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Some Extensions and Improvements of Discrete Carlson's Inequality

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Abstract In this paper, we consider Carlson type inequalities and discuss their possible improvement. First, we obtain two different types of generalizations of discrete Carlson's inequality by using the Hölder inequality and the method of real analysis, then we combine the obtained results with a summation formula of infinite series and some Mathieu type inequalities to establish some improvements of discrete Carlson's inequality and some Carlson type inequalities which are equivalent to the Mathieu type inequalities. Finally, we prove an integral inequality that enables us to deduce an improvement of the Nagy-Hardy-Carlson inequality.

Keywords infinite series; Carlson's inequality; Mathieu's inequality

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1. Introduction

Suppose that $a_n \geq 0$ ($n \in \mathbb{Z}^+$) and $0 < \sum_{n=1}^{\infty} n^2 a_n < \infty$, Carlson [1] proved the following famous inequality

$$\Big(\sum_{n=1}^{\infty} a_n\Big)^4 \leq \pi^2 \sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} n^2 a_n^2. \tag{1.1}$$

Afterwards, Landau [2] obtained a strengthened form of (1.1) as follows:

$$\Big(\sum_{n=1}^{\infty} a_n\Big)^4 \leq \pi^2 \sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} (n-\frac{1}{2})^2 a_n^2. \tag{1.2}$$

Making use of an integral inequality, Nagy [3] proved an improvement of (1.1), which reads

$$\Big(\sum_{n=1}^{\infty} a_n\Big)^2 + \Big(\sum_{n=1}^{\infty} (-1)^n a_n\Big)^2 \leq \pi \Big[\sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} n^2 a_n^2\Big]^{\frac{1}{2}}. \tag{1.3}$$

Since (1.1) and its integral version, as well as their generalizations and improvements, turn out to have applications in various branches of mathematics, such as fourier analysis, interpolation theory, harmonic analysis, etc. [4,5], many authors have paid a warm attention to them. For instances, Barza et al. obtained a multiplicative inequality for inner product and deduced some improvements of inequalities of Carlson type on the basis of the results obtained in [6]; Kuang et

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al. presented an extension and improvement of (1.1) by using the Euler-Maclaurin summation formula in [7]; Leo et al. established some new inequalities of Carlson type for finite sums in [8]. For more improvements and generalizations of Carlson's inequality, we can refer to [2,9] and the references therein.

In order to extend and improve discrete Carlson's inequality, the Hölder inequality and some real analysis techniques are employed to establish two different types of variations of (1.1) and an integral inequality. Moreover, combining the obtained results with the summation formula of infinite series $\sum_{n=-\infty}^{+\infty}\frac{1}{(n+x)^2+t^2}$, some new improvements of (1.1), (1.2) and (1.3) are constructed. Meanwhile, utilizing the obtained results and some inequalities of Matheiu type [10,11], some new Carlson type inequalities, which are equivalent to Mathieu type inequalities, are also derived.

2. Some preliminary lemmas

In this section, we state the following lemmas, which are useful in the proofs of our results.

Lemma 2.1 ([12]) *If $x\in\mathbb{R}$ and $t\in\mathbb{R}^+$, then*

$$\sum_{n=-\infty}^{\infty}\frac{t}{(n+x)^2+t^2}=\frac{\pi(1-e^{-4\pi t})}{1-2e^{-2\pi t}\cos 2\pi x+e^{-4\pi t}}. \tag{2.1}$$

Lemma 2.2 ([10]) *Let $x\in\mathbb{R}\setminus\{0\}$. Then*

$$\sum_{n=1}^{\infty}\frac{2n}{(n^2+x^2)^2}<\frac{1}{x^2+\frac{1}{6}}. \tag{2.2}$$

Lemma 2.3 *Let $t\in\mathbb{R}^+$. Then*

$$\sum_{n=1}^{\infty}\frac{2n}{(n^2+t^2-\frac{1}{6})^2}<\frac{1}{t^2}. \tag{2.3}$$

Proof If $t^2>\frac{1}{6}$, setting $x=\sqrt{t^2-\frac{1}{6}}$ in (2.2) yields (2.3); if $t^2\le\frac{1}{6}$, then

$$\sum_{n=1}^{\infty}\frac{2n}{(n^2+t^2-\frac{1}{6})^2}<\sum_{n=1}^{\infty}\frac{2n}{(n^2-\frac{1}{6})^2}<\sum_{n=1}^{\infty}\frac{2n}{(n^2-\frac{n^2}{6})^2}=\frac{72}{25}\sum_{n=1}^{\infty}\frac{1}{n^3}<6\le\frac{1}{t^2}.\ \Box \tag{2.4}$$

Lemma 2.4 ([11]) *If $\mu,t\in\mathbb{R}^+$, then*

$$\sum_{n=1}^{\infty}\frac{2n}{(n^2+t^2)^{\mu+1}}<\frac{1}{\mu t^{2\mu}}. \tag{2.5}$$

In Lemma 2.1, taking $x=0$ or $x=-\frac{1}{2}$, respectively, will readily yield the following lemma.

Lemma 2.5 *Let $t\in\mathbb{R}^+$. Then*

$$\sum_{n=1}^{\infty}\frac{1}{n^2+t^2}=\frac{\pi}{2t}\frac{e^{2\pi t}+1}{e^{2\pi t}-1}-\frac{1}{2t^2} \tag{2.6}$$

and

$$\sum_{n=1}^{\infty}\frac{1}{(n-\frac{1}{2})^2+t^2}=\frac{\pi}{2t}\frac{e^{2\pi t}-1}{e^{2\pi t}+1}. \tag{2.7}$$

Lemma 2.6 *If $t \in \mathbb{R}^+$, then*

$$\sum_{n=1}^{\infty} \frac{1}{n^2+t^2} \leq \frac{\pi^2}{2(\pi t+1)}. \tag{2.8}$$

Proof From (2.6), it follows that

$$\sum_{n=1}^{\infty} \frac{1}{n^2+t^2} = \frac{\pi}{2t}\frac{e^{2\pi t}+1}{e^{2\pi t}-1} - \frac{1}{2t^2} = \frac{\pi}{2t}\big(1+\frac{2}{e^{2\pi t}-1}-\frac{1}{\pi t}\big). \tag{2.9}$$

Taylor expansion of e^x yields $e^x > 1+x+\frac{x^2}{2}$ for $x>0$. Therefore we have

$$e^{2\pi t}-1 > 2\pi t + 2\pi^2 t^2. \tag{2.10}$$

From (2.9) and (2.10), by simple computation, we can get (2.8). □

Lemma 2.7 *If $t, \delta \in \mathbb{R}$ satisfy $0 < t \leq \delta$, then*

$$\sum_{n=1}^{\infty} \frac{1}{(n-\frac{1}{2})^2+t^2} \leq \frac{\pi^2 e^{2\pi\delta}}{2(\pi t e^{2\pi\delta}+1)}. \tag{2.11}$$

Proof From(2.7), it follows that

$$\sum_{n=1}^{\infty} \frac{1}{(n-\frac{1}{2})^2+t^2} = \frac{\pi}{2t}\frac{e^{2\pi t}-1}{e^{2\pi t}+1} = \frac{\pi}{2t}\big(1-\frac{2}{e^{2\pi t+1}}\big). \tag{2.12}$$

By mean-value theorem, we have

$$e^{2\pi t}-1 = 2\pi t e^{2\pi\xi}, \quad \xi \in (0,t). \tag{2.13}$$

Therefore, we have

$$e^{2\pi t} < 1+2\pi t e^{2\pi\delta}, \quad \text{if} \ \ 0<t\leq\delta. \tag{2.14}$$

From (2.12) and (2.14), by simple calculation, we can get (2.11). □

Making use of the differential method, we have the following lemma.

Lemma 2.8 *If $A, B, a, b \in \mathbb{R}$ satisfy $A, B, a > 0$ and $b \geq 0$, then*

$$\min\big\{\frac{At^2+B}{at+b}\big|0<t\leq\sqrt{\frac{B}{A}}\big\} = \frac{At_0^2+B}{at_0+b} = \frac{2\sqrt{A^2b^2+a^2AB}-2Ab}{a^2}, \tag{2.15}$$

where $t_0 = \frac{\sqrt{A^2b^2+a^2AB}-Ab}{aA}$.

3. Main results

In this section we give our main results.

Theorem 3.1 *Let $a, b \in \mathbb{R}$ satisfy $a>0$ and $b \geq 0$. Let $\{a_n\}, \{\alpha(n)\}, \{\beta(n)\}$ be non-negative sequences such that $\sum_{n=1}^{\infty}\alpha(n)a_n^2 < \infty$ and $0 < \sum_{n=1}^{\infty}\beta(n)a_n^2 < \infty$. And suppose that*

$$\sum_{n=1}^{\infty} \frac{1}{\alpha(n)+\beta(n)t^2} \leq \frac{1}{at+b} \quad \textit{for} \ \ t \in \left(0, \sqrt{\frac{\sum_{n=1}^{\infty}\alpha(n)a_n^2}{\sum_{n=1}^{\infty}\beta(n)a_n^2}}\right].$$

Then

$$\Big(\sum_{n=1}^{\infty} a_n\Big)^4 \le \frac{4\sum_{n=1}^{\infty}\alpha(n)a_n^2\sum_{n=1}^{\infty}\beta(n)a_n^2 - 4b(\sum_{n=1}^{\infty}a_n)^2\sum_{n=1}^{\infty}\beta(n)a_n^2}{a^2}. \tag{3.1}$$

Proof Let $t \in \big(0, \sqrt{\frac{\sum_{n=1}^{\infty}\alpha(n)a_n^2}{\sum_{n=1}^{\infty}\beta(n)a_n^2}}\big]$. By Cauchy-Schwarz inequality, we have

$$\begin{aligned}\Big(\sum_{n=1}^{\infty} a_n\Big)^2 &= \Big(\sum_{n=1}^{\infty} a_n\sqrt{\alpha(n)+\beta(n)t^2}\frac{1}{\sqrt{\alpha(n)+\beta(n)t^2}}\Big)^2\\ &\le \Big[(\sum_{n=1}^{\infty}\beta(n)a_n^2)t^2 + \sum_{n=1}^{\infty}\alpha(n)a_n^2\Big]\frac{1}{at+b}.\end{aligned} \tag{3.2}$$

Combining (2.15) with (3.2) yields that

$$\begin{aligned}\Big(\sum_{n=1}^{\infty} a_n\Big)^2 &\le \min\left\{\frac{t^2\sum_{n=1}^{\infty}\beta(n)a_n^2+\sum_{n=1}^{\infty}\alpha(n)a_n^2}{at+b}\Big|t\in\left(0,\sqrt{\frac{\sum_{n=1}^{\infty}\alpha(n)a_n^2}{\sum_{n=1}^{\infty}\beta(n)a_n^2}}\right]\right\}\\ &= \frac{2\sqrt{(\sum_{n=1}^{\infty}\beta(n)a_n^2)^2b^2+a^2\sum_{n=1}^{\infty}\alpha(n)a_n^2\sum_{n=1}^{\infty}\beta(n)a_n^2}-2b\sum_{n=1}^{\infty}\beta(n)a_n^2}{a^2}.\end{aligned} \tag{3.3}$$

It follows from (3.3) that

$$\Big(a^2\Big(\sum_{n=1}^{\infty}a_n\Big)^2+2b\sum_{n=1}^{\infty}\beta(n)a_n^2\Big)^2 \le 4b^2\Big(\sum_{n=1}^{\infty}\beta(n)a_n^2\Big)^2+4a^2\sum_{n=1}^{\infty}\alpha(n)a_n^2\sum_{n=1}^{\infty}\beta(n)a_n^2. \tag{3.4}$$

A direct calculation in (3.4) leads to (3.1). □

Setting $\alpha(n) = n^2$, $\beta(n) = 1$, combining Theorem 3.1 with (2.8), the following corollaries can be obtained naturally.

Corollary 3.2 *Let $a_n \ge 0$, $n \in \mathbb{N}$ such that $\sum_{n=1}^{\infty} n^2a_n^2 < \infty$. Then*

$$\Big(\sum_{n=1}^{\infty}a_n\Big)^4 \le \pi^2\sum_{n=1}^{\infty}a_n^2\sum_{n=1}^{\infty}n^2a_n^2 - 2\Big(\sum_{n=1}^{\infty}a_n\Big)^2\sum_{n=1}^{\infty}a_n^2. \tag{3.5}$$

Similarly, taking $\alpha(n) = (n-\frac{1}{2})^2, \beta(n) = 1$, gathering Theorem 3.1 with (2.11), we also have the following corollary.

Corollary 3.3 *Let $a_n \ge 0$, $n \in \mathbb{N}$ with $\sum_{n=1}^{\infty} n^2a_n^2 < \infty$. Then*

$$\Big(\sum_{n=1}^{\infty}a_n\Big)^4 \le \pi^2\sum_{n=1}^{\infty}a_n^2\sum_{n=1}^{\infty}(n-\frac{1}{2})^2a_n^2 - 2e^{-2\pi\delta}\Big(\sum_{n=1}^{\infty}a_n\Big)^2\sum_{n=1}^{\infty}a_n^2, \tag{3.6}$$

where $\delta = \sqrt{\frac{\sum_{n=1}^{\infty}(n-\frac{1}{2})^2a_n^2}{\sum_{n=1}^{\infty}a_n^2}}$.

Theorem 3.4 *Suppose that $a, b \in \mathbb{R}$ and $a > b > 0$. Let $\{\alpha(n)\}, \{\beta(n)\}$ be non-negative sequences with $\sum_{n=1}^{\infty}\frac{1}{(\alpha(n)+\beta(n)t^2)^a} < \infty$ for $t > 0$. Set $C = (\frac{a}{a-b})^a(\frac{a-b}{b})^b$. Then the inequality*

$$\Big(\sum_{n=1}^{\infty}a_n\Big)^{a+1} \le C\Big(\sum_{n=1}^{\infty}\alpha(n)a_n^{\frac{a+1}{a}}\Big)^{a-b}\Big(\sum_{n=1}^{\infty}\beta(n)a_n^{\frac{a+1}{a}}\Big)^b \tag{3.7}$$

holds for all non-negative sequences $\{a_n\}$ with $0 < \sum_{n=1}^{\infty}\alpha(n)a_n^{\frac{a+1}{a}} < \infty$ and $0 < \sum_{n=1}^{\infty}\beta(n)a_n^{\frac{a+1}{a}} <$

∞ *if and only if the inequality*

$$\sum_{n=1}^{\infty} \frac{1}{(\alpha(n)+\beta(n)t^2)^a} < \frac{1}{t^{2b}} \tag{3.8}$$

holds for $t>0$.

Proof Suppose that (3.8) holds for $t>0$. In view of Hölder inequality, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{\infty} a_n (\alpha(n)+\beta(n)t^2)^{\frac{a}{a+1}} (\alpha(n)+\beta(n)t^2)^{-\frac{a}{a+1}} \\
&\le \Big(\sum_{n=1}^{\infty} a_n^{\frac{a+1}{a}} (\alpha(n)+\beta(n)t^2)\Big)^{\frac{a}{a+1}} \Big(\sum_{n=1}^{\infty} \frac{1}{(\alpha(n)+\beta(n)t^2)^a}\Big)^{\frac{1}{a+1}} \\
&\le \Big(\sum_{n=1}^{\infty} \alpha(n) a_n^{\frac{a+1}{a}} + t^2 \sum_{n=1}^{\infty} \beta(n) a_n^{\frac{a+1}{a}}\Big)^{\frac{a}{a+1}} \frac{1}{t^{\frac{2b}{a+1}}} \\
&= \Big(t^{-\frac{2b}{a}} \sum_{n=1}^{\infty} \alpha(n) a_n^{\frac{a+1}{a}} + t^{2-\frac{2b}{a}} \sum_{n=1}^{\infty} \beta(n) a_n^{\frac{a+1}{a}}\Big)^{\frac{a}{a+1}} =: f(t).
\end{aligned} \tag{3.9}$$

By using the differential method, we get

$$\min\{f(t): t>0\} = (\frac{a}{a-b})^{\frac{a}{a+1}} (\frac{a-b}{b})^{\frac{b}{a+1}} \Big(\sum_{n=1}^{\infty} \alpha(n) a_n^{\frac{a+1}{a}}\Big)^{\frac{a-b}{a+1}} \Big(\sum_{n=1}^{\infty} \beta(n) a_n^{\frac{a+1}{a}}\Big)^{\frac{b}{a+1}}. \tag{3.10}$$

Combining (3.9) with (3.10) yields

$$\sum_{n=1}^{\infty} a_n \le \min\{f(t): t>0\} = C^{\frac{1}{a+1}} \Big(\sum_{n=1}^{\infty} \alpha(n) a_n^{\frac{a+1}{a}}\Big)^{\frac{a-b}{a+1}} \Big(\sum_{n=1}^{\infty} \beta(n) a_n^{\frac{a+1}{a}}\Big)^{\frac{b}{a+1}}. \tag{3.11}$$

Obviously, (3.7) follows from (3.11).

Conversely, suppose that (3.7) holds for all non-negative real sequences $\{a_n\}$ with $0 < \sum_{n=1}^{\infty} \alpha(n)\ a_n^{\frac{a+1}{a}} < \infty$ and $0 < \sum_{n=1}^{\infty} \beta(n) a_n^{\frac{a+1}{a}} < \infty$. Set $a_n = \frac{1}{(\alpha(n)+\beta(n)t^2)^a}$, $n \in \mathbb{N}$, where $t \in \mathbb{R}^+$. Simple calculation shows that

$$\alpha(n) a_n^{\frac{a+1}{a}} = \frac{\alpha(n)}{(\alpha(n)+\beta(n)t^2)^{a+1}} < \frac{1}{(\alpha(n)+\beta(n)t^2)^a} \tag{3.12}$$

and

$$\beta(n) a_n^{\frac{a+1}{a}} = \frac{\beta(n)}{(\alpha(n)+\beta(n)t^2)^{a+1}} < \frac{1}{t^2(\alpha(n)+\beta(n)t^2)^a}. \tag{3.13}$$

It is easy to see that $a_n = \frac{1}{(\alpha(n)+\beta(n)t^2)^a}$, $n \in \mathbb{N}$ satisfies $\sum_{n=1}^{\infty} \alpha(n) a_n^{\frac{a+1}{a}} < \infty$ and $\sum_{n=1}^{\infty} \beta(n) a_n^{\frac{a+1}{a}} < \infty$. Substituting $a_n = \frac{1}{(\alpha(n)+\beta(n)t^2)^a}$ into (3.7) yields

$$\Big(\sum_{n=1}^{\infty} \frac{1}{(\alpha(n)+\beta(n)t^2)^a}\Big)^{a+1} \le C \Big(\sum_{n=1}^{\infty} \frac{\alpha(n)}{(\alpha(n)+\beta(n)t^2)^{a+1}}\Big)^{a-b} \Big(\sum_{n=1}^{\infty} \frac{\beta(n)}{(\alpha(n)+\beta(n)t^2)^{a+1}}\Big)^{b}. \tag{3.14}$$

In Young inequality

$$x^{\lambda} y^{1-\lambda} \le \lambda x + (1-\lambda) y \quad \text{for} \quad x>0,\ y>0,\ 0<\lambda<1, \tag{3.15}$$

setting $\lambda=\frac{a-b}{a}$, $x=\frac{a}{a-b}\sum_{n=1}^{\infty}\frac{\alpha(n)}{(\alpha(n)+\beta(n)t^2)^{a+1}}$ and $y=\frac{a}{b}\sum_{n=1}^{\infty}\frac{\beta(n)t^2}{(\alpha(n)+\beta(n)t^2)^{a+1}}$ gives

$$\begin{aligned}&(\frac{a}{a-b})^{\frac{a-b}{a}}(\frac{a}{b})^{\frac{b}{a}}\Big(\sum_{n=1}^{\infty}\frac{\alpha(n)}{(\alpha(n)+\beta(n)t^2)^{a+1}}\Big)^{\frac{a-b}{a}}\Big(\sum_{n=1}^{\infty}\frac{\beta(n)t^2}{(\alpha(n)+\beta(n)t^2)^{a+1}}\Big)^{\frac{b}{a}}\\&\le\sum_{n=1}^{\infty}\frac{1}{(\alpha(n)+\beta(n)t^2)^{a}}.\end{aligned}\tag{3.16}$$

By (3.14) and (3.16), we have

$$\Big(\sum_{n=1}^{\infty}\frac{1}{(\alpha(n)+\beta(n)t^2)^{a}}\Big)^{a+1}\le\frac{1}{t^{2b}}\Big(\sum_{n=1}^{\infty}\frac{1}{(\alpha(n)+\beta(n)t^2)^{a}}\Big)^{a}.\tag{3.17}$$

It follows from (3.17) that $\sum_{n=1}^{\infty}\frac{1}{(\alpha(n)+\beta(n)t^2)^{a}}\le\frac{1}{t^{2b}}$. $\Box$

Setting $\alpha(n)=\frac{n^2-\frac{1}{6}}{\sqrt{2n}}$, $\beta(n)=\frac{1}{\sqrt{2n}}$, $a=2$, $b=1$ in Theorem 3.4 and using (2.3), we get the following results.

Corollary 3.5 *Let $a_n\ge0$ with $\sum_{n=1}^{\infty}n^{\frac{3}{2}}a_n^{\frac{3}{2}}<\infty$. Then*

$$\Big(\sum_{n=1}^{\infty}a_n\Big)^3\le2\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}a_n^{\frac{3}{2}}\sum_{n=1}^{\infty}(n^{\frac{3}{2}}-\frac{1}{6\sqrt{n}})a_n^{\frac{3}{2}}.\tag{3.18}$$

Similarly, taking $\alpha(n)=\frac{n^2}{(2n\mu)^{\frac{1}{\mu+1}}}$, $\beta(n)=\frac{1}{(2n\mu)^{\frac{1}{\mu+1}}}$, $a=\mu+1$, $b=\mu$ in Theorem 3.4 and using (2.5), the following corollaries can be obtained naturally.

Corollary 3.6 *Let $a_n\ge0,\mu>0$ with $\sum_{n=1}^{\infty}n^{\frac{2\mu+1}{\mu+1}}a_n^{\frac{\mu+2}{\mu+1}}<\infty$. Then*

$$\Big(\sum_{n=1}^{\infty}a_n\Big)^{\mu+2}\le K\Big(\sum_{n=1}^{\infty}n^{\frac{2\mu+1}{\mu+1}}a_n^{\frac{\mu+2}{\mu+1}}\Big)\Big(\sum_{n=1}^{\infty}n^{-\frac{1}{\mu+1}}a_n^{\frac{\mu+2}{\mu+1}}\Big)^{\mu},\tag{3.19}$$

where $K=\frac{1}{2}(\frac{\mu+1}{\mu})^{\mu+1}$.

Theorem 3.7 *Let $f:[a,b]\to\mathbb{R}$ be a function whose derivative $f'(x)$ is continuous on the closed interval $[a,b]$ and $\int_a^b f(x)\mathrm{d}x=0$. Then*

$$(f^2(a)+f^2(b))^2\le4\int_a^b f^2(x)\mathrm{d}x\int_a^b(f'(x))^2\mathrm{d}x-\frac{4}{b-a}(f(a)+f(b))^2\int_a^b f^2(x)\mathrm{d}x.\tag{3.20}$$

Proof We need only consider the case $\int_a^b(f'(x))^2\mathrm{d}x>0$. Since $\int_a^b f(x)\mathrm{d}x=0$, it follows that there exists $c\in(a,b)$ such that $f(c)=0$. Consequently, for all $t\in\mathbb{R}$, one can find that

$$\int_a^c f'(x)(f(x)+t)\mathrm{d}x=-\frac{f^2(a)}{2}-f(a)t,\quad\int_c^b f'(x)(f(x)+t)\mathrm{d}x=\frac{f^2(b)}{2}+f(b)t.\tag{3.21}$$

It follows from (3.21) that

$$|f^2(a)+f^2(b)+2(f(a)+f(b))t|\le2\int_a^b|f'(x)(f(x)+t)|\mathrm{d}x.\tag{3.22}$$

By (3.22) and Cauchy-Schwarz inequality, we have

$$(f^2(a)+f^2(b)+2(f(a)+f(b))t)^2\le4\int_a^b(f'(x))^2\mathrm{d}x\int_a^b[f(x)+t]^2\mathrm{d}x$$

$$= 4\int_a^b f^2(x)\mathrm{d}x\int_a^b (f'(x))^2\mathrm{d}x + 4(b-a)t^2\int_a^b (f'(x))^2\mathrm{d}x. \tag{3.23}$$

By (3.23), we get

$$\Big[4(f(a)+f(b))^2 - 4(b-a)\int_a^b (f'(x))^2\mathrm{d}x\Big]t^2 + 4[f(a)+f(b)][f^2(a)+f^2(b)]t+$$
$$[f^2(a)+f^2(b)]^2 - 4\int_a^b f^2(x)\mathrm{d}x\int_a^b (f'(x))^2\mathrm{d}x \le 0. \tag{3.24}$$

It follows from (3.24) and properties of quadratic function that

$$\triangle = 16[f(a)+f(b)]^2[f^2(a)+f^2(b)]^2 - 16\Big[(f(a)+f(b))^2 - (b-a)\int_a^b (f'(x))^2\mathrm{d}x\Big]\times$$
$$\Big[(f^2(a)+f^2(b))^2 - 4\int_a^b f^2(x)\mathrm{d}x\int_a^b (f'(x))^2\mathrm{d}x\Big] \le 0. \tag{3.25}$$

A direct calculation in (3.25) leads to (3.20). □

The following corollary of Theorem 3.7 is an improvement of (1.3).

Corollary 3.8 *Let $a_n \ge 0$ with $\sum_{n=1}^{\infty} n^2a_n^2 < \infty$. Then*

$$\Big(\sum_{n=1}^{\infty} a_n\Big)^2 + \Big(\sum_{n=1}^{\infty} (-1)^n a_n\Big)^2 \le \Big[\pi^2\sum_{n=1}^{\infty} a_n^2\sum_{n=1}^{\infty} n^2a_n^2 - 8\Big(\sum_{n=1}^{\infty} a_{2n}\Big)^2\sum_{n=1}^{\infty} a_n^2\Big]^{\frac{1}{2}}. \tag{3.26}$$

Proof By setting $f(x) = \sum_{n=1}^{\infty} a_n\cos(nx)$, $a = 0$, $b = \pi$ in Theorem 3.7, we have

$$f(0) = \sum_{n=1}^{\infty} a_n,\ f(\pi) = \sum_{n=1}^{\infty} (-1)^n a_n,\ \int_0^{\pi} f(x)\mathrm{d}x = 0 \tag{3.27}$$

and

$$\int_0^{\pi} f^2(x)\mathrm{d}x = \frac{\pi}{2}\sum_{n=1}^{\infty} a_n^2,\ \ \int_0^{\pi} (f'(x))^2\mathrm{d}x = \frac{\pi}{2}\sum_{n=1}^{\infty} n^2a_n^2. \tag{3.28}$$

From (3.20), (3.27) and (3.28), we have (3.26). □

4. Concluding remarks

Remark 4.1 The comparison between (3.5) and (3.6) is as follows. Both (3.5) and (3.6) are improvements of Carlson's inequality (1.1). It is easy to observe that neither the right side of the inequality (3.5) nor the right side of the inequality (3.6) is uniformly better than the other. For example, let

$$a_n = \begin{cases} 1, & n = 1, \\ 0, & n > 1. \end{cases}$$

It is clear that real sequence $\{a_n\}$ satisfies $\sum_{n=1}^{\infty} n^2a_n^2 < \infty$. Then the right hand side of (3.5) and (3.6) becomes

$$\pi^2\sum_{n=1}^{\infty} a_n^2\sum_{n=1}^{\infty}(n-\frac{1}{2})^2a_n^2 - 2e^{-2\pi\delta}\Big(\sum_{n=1}^{\infty} a_n\Big)^2\sum_{n=1}^{\infty} a_n^2 = \frac{\pi^2}{4} - 2e^{-\pi},$$

and

$$\pi^2\sum_{n=1}^{\infty}a_n^2\sum_{n=1}^{\infty}n^2a_n^2-2\Big(\sum_{n=1}^{\infty}a_n\Big)^2\sum_{n=1}^{\infty}a_n^2=\pi^2-2,$$

respectively. Therefore

$$\begin{aligned}&\pi^2\sum_{n=1}^{\infty}a_n^2\sum_{n=1}^{\infty}(n-\frac{1}{2})^2a_n^2-2e^{-2\pi\delta}\Big(\sum_{n=1}^{\infty}a_n\Big)^2\sum_{n=1}^{\infty}a_n^2\\&<\pi^2\sum_{n=1}^{\infty}a_n^2\sum_{n=1}^{\infty}n^2a_n^2-2\Big(\sum_{n=1}^{\infty}a_n\Big)^2\sum_{n=1}^{\infty}a_n^2.\end{aligned}$$

On the other aspect, for every real number $\alpha>1$, taking a positive integer n_α such that $\ln^{\alpha-1}(n_\alpha+2)>2\ln^{\alpha-1}2$, and setting

$$a_n=\begin{cases}\frac{1}{(n+1)\ln^{\alpha}(n+1)}, & n\le n_\alpha,\\ 0, & n>n_\alpha,\end{cases}$$

it is evident that real sequence $\{a_n\}$ satisfies $\sum_{n=1}^{\infty}n^2a_n^2<\infty$. Simple computation shows that

$$\lim_{\alpha\to1^+}(1-e^{-2\pi\delta})\Big(\sum_{n=1}^{\infty}a_n\Big)^2=\infty, \tag{4.1}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty}na_n^2&=\sum_{n=1}^{n_\alpha}\frac{n}{(n+1)^2\ln^{2\alpha}(n+1)}<\sum_{n=1}^{\infty}\frac{1}{(n+1)\ln^2(n+1)}\\&<\frac{1}{2\ln^2(2)}+\int_1^{\infty}\frac{\mathrm{d}x}{(x+1)\ln^2(x+1)}=\frac{1}{2\ln^2 2}+\frac{1}{\ln 2}.\end{aligned}\tag{4.2}$$

From (4.1) and (4.2) it is easy to verify that

$$\begin{aligned}&\pi^2\sum_{n=1}^{\infty}a_n^2\sum_{n=1}^{\infty}n^2a_n^2-2\Big(\sum_{n=1}^{\infty}a_n\Big)^2\sum_{n=1}^{\infty}a_n^2\\&<\pi^2\sum_{n=1}^{\infty}a_n^2\sum_{n=1}^{\infty}(n-\frac{1}{2})^2a_n^2-2e^{-2\pi\delta}\Big(\sum_{n=1}^{\infty}a_n\Big)^2\sum_{n=1}^{\infty}a_n^2,\end{aligned}\tag{4.3}$$

if α is sufficiently close to 1 and $\alpha>1$.

Remark 4.2 The superiority of the corresponding constants in (3.18) and (3.19) is as follows. Theorem 3.4 implies that the inequalities (3.18) and (3.19) are respectively equivalent to (2.3) and (2.5). Accordingly, it is easy to verify that the constant factor 2 on the right hand side of (3.18) and the constant factor $K=\frac{1}{2}(\frac{\mu+1}{\mu})^{\mu+1}$ on the right hand side of (3.19) are sharp, respectively.

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