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Inequalities on the Triangle

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Abstract The article deals with generalizations of the inequalities for convex functions on the triangle. The Jensen and the Hermite-Hadamard inequality are included in the study. Considering a convex function on the triangle, we obtain a generalization of the Jensen-Mercer inequality, and a refinement of the Hermite-Hadamard inequality.

Keywords convex function; convex combination; triangle; barycenter

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1. Introduction

Let $\mathbb{X}$ be a real linear space. A liner combination $\alpha A+\beta B$ of points $A, B \in \mathbb{X}$ and coefficients $\alpha, \beta \in \mathbb{R}$ is affine if $\alpha+\beta = 1$. A set $\mathcal{X} \subseteq \mathbb{X}$ is affine if it contains all binomial affine combinations of its points. A function $h : \mathcal{X} \to \mathbb{R}$ is affine if the equality

$$h(\alpha A + \beta B) = \alpha h(A) + \beta h(B) \tag{1}$$

holds for every binomial affine combination $\alpha A + \beta B$ of the affine set $\mathcal{X}$.

Convex combinations and sets are introduced by restricting to affine combinations with nonnegative coefficients. A function $f : \mathcal{X} \to \mathbb{R}$ is convex if the inequality

$$f(\alpha A + \beta B) \leq \alpha f(A) + \beta f(B) \tag{2}$$

holds for every binomial convex combination $\alpha A + \beta B$ of the convex set $\mathcal{X}$.

Using mathematical induction, the above concept can be extended to n-membered affine or convex combinations.

In this paper, we use the Euclidean plane $\mathbb{X} = \mathbb{R}^2$. Besides planar convex and affine combinations, we will use barycenters of the planar sets, especially triangles. If μ is a positive measure on $\mathbb{R}^2$, and if $\mathcal{X} \subseteq \mathbb{R}^2$ is a measurable set such that $\mu(\mathcal{X}) > 0$, then the integral mean point

$$M = \Big(\frac{\int_{\mathcal{X}} x\mathrm{d}\mu}{\mu(\mathcal{X})}, \frac{\int_{\mathcal{X}} y\mathrm{d}\mu}{\mu(\mathcal{X})}\Big) \tag{3}$$

is called the barycenter of the set $\mathcal{X}$ respecting the measure μ, or just the set barycenter. The barycenter M belongs to the convex hull of the set $\mathcal{X}$, as the smallest convex set containing $\mathcal{X}$.

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Given the measurable set $\mathcal{X}$ of positive measure, every affine function $h:\mathbb{R}^2\to\mathbb{R}$ satisfies the equality

$$h\Big(\frac{\int_{\mathcal{X}} x\mathrm{d}\mu}{\mu(\mathcal{X})},\frac{\int_{\mathcal{X}} y\mathrm{d}\mu}{\mu(\mathcal{X})}\Big)=\frac{\int_{\mathcal{X}} h(x,y)\mathrm{d}\mu}{\mu(\mathcal{X})}. \tag{4}$$

If $\mathcal{X}$ is convex, then every integrable convex function $f:\mathcal{X}\to\mathbb{R}$ satisfies the inequality

$$f\Big(\frac{\int_{\mathcal{X}} x\mathrm{d}\mu}{\mu(\mathcal{X})},\frac{\int_{\mathcal{X}} y\mathrm{d}\mu}{\mu(\mathcal{X})}\Big)\le\frac{\int_{\mathcal{X}} f(x,y)\mathrm{d}\mu}{\mu(\mathcal{X})}. \tag{5}$$

The above inequality presents the simple integral form of the Jensen inequality for planar sets. For the purpose of the paper, the set $\mathcal{X}$ will be used as a triangle. Convex function on the triangle is consequently integrable.

2. The Jensen and the Hermite-Hadamard Inequality on the Triangle

Through the paper we use triangles. The triangle with vertices $A,B,C\in\mathbb{R}^2$, that is, the convex hull $\mathrm{conv}\{A,B,C\}$ will be denoted with ABC. We use proper triangles assuming that its vertices do not belong to one line.

Using the triangle area, each point $X\in ABC$ can be presented by the unique trinomial convex combination

$$X=\alpha A+\beta B+\gamma C, \tag{6}$$

where

$$\alpha=\frac{\mathrm{ar}(XBC)}{\mathrm{ar}(ABC)},\quad \beta=\frac{\mathrm{ar}(XAC)}{\mathrm{ar}(ABC)},\quad \gamma=\frac{\mathrm{ar}(XAB)}{\mathrm{ar}(ABC)}. \tag{7}$$

The next two lemmas present the properties of a convex function $f:ABC\to\mathbb{R}$ concerning its supporting and secant plane.

The discrete version refers to the connection of the given convex combination of triangle points with triangle vertices.

Lemma 2.1 *Let ABC be a triangle in plane $\mathbb{R}^2$, and let $\sum_{i=1}^n\lambda_iX_i$ be a convex combination of points $X_i\in ABC$. Let $\alpha A+\beta B+\gamma C$ be the unique vertices convex combination such that*

$$\sum_{i=1}^n\lambda_iX_i=\alpha A+\beta B+\gamma C. \tag{8}$$

Then every convex function $f:ABC\to\mathbb{R}$ satisfies the double inequality

$$f(\alpha A+\beta B+\gamma C)\le\sum_{i=1}^n\lambda_if(X_i)\le\alpha f(A)+\beta f(B)+\gamma f(C). \tag{9}$$

Proof Putting $M=\sum_{i=1}^n\lambda_iX_i$, we have the following three cases depending on the position of the point M.

If M belongs to the triangle interior, then using a supporting plane $y=h_1(x,y)$ of the convex surface $z=f(x,y)$ at the graph point $(M,f(M))$, and the secant plane $z=h_2(x,y)$ passing through the graph points $(A,f(A))$, $(B,f(B))$ and $(C,f(C))$, we get

$$f(\alpha A+\beta B+\gamma C)=h_1(M)=\sum_{i=1}^n\lambda_ih_1(X_i)\le\sum_{i=1}^n\lambda_if(X_i)$$

$$\leq \sum_{i=1}^{n} \lambda_i h_2(X_i) = h_2(M) = \alpha f(A) + \beta f(B) + \gamma f(C) \tag{10}$$

because h_2 and f coincide at vertices.

If M belongs to the side relative interior (assume that M belongs to the relative interior of the side AB in which case $\gamma = 0$), then we can apply the previous procedure to the restriction of the convex surface $z = f(x, y)$ to the convex curve on the line segment AB using a supporting line and the secant line.

If M is the triangle vertex (assume that $M = A$ in which case $\beta = \gamma = 0$), then equation (9) is reduced to the trivial double inequality $f(A) \leq f(A) \leq f(A)$. $\square$

The discrete-integral version refers to the connection of the triangle barycenter with triangle vertices.

Lemma 2.2 *Let ABC be a triangle in plane $\mathbb{R}^2$, and let μ be a positive measure on $\mathbb{R}^2$ such that $\mu(ABC) > 0$. Let $\alpha A + \beta B + \gamma C$ be the unique vertices convex combination such that*

$$\Big(\frac{\int_{ABC} x \mathrm{d}\mu}{\mu(ABC)}, \frac{\int_{ABC} y \mathrm{d}\mu}{\mu(ABC)}\Big) = \alpha A + \beta B + \gamma C. \tag{11}$$

Then every convex function $f : ABC \to \mathbb{R}$ satisfies the double inequality

$$f(\alpha A + \beta B + \gamma C) \leq \frac{\int_{ABC} f(x, y) \mathrm{d}\mu}{\mu(ABC)} \leq \alpha f(A) + \beta f(B) + \gamma f(C). \tag{12}$$

Proof Assuming that M is the barycenter of the triangle ABC, the proof can be done as in Lemma 2.1. Utilizing equation (10), we use the integral means instead of the n-membered convex combinations. $\square$

Using the Riemann integral in Lemma 2.2, the condition in (11) gives the barycentric point

$$\Big(\frac{\int_{ABC} x \mathrm{d}x\mathrm{d}y}{\mathrm{ar}(ABC)}, \frac{\int_{ABC} y \mathrm{d}x\mathrm{d}y}{\mathrm{ar}(ABC)}\Big) = \frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C, \tag{13}$$

and its use in equation (12) implies the Hermite-Hadamard inequality for convex functions on the triangle,

$$f(\frac{A + B + C}{3}) \leq \frac{\int_{ABC} f(x, y) \mathrm{d}x\mathrm{d}y}{\mathrm{ar}(ABC)} \leq \frac{f(A) + f(B) + f(C)}{3}. \tag{14}$$

In fact, the above inequality holds for all integrable functions $f : ABC \to \mathbb{R}$ that admit a supporting plane at the barycenter $M = (A + B + C)/3$, and satisfy the supporting-secant plane inequality

$$h_1(X) \leq f(X) \leq h_2(X), \quad X \in ABC. \tag{15}$$

Moreover, the inequality in equation (14) follows by integrating the inequality in equation (15) over the triangle ABC.

Convex functions play an important role in pure and applied mathematics, and have an essential place in inequalities. Among many significant results about convex functions, the Jensen and the Hermite-Hadamard inequality are fundamental. In 1905, applying mathematical induction, Jensen [1] extended the inequality in equation (2) to n-membered convex combinations. In

1906, Jensen [2] stated the integral form of the aforementioned extended inequality. In 1883, studying convex functions, Hermite [3] attained the inequality in equation (14). In 1893, not knowing Hermite's result, Hadamard [4] got the left-hand side of equation (14). For information as regards the Jensen and the Hermite-Hadamard inequality, one may refer to papers [5–10].

3. Main results

In Theorem 3.1, we improve the Hermite-Hadamard inequality in equation (14) by using the convex combination of barycenters of subtriangles. In Theorem 3.8, we establish the Jensen-Mercer inequality on the triangle by applying special affine combinations of triangle points.

The following is a refinement of the Hermite-Hadamard inequality. We use the designation $f(X)\mathrm{d}S$ for $f(x,y)\mathrm{d}x\mathrm{d}y$.

Theorem 3.1 *Let ABC be a triangle in plane $\mathbb{R}^2$. Then every convex function $f : ABC \to \mathbb{R}$ satisfies the series of inequalities*

$$\begin{aligned} f(\frac{A+B+C}{3}) &\le \frac{1}{3}f(\frac{4A+4B+C}{9}) + \frac{1}{3}f(\frac{4A+B+4C}{9}) + \frac{1}{3}f(\frac{A+4B+4C}{9}) \\ &\le \frac{\int_{ABC} f(X)\mathrm{d}S}{\mathrm{ar}(ABC)} \le \frac{1}{3}f(\frac{A+B+C}{3}) + \frac{2}{3}\frac{f(A)+f(B)+f(C)}{3} \\ &\le \frac{f(A)+f(B)+f(C)}{3}. \end{aligned} \tag{16}$$

Proof Denoting the barycenter of the triangle ABC with M, we have the vertices convex combination

$$M = \frac{A+B+C}{3}, \tag{17}$$

and the barycenters (of triangles MAB, MAC and MBC) convex combination

$$M = \frac{1}{3}(\frac{M+A+B}{3}) + \frac{1}{3}(\frac{M+A+C}{3}) + \frac{1}{3}(\frac{M+B+C}{3}). \tag{18}$$

Applying the Jensen inequality to the above convex combination, and the Hermite-Hadamard inequality to each subtriangle barycenter, we get

$$\begin{aligned} f(\frac{A+B+C}{3}) &\le \frac{1}{3}f(\frac{M+A+B}{3}) + \frac{1}{3}f(\frac{M+A+C}{3}) + \frac{1}{3}f(\frac{M+B+C}{3}) \\ &\le \frac{\int_{MAB} f(X)\mathrm{d}S}{3\,\mathrm{ar}(MAB)} + \frac{\int_{MAC} f(X)\mathrm{d}S}{3\,\mathrm{ar}(MAC)} + \frac{\int_{MBC} f(X)\mathrm{d}S}{3\,\mathrm{ar}(MBC)} \\ &\le \frac{3f(M)+2f(A)+2f(B)+2f(C)}{9} \\ &= \frac{1}{3}f(\frac{A+B+C}{3}) + \frac{2}{3}\frac{f(A)+f(B)+f(C)}{3} \\ &\le \frac{f(A)+f(B)+f(C)}{3}. \end{aligned} \tag{19}$$

Since $\mathrm{ar}(MAB) = \mathrm{ar}(MAC) = \mathrm{ar}(MBC) = (1/3)\,\mathrm{ar}(ABC)$, it follows that

$$\frac{\int_{MAB} f(X)\mathrm{d}S}{3\,\mathrm{ar}(MAB)} + \frac{\int_{MAC} f(X)\mathrm{d}S}{3\,\mathrm{ar}(MAC)} + \frac{\int_{MBC} f(X)\mathrm{d}S}{3\,\mathrm{ar}(MBC)} = \frac{\int_{ABC} f(X)\mathrm{d}S}{\mathrm{ar}(ABC)}. \tag{20}$$

Inserting the right side of equation (20) into equation (19), we cover the inequality in equation (16). □

Corollary 3.2 *Let ABC be a triangle in plane $\mathbb{R}^2$. Then every convex function $f : ABC \to \mathbb{R}$ satisfies the double inequality*

$$f(\frac{A+B+C}{3}) \le \frac{1}{3}\Big(\frac{\int_{AB} f(X)\mathrm{d}s}{\mathrm{d}(A,B)} + \frac{\int_{AC} f(X)\mathrm{d}s}{\mathrm{d}(A,C)} + \frac{\int_{BC} f(X)\mathrm{d}s}{\mathrm{d}(B,C)}\Big)$$
$$\le \frac{f(A)+f(B)+f(C)}{3}. \quad (21)$$

Proof Applying the Jensen inequality to the convex combination

$$M = \frac{1}{3}(\frac{A+B}{2}) + \frac{1}{3}(\frac{A+C}{2}) + \frac{1}{3}(\frac{B+C}{2}), \quad (22)$$

we get

$$f(M) \le \frac{1}{3}f(\frac{A+B}{2}) + \frac{1}{3}f(\frac{A+C}{2}) + \frac{1}{3}f(\frac{B+C}{2}), \quad (23)$$

and using the classic Hermite-Hadamard inequality for the line segment, we obtain the inequality in equation (21). □

Combining the inequalities in equations (16) and (21), we can take out the inequality

$$\frac{\int_{ABC} f(X)\mathrm{d}S}{\mathrm{ar}(ABC)} \le \frac{1}{3}\Big(\frac{\int_{AB} f(X)\mathrm{d}s}{3\mathrm{d}(A,B)} + \frac{\int_{AC} f(X)\mathrm{d}s}{3\mathrm{d}(A,C)} + \frac{\int_{BC} f(X)\mathrm{d}s}{3\mathrm{d}(B,C)}\Big) +$$
$$\frac{2}{3}\frac{f(A)+f(B)+f(C)}{3}. \quad (24)$$

Conjecture 3.3 *Let ABC be a triangle in plane $\mathbb{R}^2$. Then every convex function $f : ABC \to \mathbb{R}$ satisfies the inequality*

$$\frac{\int_{ABC} f(X)\mathrm{d}S}{\mathrm{ar}(ABC)} \le \frac{1}{3}\Big(\frac{\int_{AB} f(X)\mathrm{d}s}{\mathrm{d}(A,B)} + \frac{\int_{AC} f(X)\mathrm{d}s}{\mathrm{d}(A,C)} + \frac{\int_{BC} f(X)\mathrm{d}s}{\mathrm{d}(B,C)}\Big). \quad (25)$$

Now, we replace the barycenter M by a triangle PQR containing in the interior of the triangle ABC, and satisfying the barycenter equality

$$\frac{P+Q+R}{3} = \frac{A+B+C}{3}. \quad (26)$$

Such two triangles are presented in Figure 1. Applying the right-hand side of the inequality in equation (9) to the above assumption, and multiplying by 3, we obtain the simple inequality

$$f(P)+f(Q)+f(R) \le f(A)+f(B)+f(C) \quad (27)$$

that will be used in this section.

Corollary 3.4 *Let ABC be a triangle in plane $\mathbb{R}^2$, and let $PQR \subset ABC$ be a subtriangle contained in the interior of ABC, and sharing the common barycenter with ABC. Then every convex function $f : ABC \to \mathbb{R}$ satisfies the series of inequalities*

$$f(\frac{A+B+C}{3}) \le \frac{1}{3}f(M) + \frac{1}{9}\sum_{i=1}^{6} f(M_i) \le \frac{1}{3}\frac{\int_{\triangle_0} f(X)\mathrm{d}X}{\mathrm{ar}(\triangle_0)} + \frac{1}{9}\sum_{i=1}^{6}\frac{\int_{\triangle_i} f(X)\mathrm{d}X}{\mathrm{ar}(\triangle_i)}$$

$$\leq \frac{2}{3}\frac{f(P)+f(Q)+f(R)}{3}+\frac{1}{3}\frac{f(A)+f(B)+f(C)}{3}$$
$$\leq \frac{f(A)+f(B)+f(C)}{3}, \tag{28}$$

where

$$M=\frac{P+Q+R}{3}, M_1=\frac{A+P+Q}{3}, \ldots, M_6=\frac{B+C+R}{3} \tag{29}$$

and

$$\triangle_0 = PQR, \triangle_1 = APQ, \ldots, \triangle_6 = BCR. \tag{30}$$

Proof Applying the Hermite-Hadamard inequality to the barycenter of the triangle PQR written as

$$M=\frac{1}{3}P+\frac{1}{3}Q+\frac{1}{3}R, \tag{31}$$

we have

$$f(M)=f(\frac{P+Q+R}{3}) \leq \frac{\int_{PQR} f(X)\mathrm{d}X}{\mathrm{ar}(PQR)} \leq \frac{f(P)+f(Q)+f(R)}{3}. \tag{32}$$

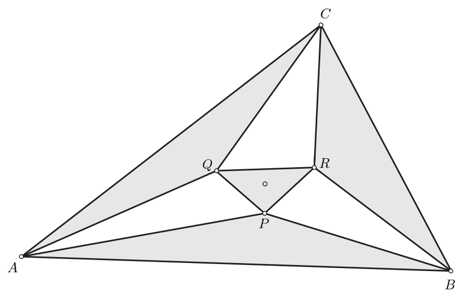

Figure 1 Triangles ABC and PQR with the common barycenter

Applying the same procedure to the convex combination of the barycenters $(A+P+Q)/3$, $(B+P+R)/3$ and $(C+Q+R)/3$ given as

$$M=\frac{1}{3}(\frac{A+P+Q}{3})+\frac{1}{3}(\frac{B+P+R}{3})+\frac{1}{3}(\frac{C+Q+R}{3}), \tag{33}$$

we get

$$f(M) \leq \frac{1}{3}f(\frac{A+P+Q}{3})+\frac{1}{3}f(\frac{B+P+R}{3})+\frac{1}{3}f(\frac{C+Q+R}{3})$$
$$\leq \frac{\int_{APQ} f(X)\mathrm{d}X}{3\,\mathrm{ar}(APQ)}+\frac{\int_{BPR} f(X)\mathrm{d}X}{3\,\mathrm{ar}(BPR)}+\frac{\int_{CQR} f(X)\mathrm{d}X}{3\,\mathrm{ar}(CQR)}$$
$$\leq \frac{2f(P)+2f(Q)+2f(R)+f(A)+f(B)+f(C)}{9}. \tag{34}$$

Similarly, using the convex combination

$$M=\frac{1}{3}(\frac{A+B+P}{3})+\frac{1}{3}(\frac{A+C+Q}{3})+\frac{1}{3}(\frac{B+C+R}{3}), \tag{35}$$

we get

$$f(M) \leq \frac{1}{3}f(\frac{A+B+P}{3})+\frac{1}{3}f(\frac{A+C+Q}{3})+\frac{1}{3}f(\frac{B+C+R}{3})$$

$$\leq \frac{\int_{ABP} f(X)\mathrm{d}X}{3\,\mathrm{ar}(ABP)} + \frac{\int_{ACQ} f(X)\mathrm{d}X}{3\,\mathrm{ar}(ACQ)} + \frac{\int_{BCR} f(X)\mathrm{d}X}{3\,\mathrm{ar}(BCR)}$$
$$\leq \frac{f(P)+f(Q)+f(R)+2f(A)+2f(B)+2f(C)}{9}. \tag{36}$$

Taking the arithmetic means of the inequalities in equations (32), (34) and (36), using equation (27) and rearranging, we obtain the inequality in equation (28). □

One variant of Jensen's inequality is interesting for more than ten years. The variant states that every convex function f on the line segment $[a,b]$ satisfies the inequality

$$f\Big(a+b-\sum_{i=1}^{n}\lambda_i c_i\Big) \leq f(a)+f(b)-\sum_{i=1}^{n}\lambda_i f(c_i) \tag{37}$$

for all convex combinations $\sum_{i=1}^{n}\lambda_i c_i$ of points $c_i \in [a,b]$. This inequality is obtained in [11], and it is usually called the Jensen-Mercer inequality. Some generalizations can be found in [9] and [12].

What is the basis of the inequality in equation (37). Taking one point $c \in [a,b]$, we have the initial inequality

$$f(a+b-c) \leq f(a)+f(b)-f(c). \tag{38}$$

If $c \in [a,b]$, then $d = a+b-c \in [a,b]$, and the intervals $[a,b]$ and $[c,d]$ have the same barycenter $(a+b)/2$.

We want to transfer the inequality in equation (37) to convex functions on the triangle. At first, an example is given to indicate the difference in relation to the line segment.

Example 3.5 We take the triangle ABC with vertices $A(0,0)$, $B(6,0)$ and $C(0,6)$, and points $P(1,0)$ and $Q(0,1)$ which belong to ABC. The point

$$R = A+B+C-P-Q = (5,5) \tag{39}$$

does not belong to the triangle ABC. Triangles ABC and PQR share the common barycenter $M(2,2)$.

The following are conditions which ensure that the point R belongs to the triangle.

Lemma 3.6 *Let ABC be a triangle in plane $\mathbb{R}^2$, and let $P,Q \in ABC$ be points given as the convex combinations*

$$P = \alpha_1 A+\beta_1 B+\gamma_1 C, \quad Q = \alpha_2 A+\beta_2 B+\gamma_2 C. \tag{40}$$

Then the point

$$R = A+B+C-P-Q \tag{41}$$

belongs to the triangle ABC if, and only if,

$$\alpha_1+\alpha_2 \leq 1, \quad \beta_1+\beta_2 \leq 1, \quad \gamma_1+\gamma_2 \leq 1. \tag{42}$$

Proof The affine combination

$$R = (1-\alpha_1-\alpha_2)A+(1-\beta_1-\beta_2)B+(1-\gamma_1-\gamma_2)C \tag{43}$$

is convex if and only if the coefficients condition in equation (42) is valid. □

Leaving out the convex combinations of points P and Q, below we simply assume that the point R belongs to the triangle.

Lemma 3.7 *Let ABC be a triangle in plane $\mathbb{R}^2$, and let $P, Q \in ABC$ be points such that the point $A+B+C-P-Q$ belongs to ABC. Then every convex function $f : ABC \to \mathbb{R}$ satisfies the inequality*

$$f(A+B+C-P-Q) \le f(A)+f(B)+f(C)-f(P)-f(Q). \tag{44}$$

Proof Taking the point $R = A+B+C-P-Q$, we have the convex combinations equality

$$\frac{1}{3}P+\frac{1}{3}Q+\frac{1}{3}R = \frac{1}{3}A+\frac{1}{3}B+\frac{1}{3}C \tag{45}$$

which can be taken as the condition in equation (8). So, applying the right-hand side of the inequality in equation (9), we get

$$\frac{1}{3}f(P)+\frac{1}{3}f(Q)+\frac{1}{3}f(R) \le \frac{1}{3}f(A)+\frac{1}{3}f(B)+\frac{1}{3}f(C), \tag{46}$$

and rearranging, we obtain the inequality in equation (44). □

The following is an extension of the Jensen-Mercer inequality in equation (37) to the triangle.

Theorem 3.8 *Let ABC be a triangle in plane $\mathbb{R}^2$, and let $\sum_{i=1}^n \lambda_i P_i$ and $\sum_{i=1}^n \lambda_i Q_i$ be convex combinations of points $P_i, Q_i \in ABC$ such that the points $A+B+C-P_i-Q_i$ belong to ABC. Then every convex function $f : ABC \to \mathbb{R}$ satisfies the inequality*

$$f\Big(A+B+C-\sum_{i=1}^n \lambda_i P_i-\sum_{i=1}^n \lambda_i Q_i\Big) \le f(A)+f(B)+f(C)-\sum_{i=1}^n \lambda_i f(P_i)-\sum_{i=1}^n \lambda_i f(Q_i). \tag{47}$$

Proof We have the equality

$$A+B+C-\sum_{i=1}^n \lambda_i P_i-\sum_{i=1}^n \lambda_i Q_i = \sum_{i=1}^n \lambda_i (A+B+C-P_i-Q_i) \tag{48}$$

whose right side is a convex combination. Using the above equality, applying Jensen's inequality, and the inequality in equation (28), we get

$$\begin{aligned} f\Big(A+B+C-\sum_{i=1}^n \lambda_i P_i-\sum_{i=1}^n \lambda_i Q_i\Big) &= f\Big(\sum_{i=1}^n \lambda_i (A+B+C+P_i-Q_i)\Big) \\ \le \sum_{i=1}^n \lambda_i f(A+B+C-P_i-Q_i) &\le \sum_{i=1}^n \lambda_i (f(A)+f(B)+f(C)-f(P_i)-f(Q_i)) \\ = f(A)+f(B)+f(C)-\sum_{i=1}^n \lambda_i f(P_i)-\sum_{i=1}^n \lambda_i f(Q_i) \end{aligned} \tag{49}$$

concluding the proof. □

4. Generalization

Theorems 3.1 and 3.8 can be generalized to convex functions on the simplex.

Let $A_1, \ldots, A_{m+1} \in \mathbb{R}^m$ be points such that $A_1 - A_{m+1}, \ldots, A_m - A_{m+1}$ are linearly independent. The convex hull $\operatorname{conv}\{A_1, \ldots, A_{m+1}\}$ is called the m-simplex with vertices $A_1, \ldots, A_{m+1}$. In accordance with the previous section, we use the designation $A_1, \ldots, A_{m+1}$.

Every convex function $f : A_1, \ldots, A_{m+1} \to \mathbb{R}$ satisfies the double inequality

$$f\Big(\frac{\sum_{k=1}^{m+1} A_k}{m+1}\Big) \leq \frac{\int_{A_1,\ldots,A_{m+1}} f(X)\mathrm{d}V}{\operatorname{vol}(A_1, \ldots, A_{m+1})} \leq \frac{\sum_{k=1}^{m+1} f(A_k)}{m+1}, \tag{50}$$

known as the Hermite-Hadamard inequality on the m-simplex. The designations vol and $\mathrm{d}V$ refer to the volume in space $\mathbb{R}^m$. Variants like this can be found in [13–15].

Theorem 4.1 *Let $A_1, \ldots, A_{m+1}$ be an m-simplex in space $\mathbb{R}^m$. Then every convex function $f : A_1, \ldots, A_{m+1} \to \mathbb{R}$ satisfies the series of inequalities*

$$\begin{aligned} f\Big(\frac{\sum_{k=1}^{m+1} A_k}{m+1}\Big) &\leq \frac{1}{m+1}\sum_{k=1}^{m+1} f\Big(\frac{A_k + (m+2)(A_1 + \cdots + A_{k-1} + A_{k+1} + \cdots + A_{m+1})}{(m+1)^2}\Big) \\ &\leq \frac{\int_{A_1,\ldots,A_{m+1}} f(X)\mathrm{d}V}{\operatorname{vol}(A_1, \ldots, A_{m+1})} \leq \frac{1}{m+1} f\Big(\frac{\sum_{k=1}^{m+1} A_k}{m+1}\Big) + \frac{m}{m+1}\frac{\sum_{k=1}^{m+1} f(A_k)}{m+1} \\ &\leq \frac{\sum_{k=1}^{m+1} f(A_k)}{m+1}. \end{aligned} \tag{51}$$

Proof We use the barycenter

$$M = \frac{\sum_{k=1}^{m+1} A_k}{m+1} \tag{52}$$

as the convex combination

$$M = \frac{1}{m+1}\sum_{k=1}^{m+1}\Big(\frac{M + A_1 + \cdots + A_{k-1} + A_{k+1} + \cdots + A_{m+1}}{m+1}\Big). \tag{53}$$

Applying the Jensen inequality, and the Hermite-Hadamard inequality in equation (50), we can prove the inequality in equation (51). □

Corollary 4.2 *Let $A_1, \ldots, A_{m+1}$ be an m-simplex in space $\mathbb{R}^m$. Then every convex function $f : A_1, \ldots, A_{m+1} \to \mathbb{R}$ satisfies the double inequality*

$$\begin{aligned} f\Big(\frac{\sum_{k=1}^{m+1} A_k}{m+1}\Big) &\leq \frac{1}{m+1}\sum_{k=1}^{m+1} \frac{\int_{A_1,\ldots,A_{k-1}A_{k+1},\ldots,A_{m+1}} f(X)\mathrm{d}V'}{\operatorname{vol}'(A_1, \ldots, A_{k-1}A_{k+1}, \ldots, A_{m+1})} \\ &\leq \frac{\sum_{k=1}^{m+1} f(A_k)}{m+1}, \end{aligned} \tag{54}$$

where the designations vol' *and* $\mathrm{d}V'$ *refer to the volume in space* $\mathbb{R}^{m-1}$.

Conjecture 4.3 *Let $A_1, \ldots, A_{m+1}$ be an m-simplex in space $\mathbb{R}^m$. Then every convex function $f : A_1, \ldots, A_{m+1} \to \mathbb{R}$ satisfies the inequality*

$$\frac{\int_{A_1,\ldots,A_{m+1}} f(X)\mathrm{d}V}{\operatorname{vol}(A_1, \ldots, A_{m+1})} \leq \frac{1}{m+1}\sum_{k=1}^{m+1} \frac{\int_{A_1,\ldots,A_{k-1}A_{k+1},\ldots,A_{m+1}} f(X)\mathrm{d}V'}{\operatorname{vol}'(A_1, \ldots, A_{k-1}A_{k+1}, \ldots, A_{m+1})}. \tag{55}$$

In the case $m = 1$, the above conjecture inequality can be read as

$$\frac{\int_a^b f(x)\mathrm{d}x}{b-a} \le \frac{f(a)+f(b)}{2}, \tag{56}$$

that is, the right-hand side of the classic Hermite-Hadamard inequality.

Theorem 4.4 *Let $A_1, \ldots, A_{m+1}$ be an m-simplex in space $\mathbb{R}^m$, and let $\sum_{i=1}^n \lambda_i P_{ij}$ be convex combinations of points $P_{ij} \in A_1, \ldots, A_{m+1}$ such that the points $\sum_{k=1}^{m+1} A_k - \sum_{j=1}^m P_{ij}$ belong to $A_1, \ldots, A_{m+1}$. Then every convex function $f : A_1, \ldots, A_{m+1} \to \mathbb{R}$ satisfies the inequality*

$$f\Big(\sum_{k=1}^{m+1} A_k - \sum_{j=1}^{m}\sum_{i=1}^{n} \lambda_i P_{ij}\Big) \le \sum_{k=1}^{m+1} f(A_k) - \sum_{j=1}^{m}\sum_{i=1}^{n} \lambda_i f(P_{ij}). \tag{57}$$

Proof The basis of the proof is the equality

$$\sum_{k=1}^{m+1} A_k - \sum_{j=1}^{m}\sum_{i=1}^{n} \lambda_i P_{ij} = \sum_{i=1}^{n} \lambda_i \Big(\sum_{k=1}^{m+1} A_k - \sum_{j=1}^{m} P_{ij}\Big) \tag{58}$$

whose right side is the convex combination of points $P_i = \sum_{k=1}^{m+1} A_k - \sum_{j=1}^m P_{ij}$ belonging to the m-simplex $A_1, \ldots, A_{m+1}$. $\square$

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