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On Skew Strongly Reversible Rings Relative to a Monoid

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Abstract For a monoid M, we introduce the concept of skew strongly M-reversible rings which is a generalization of strongly M-reversible rings, and investigate their properties. It is shown that if G is a finitely generated Abelian group, then G is torsion-free if and only if there exists a ring R with $|R| \geq 2$ such that R is skew strongly G-reversible. Moreover, we prove that if R is a right Ore ring with classical right quotient ring Q, then R is skew strongly M-reversible if and only if Q is skew strongly M-reversible.

Keywords reversible rings; skew strongly M-reversible rings; skew monoid rings

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1. Introduction

Throughout this article, R denotes an associative ring with identity and M denotes a monoid, respectively. In [1], Cohn introduced the notion of a reversible ring. A ring R is reversible if $a, b \in R$ with $ab = 0$ implies $ba = 0$. Anderson and Camillo [2] used the term of ZC_2 for what is called reversible. A ring R is called symmetric, whenever $abc = 0$ implies $acb = 0$ for all $a, b, c \in R$. Moreover, a ring R is reduced if $a^2 = 0$ implies $a = 0$ for all $a \in R$. Huh and Lee studied a generalization of commutative rings, which is called semicommutative in [3], if $ab = 0$ implies $aRb = 0$ for all $a, b \in R$. In general, we have the following implications:

reduced (resp., commutative) rings $\Rightarrow$ symmetric rings $\Rightarrow$ reversible rings $\Rightarrow$ semicommutative rings. But none of them is irreversible.

In [4], Kim and Lee showed that polynomial rings over reversible rings need not be reversible. Later in 2008, Yang and Liu [5] introduced the notion of strongly reversible rings. A ring R is called strongly reversible, whenever polynomials $f(x), g(x) \in R[x]$ with $f(x)g(x) = 0$ implies $g(x)f(x) = 0$. It is well-known that every reduced ring is strongly reversible and the inverse is not true. Rage and Chhawchharia [6], presented the concept of an Armendariz ring. They called a ring R an Armendariz ring, if polynomials $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_mx^m$ are in $R[x]$ and satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for all i, j. In the following, we denote by $R[M]$ the monoid ring constructed from a ring R and a monoid M, e will always stand for the identity of M. Liu [7] called a ring R an M-Armendariz

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ring (an Armendariz ring relative to a monoid M), if whenever $\alpha = a_1g_1 + a_2g_2 + \cdots + a_ng_n$, $\beta = b_1h_1 + b_2h_2 + \cdots + b_mg_m \in R[M]$ satisfy $\alpha\beta = 0$, then $a_ib_j = 0$, for all i, j. As mentioned in [8], Singh, Juyal and Khan studied a generalization of a strongly reversible ring, which is called strongly M-reversible, whenever $\alpha\beta = 0$ implies $\beta\alpha = 0$ with $\alpha, \beta \in R[M]$.

Motivated by the results of [5,7,9,10], we propose a unified approach to generalize strongly reversible rings and strongly M-reversible rings. The idea is to study the reversible condition defined for the skew monoid ring $R * M$, where R is a ring and M is a monoid. Assume that there exists a monoid homomorphism $\omega : M \to \mathrm{End}(R)$. We denote $\omega(g)$ by ω_g, for each $g \in M$. According to [11], we can form a skew monoid ring $R*M$ (induced by the monoid homomorphism ω) by taking its elements to be finite formal combinations $\sum_{i=1}^{n} a_ig_i$, with multiplication induced by $(ag)(bh) = (a\omega_g(b))(gh)$. Note that the trivial monoid homomorphism is $\omega : M \to \mathrm{End}(R)$ defined by $\omega_g(r) = r$ for each $g \in M$ and $r \in R$. We say that R is a skew strongly M-reversible ring relative to M (or simply skew strongly M-reversible ring), whenever $\alpha\beta = 0$ implies $\beta\alpha = 0$, where $\alpha, \beta \in R * M$. If $M = (\mathbb{N} \cup \{0\}, +)$ and the monoid homomorphism $\omega : M \to \mathrm{End}(R)$ is trivial, it is clear that a ring R is skew strongly M-reversible if and only if R is strongly M-reversible if and only if R is strongly reversible. Therefore, our results will unify some results on strongly reversible rings and strongly M-reversible rings.

2. Main results

In this section, we introduce the notion of a skew strongly M-reversible ring and investigate its properties. We begin with the following definition.

Definition 2.1 *Let R be a ring, M a monoid and $\omega : M \to \mathrm{End}(R)$ a monoid homomorphism. A ring R is called skew strongly M-reversible ring relative to M (or simply skew strongly M-reversible ring), if $\alpha\beta = 0$ implies $\beta\alpha = 0$ for all $\alpha, \beta \in R * M$.*

Example 2.2 *Here are some special cases of skew strongly M-reversible rings:*

(1) Let R be an arbitrary ring and $M = \{e\}$. Then the trivial monoid homomorphism $\omega : M \to \mathrm{End}(R)$ is the only monoid homomorphism and clearly R is skew strongly M-reversible if and only if R is strongly M-reversible.

(2) If $M = (\mathbb{N} \cup \{0\}, +)$ and the monoid homomorphism $\omega : M \to \mathrm{End}(R)$ is trivial, it is clear that a ring R is skew strongly M-reversible if and only if R is strongly M-reversible if and only if R is strongly reversible.

(3) Every M-invariant subring S (i.e., $\omega_g(S) \subseteq S$ for all $g \in M$) of a skew strongly M-reversible ring is also skew strongly M-reversible.

Proposition 2.3 *Let R be a ring, M a monoid and $\omega : M \to \mathrm{End}(R)$ a monoid homomorphism. If a is a central idempotent of R with $\omega_g(a) = a$ for each $g \in M$, then the following statements are equivalent:*

(1) R is a skew strongly M-reversible ring.

(2) aR and $(1-a)R$ are skew strongly M-reversible rings.

Proof (1) $\Rightarrow$ (2) is straightforward.

(2) $\Rightarrow$ (1). Let aR and $(1-a)R$ be skew strongly M-reversible rings, and let $\alpha=\sum_{i=1}^{m}a_ig_i$, $\beta=\sum_{j=1}^{n}b_jh_j$ be elements in $R*M$ with $\alpha\beta=0$. Suppose $\alpha_1=\sum_{i=1}^{m}aa_ig_i$, $\beta_1=\sum_{j=1}^{n}ab_jh_j$, $\alpha_2=\sum_{i=1}^{m}(1-a)a_ig_i$ and $\beta_2=\sum_{j=1}^{n}(1-a)b_jh_j$, then $\alpha_1,\beta_1\in(aR)*M$ and $\alpha_2,\beta_2\in((1-a)R)*M$. This implies that

$$\begin{aligned}\alpha_1\beta_1&=aa_1\omega_{g_1}(ab_1)\,g_1h_1+\cdots+aa_m\omega_{g_n}(ab_m)\,g_nh_m=a\alpha\beta=0,\\ \alpha_2\beta_2&=(1-a)\,a_1\omega_{g_1}((1-a)\,b_1)\,g_1h_1+\cdots+(1-a)\,a_m\omega_{g_n}((1-a)\,b_m)\,g_nh_m\\ &=(1-a)\,\alpha\beta=0,\end{aligned}$$

it follows that $\beta_1\alpha_1=0$ and $\beta_2\alpha_2=0$ since aR and $(1-a)R$ are skew strongly M-reversible. Therefore, $\beta\alpha=b_1\omega_{h_1}(a_1)\,h_1g_1+\cdots+b_n\omega_{h_n}(a_m)\,h_ng_m=0$. This shows that R is skew strongly M-reversible. □

According to Krempa [12], an endomorphism α of a ring R is said to be rigid if $a\alpha(a)=0$ implies $a=0$, for $a\in R$. A ring R is α-rigid if there exists a rigid endomorphism α of R. Clearly, every domain D with a monomorphism α is α-rigid. In [13], the authors introduced α-compatible rings and studied their properties. A ring R is α-compatible if for each $a,b\in R$, $ab=0$ if and only if $a\alpha(b)=0$. Clearly, this may only happen when the endomorphism α is injective. Also by [13, Lemma 2.2], a ring R is α-rigid if and only if R is α-compatible and reduced. For a ring R and a monoid M with $\omega:M\to\mathrm{End}(R)$ a monoid homomorphism, we say that R is M-compatible (resp., M-rigid) if ω_g is compatible (resp., rigid) for any $g\in M$.

Lemma 2.4 ([11, Lemma 2.11]) *Let R be a ring, M a monoid and $\omega:M\to\mathrm{End}(R)$ a monoid homomorphism. If R is M-compatible, then $\omega_g(a)=a$ for each idempotent $a\in R$ and $g\in M$.*

Corollary 2.5 *Let R be an M-compatible ring and M a monoid with $\omega:M\to\mathrm{End}(R)$ a monoid homomorphism. Then R is a skew strongly M-reversible if and only if aR and $(1-a)R$ are skew strongly M-reversible.*

A monoid M is called a u.p.-monoid (unique product monoid) if for any two nonempty finite subsets $A,B\in M$, there exists an element $g\in M$ uniquely in the form of ab with $a\in A$ and $b\in B$. The class of u.p.-monoid is quite large and important [12, 13, 14]. For example, this class includes the right or left ordered monoids, submonoids of a free group, and torsion-free nilpotent groups. Every u.p.-monoid M has no nonunity element of finite order.

Lemma 2.6 *Let M be a u.p.-monoid and R an M-rigid ring. Then $R*M$ is reduced.*

Proof Suppose $\alpha=\sum_{i=1}^{n}a_ig_i$ in $R*M$ such that $\alpha^2=a_1\omega_{g_1}(a_1)\,g_1g_1+\cdots+a_n\omega_{g_n}(a_n)\,g_ng_n=0$, where $a_i\in R$, $g_i\in M$ for all i. Then R is skew M-Armendariz by [11, Proposition 3.3]. Thus $a_i\omega_{g_i}(a_j)=0$ for all $1\le i,j\le n$. Since R is M-rigid, we have that $a_ia_j=0$. In particular ${a_i}^2=0$ for all $1\le i\le n$. Since R is M-rigid, then R is reduced. It follows that $a_i=0$ for all $1\le i\le n$ and therefore $R*M$ is reduced. □

Proposition 2.7 *Let M be a u.p.-monoid and R an M-rigid ring. Then R is skew strongly*

M-reversible.

Proof Let $\alpha = \sum_{i=1}^{n} a_i g_i$, $\beta = \sum_{j=1}^{m} b_j h_j \in R * M$ such that $\alpha\beta = a_1\omega_{g_1}(b_1)(g_1h_1) + \cdots + a_n\omega_{g_n}(b_m)(g_nh_m) = 0$. So $(\beta\alpha)^2 = (\beta\alpha)(\beta\alpha) = \beta(\alpha\beta)\alpha = 0$. Since R is M-rigid, we have $\beta\alpha = 0$ by Lemma 2.6. Hence R is a skew strongly M-reversible ring. □

Lemma 2.8 *Direct products of skew strongly M-reversible rings are skew strongly M-reversible.*

Proposition 2.9 *Let R be a ring, M a commutative cancellative monoid with $\omega : M \to \mathrm{End}(R)$ a monoid homomorphism. Suppose N is an ideal of M such that $\omega_g = id_R$ for every $g \in N$. If R is skew strongly N-reversible, then R is skew strongly M-reversible.*

Proof Let $\alpha = \sum_{i=1}^{n} a_i g_i$, $\beta = \sum_{j=1}^{m} b_j h_j$ be elements of $R * M$ with

$$\alpha\beta = a_1\omega_{g_1}(b_1)(g_1h_1) + \cdots + a_n\omega_{g_n}(b_m)(g_nh_m) = 0.$$

Take $g \in N$. Note that $gg_1, \ldots, gg_n$, $h_1g, \ldots, h_mg \in N$ and $gg_i \neq gg_j$, $h_ig \neq h_jg$ for $i \neq j$, respectively. Put $\alpha_1 = \sum_{i=1}^{n} a_i gg_i$, $\beta_1 = \sum_{j=1}^{m} b_j h_j g$, $\alpha_1, \beta_1 \in R * N$ and we have

$$\begin{aligned}\alpha_1\beta_1 &= a_1\omega_{gg_1}(b_1)(gg_1h_1g) + \cdots + a_n\omega_{gg_n}(b_m)(gg_nh_mg)\\ &= a_1\omega_{g_1}(b_1)(gg_1h_1g) + \cdots + a_n\omega_{g_n}(b_m)(gg_nh_mg) = \alpha\beta\left(g^2\right) = 0.\end{aligned}$$

Since R is skew strongly N-reversible, we obtain

$$\begin{aligned}\beta_1\alpha_1 &= b_1\omega_{h_1g}(a_1)(h_1ggg_1) + \cdots + b_m\omega_{h_mg}(a_n)(h_mggg_n)\\ &= b_1\omega_{h_1}(a_1)(h_1ggg_1) + \cdots + b_m\omega_{h_m}(a_n)(h_mggg_n) = \beta\alpha\left(g^2\right) = 0.\end{aligned}$$

Thus

$$\beta\alpha = b_1\omega_{h_1}(a_1)(h_1g_1) + \cdots + b_m\omega_{h_m}(a_n)(h_mg_n) = 0.$$

This implies that R is skew strongly M-reversible. □

Lemma 2.10 *Let M be a cyclic group of order $n \geq 2$ and R a ring with unity. Then R is not skew strongly M-reversible.*

Proof Suppose that $M = \{e, g, g^2, \ldots, g^{n-1}\}$. Let $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} e + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} g + \cdots + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} g^{n-1}$, $\beta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} e + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} g \in R * M$, and define $\omega : M \to \mathrm{End}(R)$ by $\omega_h\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ c & d \end{pmatrix}$ for all $e \neq h \in M$. Then $\alpha\beta = 0$. But

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \omega_g\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \neq 0,$$

so $\beta\alpha \neq 0$. Thus R is not skew strongly M-reversible. □

Lemma 2.11 *Let M be a monoid and N a submonoid of M. If R is a skew strongly M-reversible* *ring, then R is skew strongly N-reversible.*

Lemma 2.12 ([7, Lemma 1.13]) *If M and N are u.p.-monoids, then so is $M \times N$.*

Let $T(G)$ be set of elements of finite order in an Abelian group G. Then $T(G)$ is a fully invariant subgroup of G. G is said to be torsion-free if $T(G) = \{e\}$.

Theorem 2.13 *Let G be a finitely generated Abelian group. Then the following conditions on G are equivalent:*

(1) G is torsion-free.

(2) There exists a ring R with $|R| \geq 2$ such that R is a skew strongly G-reversible ring.

Proof (2) $\Rightarrow$ (1). If $g \in T(G)$ and $g \neq e$, then $N = \langle g \rangle$ is cyclic group of finite order. If a ring $R \neq \{0\}$ is skew strongly G-reversible. Then R is skew strongly N-reversible by Lemma 2.11, a contradiction by Lemma 2.10. Thus every ring $R \neq \{0\}$ is not skew strongly G-reversible.

(1) $\Rightarrow$ (2). Let G be a finitely generated Abelian group with $T(G) = \{e\}$. Then $G = \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ is a finite direct product of group $\mathbb{Z}$. Clearly, G is u.p.-monoid by Lemma 2.12. Now it is immediate that if R is a commutative M-rigid ring, then R is a skew strongly G-reversible ring. This completes the proof. $\Box$

Let I be an M-invariant ideal of R, M a monoid and $\omega : M \to \mathrm{End}(R)$ a monoid homomorphism. We can define $\bar{\omega} : M \to \mathrm{End}(R/I)$ with $\overline{\omega_g}(r + I) = \omega_g(r) + I$. One can easily check that $\bar{\omega}$ is a monoid homomorphism. Also for any $\alpha = \sum_{i=1}^{n} a_i g_i$ in $R * M$, we denote $\bar{\alpha} = \sum_{i=1}^{n} \overline{a_i} g_i$ in $(R/I) * M$, where $\overline{a_i} = a_i + I$, for each $1 \leq i \leq n$. It is easy to see that the mapping $\phi : R * M \to (R/I) * M$ defined by $\phi(\alpha) = \bar{\alpha}$ is a ring homomorphism.

The following example shows that there exists a ring R such that R/I is skew strongly M-reversible for a non-zero skew strongly M-reversible proper ideal I (as a ring without identity), but R is not skew strongly M-reversible.

Example 2.14 ([5, Example 3.7]) Let S be a division ring. Consider the ring

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} | a, b, c, d \in S \right\}.$$

Then R is not skew strongly M-reversible since it is not reversible. Let M be a monoid with $|M| \geq 2$. Take a non-zero proper ideal $I = \begin{pmatrix} 0 & 0 & S \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, it is easy to see that I is a skew strongly M-reversible ideal of R. If

$$\alpha = \sum_{i=1}^{n} \begin{pmatrix} a_i & b_i & 0 \\ 0 & a_i & c_i \\ 0 & 0 & a_i \end{pmatrix} g_i, \quad \beta = \sum_{j=1}^{m} \begin{pmatrix} u_i & v_j & 0 \\ 0 & u_j & w_j \\ 0 & 0 & u_j \end{pmatrix} h_j$$

are in $(R/I)*M$ satisfying $\alpha\beta=0$. Then we have

$$\begin{pmatrix} \sum a_ig_i & \sum b_ig_i & 0 \\ 0 & \sum a_ig_i & \sum c_ig_i \\ 0 & 0 & \sum a_ig_i \end{pmatrix}\begin{pmatrix} \sum u_jh_j & \sum v_jh_j & 0 \\ 0 & \sum u_jh_j & \sum w_jh_j \\ 0 & 0 & \sum u_jh_j \end{pmatrix}=0$$

which implies $(\sum_{i=1}^n a_ig_i)(\sum_{j=1}^m u_jh_j)=0$, hence $\sum_{i=1}^n a_ig_i=0$ or $\sum_{j=1}^m u_jh_j=0$ since S is a division ring, and it is easy to prove that $\beta\alpha=0$.

However, we have the following affirmative answer to this situation as in the following.

Proposition 2.15 *Suppose that R/I is skew strongly M-reversible for some ideal I of a ring R. If I is M-rigid, then R is skew strongly M-reversible.*

Proof Suppose $\alpha=\sum_{i=1}^n a_ig_i$, $\beta=\sum_{j=1}^m b_jh_j$ are elements in $R*M$ with $\alpha\beta=0$, where $\bar{\alpha}=\sum_{i=1}^n \overline{a_i}g_i$, $\bar{\beta}=\sum_{j=1}^m \overline{b_j}h_j$ are elements in $(R/I)*M$ and $\overline{a_i}=a_i+I$, $\overline{b_j}=b_j+I$. Then we have

$$\begin{aligned}\alpha\beta&=(\sum_{i=1}^n a_ig_i)(\sum_{j=1}^m b_jh_j)=\sum_{i=1}^n\sum_{j=1}^m a_i\omega_{g_i}(b_j)g_ih_j=0,\\ \overline{\alpha}\overline{\beta}&=(\sum_{i=1}^n \overline{a_i}g_i)(\sum_{j=1}^m \overline{b_j}h_j)=\sum_{i=1}^n\sum_{j=1}^m \overline{a_i}\ \overline{\omega_{g_i}}(\overline{b_j})g_ih_j\\ &=\sum_{i=1}^n\sum_{j=1}^m (a_i+I)\overline{\omega_{g_i}}(b_j+I)g_ih_j=\bar{0},\end{aligned}$$

Since R/I is skew strongly M-reversible, it follows that

$$\bar{\beta}\bar{\alpha}=(\sum_{j=1}^m \overline{b_j}h_j)(\sum_{i=1}^n \overline{a_i}g_i)=\bar{0},$$

then we have $\beta\alpha\in I*M$. Since I is M-rigid, $I*M$ is reduced by Lemma 2.5. Hence $(\beta\alpha)^2=(\beta\alpha)(\beta\alpha)=\beta(\alpha\beta)\alpha=0$ implies that $\beta\alpha=0$. Therefore, R is skew strongly M-reversible. $\square$

A ring R is called right Ore, if given $a,b\in R$ with b regular, there exist $a_1,b_1\in R$ with b_1 regular such that $ab_1=ba_1$. It is a well-known fact that a ring R is right Ore if and only if the classical right quotient ring Q of R exists. It was shown in [15, Theorem 16] and [4, Theorem 2.6] that a ring R is reduced (resp., reversible) if and only if Q is reduced (resp., reversible).

More generally, suppose that the classical right quotient ring Q of R exists. Assume that M is a monoid with $\omega: M\to \mathrm{End}(R)$ a monoid homomorphism, then the induced map $\bar{\omega}: M\to \mathrm{End}(Q)$ defined by $\bar{\omega}_g(ab^{-1})=\omega_g(a)\cdot\omega_g(b)^{-1}$ extends ω and is also a monoid homomorphism with $ab^{-1}\in Q$, where $a,b\in R$, $g\in M$ and b is regular. In the following argument, we extend this result to skew strongly M-reversible rings.

Theorem 2.16 *Let M be a monoid and R a right Ore ring with classical right quotient ring Q of R. The ring R is skew strongly M-reversible if and only if Q is skew strongly M-reversible.*

Proof Let $\alpha=\sum_{i=1}^n a_ig_i$, $\beta=\sum_{j=1}^m b_jh_j$ be elements in $Q*M$ such that $\alpha\beta=0$, where

$a_i, b_j \in R$ and $g_i, h_j \in M$ for each i, j. Since R is a right Ore ring with classical right quotient ring Q, we can assume that $a_i = p_i\omega_{g_i}(u^{-1})$, $b_j = q_j\omega_{h_j}(v^{-1})$ with $p_i, q_j \in R$ for all i, j, regular elements $u, v \in R$ and $g \in M$ such that $\omega_g \in \mathrm{End}(R)$ by [16, Proposition 2.1.16]. Also by [16, Proposition 2.1.16], for each j, there exist $c_j \in R$ and a regular element $s \in R$ such that $u^{-1}q_j = c_j s^{-1}$. Put $\alpha_1 = \sum_{i=1}^n p_i g_i$, $\beta_1 = \sum_{j=1}^m q_j h_j$, $\beta_2 = \sum_{j=1}^m c_j h_j$, then we have

$$\begin{aligned}
0 = \alpha\beta &= (\sum_{i=1}^n a_i g_i)(\sum_{j=1}^m b_j h_j) = (\sum_{i=1}^n p_i\omega_{g_i}(u^{-1})g_i)(\sum_{j=1}^m q_j\omega_{h_j}(v^{-1})h_j)\\
&= \sum_{i=1}^n\sum_{j=1}^m p_i\omega_{g_i}(u^{-1})\omega_{g_i}(q_j\omega_{h_j}(v^{-1}))g_i h_j = \sum_{i=1}^n\sum_{j=1}^m p_i\omega_{g_i}(u^{-1}q_j\omega_{h_j}(v^{-1}))g_i h_j\\
&= (\sum_{i=1}^n p_i g_i)(\sum_{j=1}^m u^{-1}q_j\omega_{h_j}(v^{-1})h_j) = (\sum_{i=1}^n p_i g_i)(\sum_{j=1}^m c_j s^{-1}\omega_{h_j}(v^{-1})h_j)\\
&= \alpha_1\beta_2(s^{-1}\omega_{h_j}(v^{-1})).
\end{aligned}$$

Hence $\alpha_1\beta_2 = 0$, and consequently $\alpha_1\beta_1 = 0$ in $R * M$. Again by [16, Proposition 2.1.16], for each i there exist $d_i \in R$ and a regular element $t \in R$ such that $v^{-1}p_i = d_i t^{-1}$. Put $\alpha_2 = \sum_{i=1}^n d_i g_i \in R * M$. Then we have

$$\begin{aligned}
0 = \alpha_1 t\beta_1 &= (\sum_{i=1}^n p_i g_i)t(\sum_{j=1}^m q_j h_j) = (\sum_{i=1}^n (p_i t)g_i)(\sum_{j=1}^m q_j h_j)\\
&= (\sum_{i=1}^n (vd_i)g_i)(\sum_{j=1}^m q_j h_j) = v\alpha_2\beta_1,
\end{aligned}$$

thus $\alpha_2\beta_1 = 0$. Since R is skew strongly M-reversible, we have $\beta_1\alpha_2 = 0$. Then

$$\begin{aligned}
\beta\alpha &= (\sum_{j=1}^m b_j h_j)(\sum_{i=1}^n a_i g_i) = (\sum_{j=1}^m q_j\omega_{h_j}(v^{-1})h_j)(\sum_{i=1}^n p_i\omega_{g_i}(u^{-1})g_i)\\
&= \sum_{j=1}^m\sum_{i=1}^n q_j\omega_{h_j}(v^{-1})\omega_{h_j}(p_i\omega_{g_i}(u^{-1}))h_j g_i = \sum_{j=1}^m\sum_{i=1}^n q_j\omega_{h_j}(v^{-1}p_i\omega_{g_i}(u^{-1}))h_j g_i\\
&= \sum_{j=1}^m\sum_{i=1}^n q_j\omega_{h_j}(d_j t^{-1}\omega_{g_i}(u^{-1}))h_j g_i = (\sum_{j=1}^m q_j h_j)(\sum_{i=1}^n d_j t^{-1}\omega_{g_i}(u^{-1})g_i)\\
&= \beta_1\alpha_2(t^{-1}\omega_{g_i}(u^{-1})) = 0.
\end{aligned}$$

Thus Q is skew strongly M-reversible.

Conversely, if Q is skew strongly M-reversible, then the result follows from Lemma 2.8. □

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