X{\neq}\emptyset\}.$$

Proposition 2.3 ([3]) *Let $(U;R)$ be an approximation space, and X be a nonempty subset of U. The following properties are obvious:*

(1) X is R-definable if and only if $\underline{R(X)}{=}\overline{R(X)}$.

(2) X is rough with respect to R if and only if $\underline{R(X)}{\neq}\overline{R(X)}$.

Definition 2.4 ([22]) *A pseudo-BCK-algebra (more precisely, reversed left-pseudo-BCK-algebra) is a structure $\mathscr{A}{=}(A,\leq,\rightarrow,\rightsquigarrow,1)$, where "$\leq$" is a binary relation on A, "$\rightarrow$" and "$\rightsquigarrow$" are binary operations on A and "1" is an element of A verifying, for all $x,y,z{\in}A$, the axioms:*

(I) $y{\rightarrow}z{\leq}(z{\rightarrow}x){\rightsquigarrow}(y{\rightarrow}x)$, $y{\rightsquigarrow}z{\leq}(z{\rightsquigarrow}x){\rightarrow}(y{\rightsquigarrow}x)$,

(II) $y{\leq}(y{\rightarrow}x){\rightsquigarrow}x$, $y{\leq}(y{\rightsquigarrow}x){\rightarrow}x$,

(III) $x{\leq}x$,

(IV) $x{\leq}1$,

(V) $x{\leq}y, y{\leq}x{\Rightarrow}x{=}y$,

(VI) $y{\leq}x{\Leftrightarrow}y{\rightarrow}x{=}1{\Leftrightarrow}y{\rightsquigarrow}x{=}1$.

Proposition 2.5 ([22]) *Let $\mathscr{A}{=}(A,\leq,\rightarrow,\rightsquigarrow,1)$ be a pseudo-BCK-algebra. Then $\mathscr{A}$ satisfies the following property: $\forall x,y{\in}A, y{\leq}x{\rightarrow}y$, $y{\leq}x{\rightsquigarrow}y$.*

Definition 2.6 ([23]) *Let $\mathscr{A}{=}(A,\leq,\rightarrow,\rightsquigarrow,1)$ be a pseudo-BCK-algebra and F be a nonempty subset of A. F is called a pseudo-filter of $\mathscr{A}$ if it satisfies:*

(F1) $1{\in}F$,

(F2) $\forall y{\in}F$, $\rightarrow(y,F){\subseteq}F$ *and* $\rightsquigarrow(y,F){\subseteq}F$,
where, $\forall y{\in}A, \forall F{\subseteq}A, \rightarrow(y,F){:=}\{x{\in}A|\ y{\rightarrow}x{\in}F\}$, $\rightsquigarrow(y,F){:=}\{x{\in}A|\ y{\rightsquigarrow}x{\in}F\}$.

According to the definitions of "$\rightarrow(y,F)$" and "$\rightsquigarrow(y,F)$", (F2) is equivalent to the following condition:

(F2$'$) $\forall y{\in}F$, $\forall x{\in}A$, $y{\rightarrow}x{\in}F{\Rightarrow}x{\in}F$ and $y{\rightsquigarrow}x{\in}F{\Rightarrow}x{\in}F$.

Proposition 2.7 *Let* $\mathscr{A}{=}(A,\leq,\rightarrow,\rightsquigarrow,1)$ *be a pseudo-BCK-algebra,* F *be a nonempty subset of* A*, and* $1{\in}F$*. The following conditions are equivalent:*

(1) $\forall y{\in}F$, $\forall x{\in}A$, $y{\rightarrow}x{\in}F{\Rightarrow}x{\in}F$;

(2) $\forall y{\in}F$, $\forall x{\in}A$, $y{\rightsquigarrow}x{\in}F{\Rightarrow}x{\in}F$.

Proof (1) $\forall y{\in}F$, $\forall x{\in}A$, such that $y{\rightsquigarrow}x{\in}F$. From $1{=}y{\rightarrow}((y{\rightsquigarrow}x){\rightarrow}x){\in}F$ follows $(y{\rightsquigarrow}x){\rightarrow}x{\in}F$, hence $x{\in}F$. Thus, (1) $\Rightarrow$ (2). Similarly, we can prove (2) $\Rightarrow$ (1). □

According to the proposition 2.7, (F2) is also equivalent to the following condition:

(F2$''$) $\forall y{\in}F$, $\rightarrow(y,F){\subseteq}F$ (or $\rightsquigarrow(y,F){\subseteq}F$).

Definition 2.8 *Let* $\mathscr{A}{=}(A,\rightarrow,\rightsquigarrow,1)$ *be a pseudo-BCK-algebra and* F *be a nonempty subset of* A*.* $(F,\rightarrow,\rightsquigarrow,1)$ *is called a subalgebra of* $\mathscr{A}$ *if it satisfies:* $\forall x,y{\in}F$, $x{\rightarrow}y{\in}F$ *and* $x{\rightsquigarrow}y{\in}F$.

Proposition 2.9 *Let* $\mathscr{A}{=}(A,\rightarrow,\rightsquigarrow,1)$ *be a pseudo-BCK-algebra and* F *be a pseudo-filter of* $\mathscr{A}$*. Then* $(F,\rightarrow,\rightsquigarrow,1)$ *must be a subalgebra of* $\mathscr{A}$*.*

Proof $\forall y{\in}F$, we have $y{\leq}x{\rightarrow}y$, $y{\leq}x{\rightsquigarrow}y$, i.e., $1{=}y{\rightarrow}(x{\rightarrow}y){=}y{\rightarrow}(x{\rightsquigarrow}y)$. From $y{\rightarrow}(x{\rightarrow}y){\in}F$, $y{\rightarrow}(x{\rightsquigarrow}y){\in}F$, $y{\in}F$ it follows $x{\rightarrow}y, x{\rightsquigarrow}y{\in}{\rightarrow}(y,F)$. From $\rightarrow(y,F){\subseteq}F$ it follows $x{\rightarrow}y, x{\rightsquigarrow}y{\in}F$. Hence $(F,\rightarrow,\rightsquigarrow,1)$ must be a subalgebra of $\mathscr{A}$. □

Definition 2.10 ([25]) *Let* $\mathscr{A}{=}(A,\rightarrow,\rightsquigarrow,1)$ *be a pseudo-BCK-algebra,* ρ *be an equivalence relation on* A*.* ρ *is called a congruence relation on* A *if* $(x,y){\in}\rho$ *and* $(u,v){\in}\rho$ *imply* $(x{\rightarrow}u, y{\rightarrow}v){\in}\rho$ *and* $(x{\rightsquigarrow}u, y{\rightsquigarrow}v){\in}\rho$*, for all* $x,y,u,v{\in}A$.

Definition 2.11 ([18]) *Let* X *denote a BCC-algebra. Then a nonempty subset* S *of* X *is called an upper (resp., a lower) rough subalgebra (or, ideal) of* X *if the upper (resp., nonempty lower) approximation of* S *is a subalgebra (or, ideal) of* X*. If* S *is both an upper and a lower rough subalgebra (or, ideal) of* X*, we say that* S *is a rough subalgebra (or ideal) of* X*.*

Proposition 2.12 ([27]) *(1) Let* A *be a structure* $(X,\rightarrow,\rightsquigarrow,1)$*, where "* $\geq$ *" is a binary relation on* A*,* $\rightarrow$ *and* $\rightsquigarrow$ *are binary operations on* A *and* 1 *is an element of* A*,* $(A,\geq,\rightsquigarrow,1)$ *and* $(A,\geq,\rightarrow,1)$ *are BCC-algebras and satisfy*

$$a \rightarrow (b \rightsquigarrow c) = b \rightsquigarrow (a \rightarrow c) \text{ for all } a,b,c \in A.$$

Then $(A,\geq,\rightarrow,\rightsquigarrow,1)$ *is a pseudo-BCK-algebra.*

(2) Let $(A,\geq,\rightarrow,\rightsquigarrow,1)$ *be a pseudo-BCK-algebra. Then* $(A,\geq,\rightarrow,1)$ *and* $(A,\geq,\rightsquigarrow,1)$ *are BCC-algebras and satisfy* $a \rightarrow (b \rightsquigarrow c) = b \rightsquigarrow (a \rightarrow c)$ *for all* $a,b,c \in A$.

3. Rough pseudo-BCK-algebra

For the application of rough set theory to pseudo-BCK-algebras, we propose the concept of algebra-like approximation space and pseudo-BCK-algebras approximation space. Then, we put forward the concept of rough pseudo-BCK-algebras and study its related properties in an algebra-like space.

Definition 3.1 *Let* (U, R) *be an approximation space, "* $\to$ *", "* $\rightsquigarrow$ *" be two binary operation on* U*, "1" be a constant of* U*. Then we call* $(U, R, \to, \rightsquigarrow, 1)$ *an algebra-like approximation space. If* $(U, \to, \rightsquigarrow, 1)$ *is a pseudo-BCK-algebra, then* $(U, R, \to, \rightsquigarrow, 1)$ *is called a pseudo-BCK-algebra approximation space.*

Definition 3.2 *Let* $(U, R, \to, \rightsquigarrow, 1)$ *be an algebra-like approximation space. For any nonempty subset* X *of* U*,* $\underline{R(X)}$ *and* $\overline{R(X)}$ *are the R-lower and the R-upper approximation, respectively. For any nonempty subset* X *of* U*.*

(1) $(X, \to, \rightsquigarrow, 1)$ *is called a lower-rough pseudo-BCK-algebra if* $(\underline{R(X)}, \to, \rightsquigarrow, 1)$ *is a pseudo-BCK-algebra.*

(2) $(X, \to, \rightsquigarrow, 1)$ *is called an upper-rough pseudo-BCK-algebra if* $(\overline{R(X)}, \to, \rightsquigarrow, 1)$ *is a pseudo-BCK-algebra.*

(3) $(X, \to, \rightsquigarrow, 1)$ *is called a rough pseudo-BCK-algebra if* $(\underline{R(X)}, \to, \rightsquigarrow, 1)$ *and* $(\overline{R(X)}, \to, \rightsquigarrow, 1)$ *are pseudo-BCK-algebras.*

For every rough pseudo-BCK-algebra, if $\overline{R(X)}=\underline{R(X)}$, then it must be a pseudo-BCK-algebra. Thus, a rough pseudo-BCK-algebra is the generalized form of pseudo-BCK-algebra in an approximation space.

If $\mathscr{X}=(X, \to, \rightsquigarrow, 1)$ is a rough pseudo-BCK-algebra and not a pseudo-BCK-algebra, we call it a proper rough pseudo-BCK-algebra.

Proposition 3.3 *Let* $(U, R, \to, \rightsquigarrow, 1)$ *be an algebra-like approximation space and* X *be a nonempty subset of* U*. If* $(X; \to, \rightsquigarrow, 1)$ *is an upper-rough pseudo-BCK-algebra, then* $(X; \to, \rightsquigarrow, 1)$ *is a rough pseudo-BCK-algebra if and only if* $(\underline{R(X)}, \to, \rightsquigarrow, 1)$ *is a subalgebra of* $(\overline{R(X)}, \to, \rightsquigarrow, 1)$*.*

Proof Since $\underline{R(X)} \subseteq \overline{R(X)}$, and $(\overline{R(X)}, \to, \rightsquigarrow, 1)$ is a pseudo-BCK-algebra, we have $(X; \to, \rightsquigarrow, 1)$ is a rough pseudo-BCK-algebra if and only if $(\underline{R(X)}, \to, \rightsquigarrow, 1)$ is a pseudo-BCK-algebra, i.e., it is a subalgebra of $(\overline{R(X)}, \to, \rightsquigarrow, 1)$. □

Example 3.4 Let $U = \{a, b, c, d, e, f, 1\}$ in which two binary operations "$\to$" and "$\rightsquigarrow$" are given by Table 1. Let $R = \{(a, e),(e, a),(a, a),(e, e),(b, c),(c, b),(b, b),(c, c),(b, d),(d, b),(d, d),(c, d),(d, c),(1, 1),(f, f)\}$. Then R is an equivalence relation on U, $(X, R, \to, \rightsquigarrow, 1)$ is an algebra-like approximation space, and $[a]=[e]=\{a, e\}$,$[b]=[c]=[d]=\{b, c, d\}$,$[1]=1$,$[f]=f$. Put $X=\{a, b, c, d, 1\}$, then $\underline{R(X)}=\{b, c, d, 1\}$, $\overline{R(X)}=\{a, b, c, d, e, 1\}$. By verification, $(X, \to, \rightsquigarrow, 1)$ is not a pseudo-BCK-algebra, but $(\underline{R(X)}, \to, \rightsquigarrow, 1)$ and $(\overline{R(X)}, \to, \rightsquigarrow, 1)$ are two pseudo-BCK-algebras, hence $(X, \to, \rightsquigarrow, 1)$ is a proper rough pseudo-BCK-algebra.

$\to$	a	b	c	d	e	f	1
a	1	b	c	d	1	1	1
b	e	1	1	1	1	1	1
c	a	b	1	d	e	1	1
d	e	c	c	1	1	1	1
e	e	b	c	d	1	1	1
f	e	b	c	d	e	1	f
1	a	b	c	d	e	f	1

$\rightsquigarrow$	a	b	c	d	e	f	1
a	1	b	c	d	1	1	1
b	e	1	1	1	1	1	1
c	a	d	1	d	e	1	1
d	e	b	c	1	1	1	1
e	e	b	c	d	1	1	1
f	e	b	c	d	e	1	e
1	a	b	c	d	e	f	1

Table 1 The operation tables of "$\to$" and "$\rightsquigarrow$"

Definition 3.5 *Let $(U, R, \to, \rightsquigarrow, 1)$ be an algebra-like approximation space, X be a nonempty subset of U and $(X, \to, \rightsquigarrow, 1)$ be a rough pseudo-BCK-algebra. For any nonempty subset F of X,*

(1) if F is the pseudo-filter of $(\underline{R(X)}, \to, \rightsquigarrow, 1)$ and $(\overline{R(X)}, \to, \rightsquigarrow, 1)$, then F is called a rough pseudo-filter of type 1;

(2) if $\overline{R(F)}$ is the pseudo-filter of $(\overline{R(X)}, \to, \rightsquigarrow, 1)$, and $\underline{R(F)}$ (if it is nonempty) is the pseudo-filter of $(\underline{R(X)}, \to, \rightsquigarrow, 1)$, then F is called a rough pseudo filter of type 2.

Example 3.6 Let (U, R) and $(X, \to, \rightsquigarrow, 1)$ be respectively the approximation space and the rough pseudo-BCK-algebra in Example 3.4. Put $F{=}\{c, 1\}$, then $\overline{R(F)}{=}\{b, c, d, 1\}$, $\underline{R(F)}{=}\{1\}$. From $b{\to}e{=}1{\in}\overline{R(F)}$,$b{\in}\overline{R(F)}$, $e{\notin}\overline{R(F)}$ it follows that $\overline{R(F)}$ is not a pseudo-filter of $(\overline{R(X)}, \to, \rightsquigarrow, 1)$. Thus, F is not a rough pseudo-filter of type 2. But F is a rough pseudo-filter of type 1.

Example 3.7 Let (U, R) and $(X, \to, \rightsquigarrow, 1)$ be respectively the approximation space and the rough pseudo-BCK-algebra in Example 3.4. Put $F{=}\{a, b, 1\}$. From $F{\not\subseteq}\underline{R(X)}$ it follows that F is not a rough pseudo-filter of type 1. From $\underline{R(F)}{=}\{1\}$, $\overline{R(F)}{=}\{a, b, c, d, e, 1\}$) it follows that F is a rough pseudo-filter of type 2.

From Examples 3.6 and 3.7, we know that two types of rough pseudo-filters are different.

Definition 3.8 *Let $(U, R, \to, \rightsquigarrow, 1)$ be an algebra-like approximation space, X be a nonempty subset of U and $(X, \to, \rightsquigarrow, 1)$ be a rough pseudo-BCK-algebra. For any nonempty subset S of X,*

(1) if $(S, \to, \rightsquigarrow, 1)$ is the subalgebra of $(\underline{R(X)}, \to, \rightsquigarrow, 1)$ and $(\overline{R(X)}, \to, \rightsquigarrow, 1)$, then $(S, \to, \rightsquigarrow, 1)$ is called a rough subalgebra of type 1;

(2) if $(\overline{R(S)}, \to, \rightsquigarrow, 1)$ is the subalgebra of $(\overline{R(X)}, \to, \rightsquigarrow, 1)$, and $(\underline{R(S)}, \to, \rightsquigarrow, 1)$ (if $\underline{R(S)}$ is nonempty) is the subalgebra of $(\underline{R(X)}, \to, \rightsquigarrow, 1)$, then $(S, \to, \rightsquigarrow, 1)$ is called a rough subalgebra of type 2.

Obviously, $\{1\}$ is a rough pseudo-filter of type 1 (or type 2), and we call it trivial rough pseudo-filter of type 1 (or type 2). $(\{1\}, \to, \rightsquigarrow, 1)$ is a rough subalgebra of type 1 (or type 2), and we call it trivial rough subalgebra of type 1 (or type 2).

In fact, a rough subalgebra of type 1 is a pseudo-BCK-algebra, a rough subalgebra of type

2 is a rough pseudo-BCK-algebra.

According to Definitions 3.5, 3.8 and Proposition 2.9, the following property is obvious:

Proposition 3.9 *(1) In a rough pseudo-BCK-algebra $(X, \to, \rightsquigarrow, 1)$, if F is a 1-type (or 2-type) rough pseudo-filter, then $(F, \to, \rightsquigarrow, 1)$ must be a 1-type (or 2-type) rough subalgebra.*

(2) If F is R-definable, i.e., $\overline{R(F)}=\underline{R(F)}=F$, two types of rough pseudo-filters (or rough subalgebras) are identical.

Thus, in Example 3.6, $(F, \to, \rightsquigarrow, 1)$ is a rough subalgebra of type 1. Although F is not a rough pseudo-filter of type 2, $(F, \to, \rightsquigarrow, 1)$ is a rough subalgebra of type 2.

4. Rough pseudo-BCK algebra in a pseudo-BCK-algebra approximation space

As what mentioned above, we talked about rough pseudo-BCK-algebras in an algebra-like approximation space. In the next part, we will research and analysis rough pseudo-BCK-algebras in a pseudo-BCK-algebra approximation space.

Definition 4.1 *Let $(X, R, \to, \rightsquigarrow, 1)$ be a pseudo-BCK-algebra approximation space, F be a nonempty subset of X.*

(1) If $\underline{R(F)}$ is a pseudo-filter of the pseudo-BCK-algebra $(X; \to, \rightsquigarrow, 1)$, F is called a lower-rough pseudo-filter of $(X; \to, \rightsquigarrow, 1)$.

(2) If $\overline{R(F)}$ is a pseudo-filter of the pseudo-BCK-algebra $(X; \to, \rightsquigarrow, 1)$, F is called an upper-rough pseudo-filter of $(X; \to, \rightsquigarrow, 1)$.

(3) If $\underline{R(F)}$ and $\overline{R(F)}$ are two pseudo-filters of the pseudo-BCK-algebra $(X; \to, \rightsquigarrow, 1)$, F is called a rough pseudo-filter of $(X; \to, \rightsquigarrow, 1)$.

Definition 4.2 *Let $(X; R, \to, \rightsquigarrow, 1)$ be a pseudo-BCK-algebra approximation space, S be a nonempty subset of X.*

(1) If $\underline{R(S)}$ is a subalgebra of the pseudo-BCK-algebra $(X; \to, \rightsquigarrow, 1)$, S is called a lower-rough subalgebra of $(X; \to, \rightsquigarrow, 1)$.

(2) If $\overline{R(S)}$ is a subalgebra of the pseudo-BCK-algebra $(X; \to, \rightsquigarrow, 1)$, S is called an upper-rough subalgebra of $(X; \to, \rightsquigarrow, 1)$.

(3) If $\underline{R(S)}$ and $\overline{R(S)}$ are two subalgebras of the pseudo-BCK-algebra $(X; \to, \rightsquigarrow, 1)$, S is called a rough subalgebra of $(X; \to, \rightsquigarrow, 1)$.

Proposition 4.3 *Let $(X; R, \to, \rightsquigarrow, 1)$ be a pseudo-BCK-algebra approximation space, S be a nonempty subset of X. Then $(S, \to, \rightsquigarrow, 1)$ is a rough pseudo-BCK-algebra if and only if it satisfies:*

(1) $(S, \to, 1)$ is a rough BCC-balgebra;

(2) $(S, \rightsquigarrow, 1)$ is a rough BCC-algebra;

(3) $\forall x, y, z \in S$, $x \rightsquigarrow (y \to z) = y \to (x \rightsquigarrow z)$.

Proposition 4.4 *Let $(X; R, \to, \rightsquigarrow, 1)$ be a pseudo-BCK-algebra approximation space, S be a* *nonempty subset of X. $(S;\rightarrow,\rightsquigarrow,1)$ is a rough pseudo-BCK-algebra if and only if $(S,\rightarrow,\rightsquigarrow,1)$ is a rough subalgebra of $(X,\rightarrow,\rightsquigarrow,1)$.*

Proof If $(S;\rightarrow,\rightsquigarrow,1)$ is a rough pseudo-BCK-algebra, $(\overline{R(S)},\rightarrow,\rightsquigarrow,1)$ and $(\underline{R(S)},\rightarrow,\rightsquigarrow,1)$ are two pseudo-BCK-algebras. Since $\overline{R(S)}\subseteq X$, $\underline{R(S)}\subseteq X$, $(\overline{R(S)},\rightarrow,\rightsquigarrow,1)$ and $(\underline{R(S)},\rightarrow,\rightsquigarrow,1)$ are two subalgebras of pseudo-BCK-algebra $(X,\rightarrow,\rightsquigarrow,1)$. So, $(S,\rightarrow,\rightsquigarrow,1)$ is a rough subalgebra of $(X,\rightarrow,\rightsquigarrow,1)$.

Conversely, if $(S,\rightarrow,\rightsquigarrow,1)$ is a rough subalgebra of $(X,\rightarrow,\rightsquigarrow,1)$, $(\overline{R(S)},\rightarrow,\rightsquigarrow,1)$ and $(\underline{R(S)},\rightarrow,\rightsquigarrow,1)$ are two subalgebras of $(X,\rightarrow,\rightsquigarrow,1)$. This means that $(\overline{R(S)},\rightarrow,\rightsquigarrow,1)$ and $(\underline{R(S)},\rightarrow,\rightsquigarrow,1)$ are two pseudo-BCK-algebras. So, $(S;\rightarrow,\rightsquigarrow,1)$ is a rough pseudo-BCK-algebra. $\square$

According to Definitions 4.1, 4.2 and Proposition 2.9, the following property is obvious:

Proposition 4.5 *In a pseudo-BCK-algebra approximation space, if F is a rough (upper-rough, or lower-rough) pseudo-filter, then $(F,\rightarrow,\rightsquigarrow,1)$ must be a rough (upper-rough, or lower-rough) subalgebra.*

$\rightarrow$	a	b	c	d	1
a	1	1	1	1	1
b	c	1	1	1	1
c	a	b	1	d	1
d	b	b	c	1	1
1	a	b	c	d	1

$\rightsquigarrow$	a	b	c	d	1
a	1	1	1	1	1
b	d	1	1	1	1
c	b	b	1	d	1
d	a	b	c	1	1
1	a	b	c	d	1

Table 2 The operation tables of "$\rightarrow$" and "$\rightsquigarrow$"

Example 4.6 Let $X=\{a,b,c,d,1\}$ in which "$\rightarrow$" and "$\rightsquigarrow$" are given by Table 2. The equivalence relation on X is defined by $R=\{(a,b),(b,a),(a,a),(b,b),(c,d),(d,c),(c,c),(d,d),(c,1),(1,c),(1,1),(d,1),(1,d)\}$. Then $\mathscr{X}=(X,\rightarrow,\rightsquigarrow,1)$ is a pseudo-BCK-algebra, and $(X,R,\rightarrow,\rightsquigarrow,1)$ is a pseudo-BCK-algebra approximation space. Put $F=\{a,c,d,1\}$. From $[a]=\{a,b\}$, $[c]=[d]=[1]=\{c,d,1\}$ follow $\underline{R(F)}=\{c,d,1\}$, $\overline{R(F)}=\{a,b,c,d,1\}$. By verification, we know that: (1) $\underline{R(F)}$ and $\overline{R(F)}$ are two pseudo-filters of $\mathscr{X}$, hence F is a rough pseudo-filter of $\mathscr{X}$. (2) $(F,\rightarrow,\rightsquigarrow,1)$ is not a subalgebra of $\mathscr{X}$, since $d\rightarrow a=b\notin F$. (3) According to Propositions 4.5 and 2.9, $(F,\rightarrow,\rightsquigarrow,1)$ is also a rough subalgebra of $\mathscr{X}$, and F is not a pseudo-filter of $\mathscr{X}$.

$\rightarrow$	a	b	c	d	1
a	1	c	1	1	1
b	d	1	1	1	1
c	d	c	1	1	1
d	c	c	c	1	1
1	a	b	c	d	1

$\rightsquigarrow$	a	b	c	d	1
a	1	d	1	1	1
b	d	1	1	1	1
c	d	d	1	1	1
d	c	b	c	1	1
1	a	b	c	d	1

Table 3 The operation tables of "$\rightarrow$" and "$\rightsquigarrow$"

Example 4.7 Let $X=\{a,b,c,d,1\}$ in which "$\rightarrow$" and "$\rightsquigarrow$" are given by Table 3. The equivalence relation on X is defined by $R=\{(a,c),(c,a),(a,a),(c,c),(b,b),(d,1),(1,d),(d,d),(1,1)\}$. Then

$\mathscr{X}=(X;\to,\rightsquigarrow,1)$ is a pseudo-BCK-algebra, and $(X,R,\to,\rightsquigarrow,1)$ is a pseudo-BCK-algebra approximation space. Put $S=\{a,d,1\}$. From $[a]=\{a,c\}$, $[d]=[1]=\{d,1\}$ follow $\underline{R(S)}=\{d,1\}$, $\overline{R(S)}=\{a,c,d,1\}$. By verification, we know that: (1) $(\underline{R(S)};\to,\rightsquigarrow,1)$ and $(\overline{R(S)};\to,\rightsquigarrow,1)$ are two subalgebras of $\mathscr{X}$, hence $(S,\to,\rightsquigarrow,1)$ is a rough subalgebra of $\mathscr{X}$. (2) S is not a rough pseudo-filter of $\mathscr{X}$, since $\overline{R(S)}$ is not a pseudo filter of $\mathscr{X}$ (for $a\in S$, $a\to b=c\in S$, $b\notin S$). (3) $(S;\to,\rightsquigarrow,1)$ is not a subalgebra of $\mathscr{X}$, since $a,d\in S$, $d\to a=c\notin S$. (4) According to Proposition 2.9, S is not a pseudo-filter of $\mathscr{X}$.

Proposition 4.8 *Let $(X,\rho,\to,\rightsquigarrow,1)$ be a pseudo-BCK-algebra approximation space, A,B be two nonempty subsets of X. If ρ is a congruence relation on X, the conclusions are drawn as follows:*

$$\overline{\rho(A)}\to\overline{\rho(B)}\subseteq\overline{\rho(A\to B)},\ \overline{\rho(A)}\rightsquigarrow\overline{\rho(B)}\subseteq\overline{\rho(A\rightsquigarrow B)}$$

where, $\forall A,B\subseteq X$, $A\to B:=\{x\to y|\ x\in A,y\in B\}$, $A\rightsquigarrow B:=\{x\rightsquigarrow y|\ x\in A,y\in B\}$.

Proof $\forall z\in\overline{\rho(A)}\to\overline{\rho(B)}$, $\exists x\in\overline{\rho(A)},y\in\overline{\rho(B)}$ such that $z=x\to y$. From $x\in\overline{\rho(A)},y\in\overline{\rho(B)}$ follow $\exists a\in A$, $\exists b\in B$, such that $(x,a)\in\rho$, $(y,b)\in\rho$. Applying Definition 2.10, we get $(x\to y,a\to b)\in\rho$, i.e., $x\to y\in[a\to b]$. From $a\to b\in A\to B$, we get $[a\to b]\subseteq\overline{\rho(A\to B)}$, hence $x\to y\in\overline{\rho(A\to B)}$. Thus, $\overline{\rho(A)}\to\overline{\rho(B)}\subseteq\overline{\rho(A\to B)}$ is valid. Similarly, we can prove that $\overline{\rho(A)}\rightsquigarrow\overline{\rho(B)}\subseteq\overline{\rho(A\rightsquigarrow B)}$ is also valid. $\square$

Note that there is no inclusion relation between $\underline{\rho(A)}\to\overline{\rho(B)}$ and $\underline{\rho(A\to B)}$ (or, between $\underline{\rho(A)}\rightsquigarrow\overline{\rho(B)}$ and $\underline{\rho(A\rightsquigarrow B)}$).

Example 4.9 Let $X=\{a,b,c,d,1\}$ in which "$\to$" and "$\rightsquigarrow$" are given by Table 4. The congruence relation on X is defined by $\rho=\{(a,a),(a,1),(1,a),(1,1),(b,b),(c,c),(d,d)\}$. Then, $(X;\to,\rightsquigarrow,1)$ is a pseudo-BCK-algebra and $(X,\rho,\to,\rightsquigarrow,1)$ is a pseudo-BCK-algebra approximation space. Put $A=\{b,1\}$, $B=\{c,d\}$. (1) By $\overline{\rho(A)}=\{a,b,1\}$, $\overline{\rho(B)}=\{c,d\}$, we get $\overline{\rho(A)}\to\overline{\rho(B)}=\{c,d,1\}$, and $\overline{\rho(A\to B)}=\overline{\rho(\{c,d,1\})}=\{a,c,d,1\}$, hence $\overline{\rho(A)}\to\overline{\rho(B)}\subset\overline{\rho(A\to B)}$. (2) By $\underline{\rho(A)}=\{b\}$, $\underline{\rho(B)}=\{c,d\}$, we get $\underline{\rho(A)}\to\underline{\rho(B)}=\{b\}\to\{c,d\}=\{1\}$, and $\underline{\rho(A\to B)}=\underline{\rho(\{c,d,1\})}=\{c,d\}$, hence $\underline{\rho(A)}\to\underline{\rho(B)}\not\subseteq\underline{\rho(A\to B)}$ and $\underline{\rho(A\to B)}\not\subseteq\underline{\rho(A)}\to\underline{\rho(B)}$.

$\to$	a	b	c	d	1
a	1	b	c	d	1
b	a	1	1	1	1
c	a	b	1	d	1
d	a	c	c	1	1
1	a	b	c	d	1

$\rightsquigarrow$	a	b	c	d	1
a	1	b	c	d	1
b	a	1	1	1	1
c	a	d	1	d	1
d	a	b	c	1	1
1	a	b	c	d	1

Table 4 The operation tables of "$\to$" and "$\rightsquigarrow$"

Proposition 4.10 *Let $\mathscr{X}=(X;\to,\rightsquigarrow,1)$ be a pseudo-BCK-algebra, and ρ be a congruence relation on X, S be a nonempty subset of X. If $(S,\to,\rightsquigarrow,1)$ is a subalgebra of $\mathscr{X}$, then $(S,\to,\rightsquigarrow,1)$ must be an upper-rough subalgebra of $\mathscr{X}$.*

Proof $\forall x, y \in \overline{\rho(S)}$, we have $x \to y \in \overline{\rho(S)} \to \overline{\rho(S)}$. Applying Proposition 4.8, we get $\overline{\rho(S)} \to \overline{\rho(S)} \subseteq \overline{\rho(S \to S)}$. Since $(S, \to, \rightsquigarrow, 1)$ is a subalgebra of $\mathscr{X}$, we get $S \to S \subseteq S$. From $x \to y \in \overline{\rho(S)} \to \overline{\rho(S)}$ and $\overline{\rho(S)} \to \overline{\rho(S)} \subseteq \overline{\rho(S)}$ it follows $x \to y \in \overline{\rho(S)}$. Similarly, we can prove $x \rightsquigarrow y \in \overline{\rho(S)}$. Thus, $(\overline{\rho(S)}, \to, \rightsquigarrow, 1)$ is a subalgebra of $\mathscr{X}$. This means that $(S, \to, \rightsquigarrow, 1)$ is an upper-rough subalgebra of $\mathscr{X}$. □

It must be pointed out that every subalgebra must not be a lower-rough subalgebra.

Example 4.11 Let $\mathscr{X} = (X, \to, \rightsquigarrow, 1)$ and ρ be respectively the pseudo-BCK-algebra and the congruence relation in Example 4.9. Put $S = \{b, 1\}$, then $\underline{\rho(S)} = \{b\}$. By $1 \notin \underline{\rho(S)}$ it follows that $(\underline{\rho(S)}, \to, \rightsquigarrow, 1)$ is not a subalgebra, hence $(S, \to, \rightsquigarrow, 1)$ is not a lower-rough subalgebra, but it is a subalgebra of $\mathscr{X}$.

5. Several programs

In this part, we put forward the procedures used in the examples what mentioned above.

Program 5.1 Let (U, R) be an approximation space and F be a nonempty subset of U. Then we can get the upper approximation and lower approximation of F by the following Matlab program.

```
function [upper, lower]=approximation(R,U,F)
% R is the matrix of a equivalence relation on U.
[a,b]=ismember(F,U);
f=sort(b);
n=length(a);
lower=[ ];upper=[ ];
for i=1:n;
    r=find(R(f(i),:));
    if all(ismember(r,f))==1
        lower=union(lower,f(i));
    end
    upper=union(upper,r);
end
upper=U(upper);
lower=U(lower);
```

Since an equivalence relation can be regarded as a binary operation $*$, namely $\forall x, y \in U$, $x * y = 1$ means $(x, y) \in R$, we express R as a matrix in the above program.

Program 5.2 Let $(U, R, \to, \rightsquigarrow, 1)$ be an algebra-like approximation space and F be a nonempty subset of U. Then we can judge whether $(F, \to, \rightsquigarrow, 1)$ is a pseudo-BCK-algebra by the following Matlab program.

```
function test=psbck_algebra(U1,U2,U,F)
%U1 and U2 are two operation tables on U.
```

```
test=0;
[a,b]=ismember(F,U);
f=sort(b);
n=length(a);
X=U1(f,f);
Y=U2(f,f);
[a,X]=ismember(X,F);
[b,Y]=ismember(Y,F);
if sum(find([a,b]==0))~=0
    return
end
for i=1:n
    if X(i,i)~=n || Y(i,i)~=n || X(i,n)~=n || Y(i,n)~=n
        return
    end
    for j=1:n
        if X(i,j)==n && X(j,i)==n && i~=j
            return
        elseif Y(i,j)==n && Y(j,i)==n && i~=j
            return
        end
      if X(i,j)==n && Y(i,j)~=n
            return
        elseif Y(i,j)==n && X(i,j)~=n
            return
        end
        if X(j,Y(X(j,i),i))~=n || X(j,X(Y(j,i),i))~=n
            return
        end
        for k=1:n
            if X(Y(j,k),X(Y(k,i),Y(j,i)))~=n || X(X(j,k),Y(X(k,i),X(j,i)))~=n
                return
            end
        end
    end
end
test=1;
```

Program 5.3 Let $(U, R, \rightarrow, \rightsquigarrow, 1)$ be an algebra-like approximation space, X be a nonempty subset of U and $\mathscr{X}=(X,\rightarrow,\rightsquigarrow,1)$ be a pseudo-BCK-algebra. For a nonempty subset F of X, we can judge whether F is a pseudo-filter of $\mathscr{X}$ by the following Matlab program.

```
function test=pseudo_filter(U1,U,F,X)
% U1 is a operation table on U.
test=0;
[a,b]=ismember(X,U);
x=sort(b);
[c,d]=ismember(F,U);
f=sort(d);
m=length(c);
for i=1:m
    if all(ismember(X(ismember(U1(f(i),x),F)),F))==0
        return
    end
end
test=1;
```

Program 5.4 Let $(U,R,\rightarrow,\rightsquigarrow,1)$ be an algebra-like approximation space, X be a nonempty subset of U and $\mathscr{X}=(X,\rightarrow,\rightsquigarrow,1)$ be a pseudo-BCK-algebra. For a nonempty subset F of X, we can judge whether $(F,\rightarrow,\rightsquigarrow,1)$ is a subalgebra $\mathscr{X}$ by the following Matlab program.

```
function test=subalgebra(U1,U2,U,F)
%U1 and U2 are two operation tables on U.
test=0;
[a,b]=ismember(F,U);
f=sort(b);
F1=U1(f,f);
F2=U2(f,f);
a=ismember(F1,F);
b=ismember(F2,F);
if sum(find([a,b]==0))~=0
    return
end
test=1;
```

The following example shows how to use the functions in Programs 5.1–5.4.

Example 5.5 Let $(U,R,\rightarrow,\rightsquigarrow,1)$ be the algebra-like approximation space in Example 3.4, and $X=\{a,b,c,d,1\}$, $F=\{a,c,1\}$.

(1) We express two binary operation $\rightarrow$ and $\rightsquigarrow$ as two matrices $U1$ and $U2$. For convenience, the equivalence relation is denoted as the matrix R. Then, "$\rightarrow$", "$\rightsquigarrow$", "R" and "U" can be entered in Matlab with the command:

```
>>U1=['1bcd111';'e111111';'ab1de11';'ecc1111';'ebcd111';'ebcde1f';'abcdef1'];
```

```
>>U2=['1bcd111';'e111111';'ad1de11';'ebc1111';'ebcd111';'ebcde1e';'abcdef1'];
>>R=[1,0,0,0,1,0,0;0,1,1,1,0,0,0;0,1,1,1,0,0,0;0,1,1,1,0,0,0;1,0,0,0,1,0,0;
0,0,0,0,0,1,0;0,0,0,0,0,0,1];
>>U='abcdef1'
```

(2) Put X={$a, b, c, d, 1$}. To generate the upper approximation and lower approximation of X, we can use the function "approximation" (see Program 5.1):

```
>>X='abcd1'; [upper,lower]=approximation(R,U,X)
upper =
    abcde1
lower =
    bcd1
```

(3) To judge whether the upper approximation and lower approximation of X are two pseudo-BCK-algebra, we can use the function "psbck_algebra" (see Program 5.2):

```
>> t1=psbck_algebra(U1,U2,U,upper);  t2=psbck_algebra(U1,U2,U,lower);
>> t0=psbck_algebra(U1,U2,U,X); [t0,  t1,  t2]
ans =
     0     1     1
```

Thus, we conclude that $\mathscr{X}=(X, \to, \rightsquigarrow, 1)$ is a proper rough pseudo-BCK-algebra.

(4) Put F={$1, c$}. To judge whether F is a rough pseudo-filter of $\mathscr{X}$, we can use the function "pseudo_filter" (see Program 5.3):

```
>>F='1c'; [pseudo_filter(U1,U,F,upper),pseudo_filter (U1,U,F,lower)]
ans =
     1     1
>>[F_upper,F_lower]=approximation(R,U,F);
>>[pseudo_filter(U1,U,F_upper,upper),pseudo_filter (U1,U,F_lower,lower)]
ans =
     0     1
```

Thus, we conclude that F is a rough pseudo-filter of type 1, and not type 2.

(5) Put F={$1, c$}. To judge whether F is a rough subalgebra of $\mathscr{X}$, we can use the function "subalgebra" (see Program 5.4):

```
>>t0=subalgebra(U1,U2,U,F);t1=subalgebra(U1,U2,U,F_upper);
>>t2=subalgebra(U1,U2,U,F_lower);[t0,t1,t2]
ans =
     1     1     1
```

Thus, we conclude that F is a rough subalgebra of type 1 (or 2).

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