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Fixed Points of Mappings Satisfying a Weakly Contractive Type Condition

Dipankar DAS*, **Nilakshi GOSWAMI**
Mathematics Department, Gauhati University, Assam, India

Abstract In this paper, we discuss a fixed point theorem for mappings derived by a pair of mappings satisfying weak $(k, k^{/})$ contractive type condition on the tensor product spaces. Let X and Y be Banach spaces and $T_1 : X \otimes_\gamma Y \to X$ and $T_2 : X \otimes_\gamma Y \to Y$ be two operators which satisfy weak $(k, k^{/})$ contractive type condition. Using T_1 and T_2, we construct an operator T on $X \otimes_\gamma Y$ and show that T has a unique fixed point in a closed and bounded subset of $X \otimes_\gamma Y$. We derive an iteration scheme converging to this unique fixed point of T. Conversely, using a weakly contractive mapping T, we construct a pair of mappings (T_1, T_2) satisfying weak $(k, k^{/})$ contractive type condition on $X \otimes_\gamma Y$ and from this pair, we also obtain two self mappings S_1 and S_2 on X and Y respectively with unique fixed points.

Keywords tensor product; weakly contractive map; projective tensor norm

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1. Introduction

In 1997, Alber and Guerre-Delabriere [1] defined weakly contractive maps. In 2001, Rhoades [2] proved some fixed point theorems for weakly contractive maps to arbitrary Banach spaces.

Let X be a Banach space and K be a closed convex subset of X. A map $T : K \to K$ is called weakly contractive if, for each $x, y \in K$,

$$\|Tx - Ty\| \leqslant \|x - y\| - \psi(\|x - y\|)$$

where $\psi : [0, \infty) \to [0, \infty)$ is continuous and non decreasing such that ψ is positive on $[0, \infty)$, $\psi(0) = 0$, and $\lim_{t\to\infty} \psi(t) = \infty$. If K is bounded, then the infinity conditions can be omitted. If $\psi(x) = kx$, where $0 < k < 1$, then the above map reduces to a contraction map. The following are some well known iteration schemes for the mapping T.

Mann iteration scheme For a Banach space X, the Mann iterative scheme [2] is defined by

$$x_0 \in X,\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n;\ n \geqslant 0$$

where $0 \leqslant \alpha_n \leqslant 1$, for each n, $\sum \alpha_n = \infty$.

Ishikawa iteration scheme The Ishikawa iteration scheme [2] is defined by

$$x_0 \in X,\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n$$

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* Corresponding author
E-mail address: dipankardasguw@yahoo.com (Dipankar DAS)

$$y_n = (1-\beta_n)x_n + \beta_n T x_n,\ n \geqslant 0$$

where $0 \leqslant \alpha_n, \beta_n \leqslant 1$, for all n.

Kirk iteration scheme For a fixed positive integer k, the map S on the Banach space X is defined by

$$S = \sum_{i=0}^{k} \alpha_i T^i, \text{ where each } \alpha_i \geqslant 0,\ \sum_{i=0}^{k} \alpha_i = 1, \text{ and } \alpha_1 \neq 0.$$

Let $x_0 \in X$. Then the Kirk iteration scheme [2] is defined by $x_{n+1} = S^n x_0$. Now, we consider the projective tensor product $D_{X\otimes_\gamma Y}$ of two Banach spaces X and Y (refers to [3]). Let D_X, D_Y and $D_{X\otimes_\gamma Y}$ denote a closed and bounded subset of X, Y and $X \otimes_\gamma Y$, respectively. Let $T_1 : D_{X\otimes_\gamma Y} \to D_X$ and $T_2 : D_{X\otimes_\gamma Y} \to D_Y$ be two mappings. The pair (T_1, T_2) is said to satisfy weak $(k, k^/)$ contractive type condition, if

$$\|T_1(u) - T_1(v)\| \leqslant \frac{1}{M_2}(k\|u-v\| - \psi_1(k\|u-v\|))$$
$$\|T_2(u) - T_2(v)\| \leqslant \frac{1}{M_1}(k^/\|u-v\| - \psi_2(k^/\|u-v\|)),\quad \forall\, u, v \in D_{X\otimes_\gamma Y}$$

where

(i) $\psi_1 : [0,\infty) \to [0,\infty)$ and $\psi_2 : [0,\infty) \to [0,\infty)$ are continuous and non-decreasing such that ψ_1 and ψ_2 are positive on $[0,\infty)$, $\psi_1(0) = 0 = \psi_2(0)$;

(ii) $\|T_1 u\| \leqslant M_1$ and $\|T_2 u\| \leqslant M_2$; $\forall u \in D_{X\otimes_\gamma Y}$. Here, $D_{X\otimes_\gamma Y}$ is bounded by $M_1 M_2$.

In the next section, we establish a fixed point theorem for a mapping T (derived by T_1 and T_2) on the tensor product space $X \otimes_\gamma Y$. Then we derive an iteration scheme for this mapping T and show the convergence of this scheme to the fixed point of T.

2. Main results

Theorem 2.1 *Let (T_1, T_2) be a pair of mappings satisfying the weak $(k, k^/)$ contractive type condition. We define a map*

$$T : D_{X\otimes_\gamma Y} \to D_{X\otimes_\gamma Y} \text{ s.t. } T(u) = T_1(u) \otimes T_2(u), u \in D_{X\otimes_\gamma Y}.$$

Then T has a unique fixed point in $D_{X\otimes_\gamma Y}$ if $k + k^/ \leqslant 1$, where k and $k^/$ are positive.

Proof Let $u, v \in D_{X\otimes_\gamma Y}$ and $k + k^/ \leqslant 1$

$$\begin{aligned}
\|Tu - Tv\| =& \|T_1 u \otimes T_2 u - T_1 v \otimes T_2 v\| \\
\leqslant& \|T_1 u - T_1 v\| \cdot \|T_2 u\| + \|T_1 v\| \cdot \|T_2 u - T_2 v\| \\
\leqslant& \frac{1}{M_2}[k\|u-v\| - \psi_1(k\|u-v\|)]M_2 + \\
& \frac{1}{M_1}[k^/\|u-v\| - \psi_2(k^/\|u-v\|)]M_1 \\
=& (k+k^/)\|u-v\| - \psi_1(k\|u-v\|) - \psi_2(k^/\|u-v\|) \\
\leqslant& \|u-v\| - \{\psi_1(k\|u-v\|) + \psi_2(k^/\|u-v\|)\}.
\end{aligned}$$

Let $u_0 = x_0 \otimes y_0 \in D_{X\otimes_\gamma Y}$. We take $u_{n+1} = Tu_n = T_1u_n \otimes T_2u_n, n \geqslant 1$. Now

$$\begin{aligned}\|u_{n+1} - u_n\| =& \|Tu_n - Tu_{n-1}\| \\ \leqslant& \|u_n - u_{n-1}\| - \{\psi_1(k\|u_n - u_{n-1}\|) + \psi_2(k^/\|u_n - u_{n-1}\|)\} \\ \leqslant& \|u_n - u_{n-1}\|.\end{aligned}$$

Hence $\{\|u_n - u_{n-1}\|\}$ is a monotonically decreasing sequence of non-negative real numbers and so, is convergent to some real, say p. Now $\|u_{n+1} - u_n\| \leqslant \|u_n - u_{n-1}\| - \{\psi_1(k\|u_n - u_{n-1}\|) + \psi_2(k^/\|u_n - u_{n-1}\|)\}$. Taking $n \to \infty$, we get

$$\begin{aligned}&p \leqslant p - \{\psi_1(kp) + \psi_2(k^/p)\} \text{ (by continuity of } \psi_1 \text{ and } \psi_2) \\ &\Rightarrow \psi_1(kp) + \psi_2(k^/p) \leqslant 0,\end{aligned}$$

this is possible only when $p = 0$. So

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{1}$$

Now, it can be easily shown that $\{u_n\}$ is a Cauchy sequence in $D_{X\otimes_\gamma Y}$. Let it converge to some $u \in D_{X\otimes_\gamma Y}$. Now

$$\begin{aligned}\|u - Tu\| \leqslant& \|u - u_{n+1}\| + \|u_{n+1} - Tu\| = \|u - u_{n+1}\| + \|Tu_n - Tu\| \\ \leqslant& \|u - u_{n+1}\| + \|u_n - u\| - [\psi_1(k\|u_n - u\|) + \psi_2(k^/\|u_n - u\|)].\end{aligned}$$

Taking $n \to \infty$, we get $\|u - Tu\| \leqslant 0 + 0 - [\psi_1(0) + \psi_2(0)] = 0$. Hence $Tu = u$. So, u is a fixed point of T in $D_{X\otimes_\gamma Y}$. To show the uniqueness:

Let u_1 and u_2 be two distinct fixed points for T in $D_{X\otimes_\gamma Y}$. Now

$$\begin{aligned}\|u_1 - u_2\| =& \|Tu_1 - Tu_2\| \leqslant \|u_1 - u_2\| - [\psi_1(k\|u_1 - u_2\|) + \psi_2(k^/\|u_1 - u_2\|)] \\ \Rightarrow& \psi_1(k\|u_1 - u_2\|) + \psi_2(k^/\|u_1 - u_2\|) \leqslant 0,\end{aligned}$$

which is contradiction. So, $u_1 = u_2$. Thus, T has a unique fixed point in the closed and bounded subset $D_{X\otimes_\gamma Y}$ of $X \otimes_\gamma Y$. □

Example 2.2 Let $D_{l^1\otimes_\gamma \mathbb{K}}$ be the closed subset of $l^1 \otimes_\gamma \mathbb{K}$ bounded by a constant c. We define $T_1 : D_{l^1\otimes_\gamma \mathbb{K}} \to D_{l^1}$ by

$$T_1(\sum_i a_i \otimes x_i) = \frac{1}{2c}\sum_i \{a_{i_n}x_i\}_n, \text{ where } a_i = \{a_{i_n}\}_n$$

and $T_2 : D_{l^1\otimes_\gamma \mathbb{K}} \to D_{\mathbb{K}}$ by $T_2(\sum_i a_i \otimes x_i) = \frac{1}{4}\sum_i \|a_i\| \cdot |x_i|$. Then

$$\begin{aligned}\|T_1(\sum_i a_i \otimes x_i)\| &= \frac{1}{2c}\|\sum_i a_i \otimes x_i\| \leqslant \frac{1}{2c} \cdot c = \frac{1}{2}(= M_1), \\ \|T_2(\sum_i a_i \otimes x_i)\| &= \frac{1}{4}\sum_i \|a_i\| \cdot |x_i| \\ &\leqslant \frac{c}{4}(= M_2) \text{ (for projective tensor norm in } D_{l^1\otimes_\gamma \mathbb{K}}).\end{aligned}$$

For $u = \sum_i a_i \otimes x_i$ and $v = \sum_i b_i \otimes y_i$ in $D_{l^1 \otimes_\gamma \mathbb{K}}$, we have

$$\begin{aligned}
\|T_1 u - T_1 v\| &= \|\frac{1}{2c}\sum_i \{a_{i_n} x_i\} - \frac{1}{2c}\sum_i \{b_{i_n} y_i\}\| = \frac{\frac{1}{4}\|\sum_i \{a_{i_n} x_i\} - \sum_i \{b_{i_n} y_i\}\|}{\frac{c}{2}} \\
&\leqslant 2\Big[\frac{\frac{1}{2}\|\sum_i \{a_{i_n} x_i\} - \sum_i \{b_{i_n} y_i\}\| - \frac{1}{2}[\frac{1}{2}\|\sum_i \{a_{i_n} x_i\} - \sum_i \{b_{i_n} y_i\}\|]}{\frac{c}{2}}\Big] \\
&\leqslant \frac{1}{M_2}\Big[\frac{1}{2}\|\sum_i \{a_{i_n} x_i\} - \sum_i \{b_{i_n} y_i\}\| - \psi_1(\frac{1}{2}\|\sum_i \{a_{i_n} x_i\} - \sum_i \{b_{i_n} y_i\}\|)\Big] \\
&= \frac{1}{M_2}\big[\frac{1}{2}\|u - v\| - \psi_1(\frac{1}{2}\|u - v\|)\big]; \text{ where } \psi_1(t) = \frac{t}{2},\ k = \frac{1}{2}.
\end{aligned}$$

Also

$$\begin{aligned}
\|T_2 u - T_2 v\| &= \|\frac{1}{4}\sum_i \|a_i\|.|x_i| - \frac{1}{4}\sum_i \|b_i\| \cdot |y_i|\| = \|\frac{1}{4}(\|u\| - \|v\|)\| \leqslant \frac{1}{4}(\|u - v\|) \\
&\leqslant \frac{\frac{1}{2}\|u - v\| - \frac{1}{2}[\frac{1}{2}\|u - v\|]}{\frac{1}{2}} \leqslant \frac{\frac{1}{2}\|u - v\| - \psi_2(\frac{1}{2}\|u - v\|)}{M_1}; \text{ where } \psi_2(t) = \frac{t}{2}, k^{/} = \frac{1}{2}.
\end{aligned}$$

Therefore, (T_1, T_2) satisfies the weak $(k, k^{/})$ contractive type condition and

$$k + k^{/} = \frac{1}{2} + \frac{1}{2} = 1.$$

So, the mapping $T : D_{l^1 \otimes_\gamma \mathbb{K}} \to D_{l^1 \otimes_\gamma \mathbb{K}}$ defined by

$$T(\sum_i a_i \otimes x_i) = \frac{1}{8c}\sum_i \{M a_{i_n} x_i\} \text{ where } M = \sum_i \|a_i\| \cdot |x_i|$$

has a unique fixed point in $D_{l^1 \otimes_\gamma \mathbb{K}}$.

Combining the Mann and Ishihawa iteration schemes, now we have

Theorem 2.3 *Let (T_1, T_2) satisfy the weak $(k, k^{/})$ contractive type condition. Then the following iteration scheme for the mapping $T : D_{X \otimes_\gamma Y} \to D_{X \otimes_\gamma Y}$ defined by*

$$\begin{aligned}
u_0 \in D_{X \otimes_\gamma Y},\ u_{n+1} &= (1 - \beta_n)u_n + \beta_n T v_n \\
v_n &= (1 - \beta_n + \gamma_n)u_n + (\beta_n - \gamma_n)T w_n \\
w_n &= (1 - \beta_n + \gamma_n + \alpha_n)u_n + (\beta_n - \gamma_n - \alpha_n)T u_n, \quad n \geqslant 0
\end{aligned}$$

with (i) $0 \leqslant \alpha_n < \gamma_n \leqslant \beta_n \leqslant 1$, (ii) $\lim_{n\to\infty} \alpha_n = 0$, (iii) $\sum \gamma_n = \infty$, (iv) $\sum \beta_n \alpha_n = \infty$ converges to u in $D_{X \otimes_\gamma Y}$.

Proof Using Theorem 2.1, we have

$$\|Tu - Tv\| \leqslant \|u - v\| - (\psi_1(k\|u - v\|) + \psi_2(k^{/}\|u - v\|)), \quad \forall\, u, v \in D_{X \otimes_\gamma Y}.$$

Now

$$\begin{aligned}
\|u_{n+1} - u\| &= \|(1 - \beta_n)u_n + \beta_n T v_n - u\| = \|(1 - \beta_n)u_n + \beta_n T v_n - Tu\| \\
&\leqslant (1 - \beta_n)\|u_n - u\| + \beta_n\|T v_n - Tu\| \leqslant (1 - \beta_n)\|u_n - u\| + \beta_n\|v_n - u\|, \qquad (2) \\
\|v_n - u\| &= \|(1 - \beta_n + \gamma_n)u_n + (\beta_n - \gamma_n)T w_n - u\|
\end{aligned}$$

$$\leqslant (1-\beta_n+\gamma_n)\|u_n-u\|+(\beta_n-\gamma_n)\|w_n-u\|, \tag{3}$$

$$\begin{aligned}\|w_n-u\| &= \|(1-\beta_n+\gamma_n+\alpha_n)u_n+(\beta_n-\gamma_n-\alpha_n)Tu_n-u\| \\ &\leqslant (1-\beta_n+\gamma_n+\alpha_n)\|u_n-u\|+(\beta_n-\gamma_n-\alpha_n)\|u_n-u\| \\ &\leqslant \|u_n-u\|. \end{aligned} \tag{4}$$

From (2), (3) and (4), it is clear that $\|u_{n+1}-u\| \leqslant \|u_n-u\|$. So, $\{\|u_n-u\|\}$ is a monotonically decreasing sequence of non-negative real numbers and so, is convergent to some real, say p. Now, it can be shown that $p=0$ (as in previous Theorem). Hence $u_n \to u$ as $n \to \infty$, and thus, we can show that the iteration converges to u in $D_{X\otimes_\gamma Y}$. □

Remark 2.4 (i) If $\beta_n=\gamma_n=1$, then the above iteration scheme becomes $u_{n+1}=Tu_n$, which converges to the unique fixed point of T, as shown in Theorem 2.1.

(ii) If $\beta_n=\gamma_n<1$, then the above iteration scheme becomes

$$u_{n+1}=(1-\gamma_n)u_n+\gamma_n Tu_n, \text{ with } \sum \gamma_n=\infty$$

which becomes the Mann iteration scheme.

(iii) If $\beta_n-\gamma_n=\alpha_n$, then the above iteration scheme becomes

$$\begin{aligned} u_{n+1} &= (1-\beta_n)u_n+\beta_n Tv_n \\ v_n &= (1-\alpha_n)u_n+\alpha_n Tu_n, \ \sum \beta_n\alpha_n=\infty, \ \lim_{n\to\infty}\alpha_n=0 \end{aligned}$$

which is the Ishikawa iteration.

Deduction 2.5 *Let (T_1,T_2) satisfy the weak $(k,k^/)$ contractive type condition with $k+k^/\leqslant 1$. Then the Mann iterative scheme:*

$$u_0\in D_{X\otimes Y},\ u_{n+1}=(1-\alpha_n)u_n+\alpha_n Tu_n;\ n\geqslant 0$$

where $0\leqslant\alpha_n\leqslant 1$, $\sum\alpha_n=\infty$, converges to the unique fixed point u of T in $D_{X\otimes Y}$ (Here, $D_{X\otimes Y}$ denotes the closed and convex subset of $X\otimes Y$ which is bounded by M_1M_2).

Deduction 2.6 *If (T_1,T_2) satisfies the weak $(k,k^/)$ contractive type condition with $k+k^/\leqslant 1$, then the Ishikawa iteration scheme:*

$$\begin{aligned} u_0\in D_{X\otimes_\gamma Y},\ u_{n+1} &= (1-\alpha_n)u_n+\alpha_n Tv_n \\ v_n &= (1-\beta_n)u_n+\beta_n Tu_n,\ n\geqslant 0 \end{aligned}$$

where $0\leqslant\alpha_n,\beta_n\leqslant 1$ and $\sum\alpha_n\beta_n=\infty$, converges to the unique fixed point u of T in $D_{X\otimes_\gamma Y}$.

Deduction 2.7 *If (T_1,T_2) satisfies the weak $(k,k^/)$ contractive type condition with $k+k^/\leqslant 1$, then the Kirk iteration scheme:*

$$u_0\in D_{X\otimes_\gamma Y},\ u_{n+1}=S^n u_0,\ \text{where } S=\sum_{i=0}^{k}\alpha_i T^i$$

where each $\alpha_i\geqslant 0$, $\sum_{i=0}^{k}\alpha_i=1$, and $\alpha_1\neq 0$ converges to the unique fixed point u of T in $D_{X\otimes_\gamma Y}$.

Next, we define the following iteration scheme for the operator T on $X \otimes_\gamma Y$:

Let $x_0 \otimes y_0 \in D_{X\otimes_\gamma Y}$. Then $x_0 \otimes y_0$ is a limit point of the sequence, say, $\{\sum_p x_{p_{j_0}} \otimes y_{p_{j_0}}\}_{j_0}$. Now,

$$\begin{aligned}
&\|T_1(\sum_p x_{p_{j_0}} \otimes y_{pj_0}) - T_1(\sum_p x_{p_{k_0}} \otimes y_{p_{k_0}})\| \\
&\leqslant \frac{1}{M_2}[k\|\sum_p x_{p_{j_0}} \otimes y_{p_{j_0}} - \sum_p x_{pk_0} \otimes y_{p_{k_0}}\| - \\
&\quad \psi_1(k\|\sum_p x_{p_{j_0}} \otimes y_{p_{j_0}} - \sum_p x_{p_{k_0}} \otimes y_{p_{k_0}}\|)] \\
&\leqslant \frac{k}{M_2}\|\sum_p x_{p_{j_0}} \otimes y_{p_{j_0}} - \sum_p x_{p_{k_0}} \otimes y_{p_{k_0}}\|. \qquad (5)
\end{aligned}$$

So, $\{T_1(\sum_p x_{p_{j_0}} \otimes y_{p_{j_0}})\}_{j_0}$ is a Cauchy sequence in D_X and so it converges to some $\hat{x}_0 \in X$. Similarly, $\{T_2(\sum_p x_{p_{j_0}} \otimes y_{p_{j_0}})\}_{j_0}$ is a Cauchy sequence in D_Y and so it converges to some $\hat{y}_0 \in Y$. Now,

$$\begin{aligned}
\hat{x}_0 \otimes \hat{y}_0 &= \lim_{j_0\to\infty} T_1(\sum_p x_{p_{j_0}} \otimes y_{p_{j_0}}) \otimes \lim_{j_0\to\infty} T_2(\sum_p x_{p_{j_0}} \otimes y_{p_{j_0}}) \\
&= \lim_{j_0\to\infty} T(\sum_p x_{p_{j_0}} \otimes y_{p_{j_0}}) = T(x_0 \otimes y_0) \text{ (by continuity of } T). \qquad (6)
\end{aligned}$$

For $\alpha_1 + \alpha_2 < 1$, $\alpha_1, \alpha_2 \geqslant 0$, we take

$$\begin{aligned}
x_1 \otimes y_1 =&(1-(\alpha_1+\alpha_2))x_0 \otimes y_0 + \alpha_1 T_1(\sum_p x_{p_{j_0}} \otimes y_{p_{j_0}}) \otimes \hat{y}_0 + \\
&\alpha_2 \hat{x}_0 \otimes T_2(\sum_p x_{p_{j_0}} \otimes y_{p_{j_0}}), \\
x_2 \otimes y_2 =&(1-(\alpha_1+\alpha_2))x_1 \otimes y_1 + \alpha_1 T_1(\sum_p x_{p_{j_0}} \otimes y_{p_{j_0}}) \otimes \hat{y}_1 + \\
&\alpha_2 \hat{x}_1 \otimes T_2(\sum_p x_{p_{j_0}} \otimes y_{p_{j_0}})
\end{aligned}$$

(where $\{T_1(\sum_p x_{p_{j_1}} \otimes y_{p_{j_1}})\}_{j_1}$ converges to $\hat{x}_1 \in X$ and $\{T_2(\sum_p x_{p_{j_1}} \otimes y_{p_{j_1}})\}_{j_1}$ converges to $\hat{y}_1 \in Y$). Proceeding in this way,

$$x_{n+1} \otimes y_{n+1} = (1-(\alpha_1+\alpha_2))x_n \otimes y_n + \alpha_1 T_1(\sum_p x_{p_{j_n}} \otimes y_{p_{j_n}}) \otimes \hat{y}_n + \alpha_2 \hat{x}_n \otimes T_2(\sum_p x_{p_{j_n}} \otimes y_{p_{j_n}}).$$

Theorem 2.8 *The above iteration scheme converges to the unique fixed point u (derived in Theorem 2.1) of T in $D_{X\otimes_\gamma Y}$ with $k + k^{/} \leqslant 1$.*

Proof We have,

$$\begin{aligned}
&\|x_{n+1} \otimes y_{n+1} - u\| \\
&= \|(1-(\alpha_1+\alpha_2))x_n \otimes y_n + \alpha_1 T_1(\sum_p x_{p_{j_n}} \otimes y_{p_{j_n}}) \otimes \hat{y}_n + \\
&\quad \alpha_2 \hat{x}_n \otimes T_2(\sum_p x_{p_{j_n}} \otimes y_{p_{j_n}}) - u\|
\end{aligned}$$

$$\leqslant |(1-(\alpha_1+\alpha_2))|\|x_n\otimes y_n-u\|+\alpha_1\|T_1(\sum_p x_{p_{j_n}}\otimes y_{p_{j_n}})\otimes \hat{y}_n-u\|+$$
$$\alpha_2\|\hat{x}_n\otimes T_2(\sum_p x_{p_{j_n}}\otimes y_{p_{j_n}})-u\|$$
$$\leqslant (1-(\alpha_1+\alpha_2))\|x_n\otimes y_n-u\|+\alpha_1\{\|T_1(\sum_p x_{p_{j_n}}\otimes y_{p_{j_n}})\otimes \hat{y}_n-$$
$$T_1(\sum_p x_{p_{j_n}}\otimes y_{p_{j_n}})\otimes T_2(\sum_p x_{p_{j_n}}\otimes y_{p_{j_n}})\|+$$
$$\|T_1(\sum_p x_{p_{j_n}}\otimes y_{p_{j_n}})\otimes T_2(\sum_p x_{p_{j_n}}\otimes y_{p_{j_n}})-u\|\}+$$
$$\alpha_2\{\|\hat{x}_n\otimes T_2(\sum_p x_{p_{j_n}}\otimes y_{p_{j_n}})-T_1(\sum_p x_{p_{j_n}}\otimes y_{p_{j_n}})\otimes T_2(\sum_p x_{p_{j_n}}\otimes y_{p_{j_n}})\|+$$
$$\|T_1(\sum_p x_{p_{j_n}}\otimes y_{p_{j_n}})\otimes T_2(\sum_p x_{p_{j_n}}\otimes y_{p_{j_n}})-u\|\}.$$

Taking limit as $j_n\to\infty$ and using (6) for $x_n\otimes y_n$, we get,

$$\begin{aligned}\|x_{n+1}\otimes y_{n+1}-u\| \leqslant&(1-(\alpha_1+\alpha_2))\|x_n\otimes y_n-u\|+\\ &\alpha_1\|T(x_n\otimes y_n)-u\|+\alpha_2\|T(x_n\otimes y_n)-u\|\\ =&(1-(\alpha_1+\alpha_2))\|x_n\otimes y_n-u\|+\\ &(\alpha_1+\alpha_2)\|T(x_n\otimes y_n)-Tu\|\\ \leqslant&(1-(\alpha_1+\alpha_2))\|x_n\otimes y_n-u\|+(\alpha_1+\alpha_2)[\|x_n\otimes y_n-u\|-\\ &\{\psi_1(k\|x_n\otimes y_n-u\|)+\psi_2(k^/\|x_n\otimes y_n-u\|)\}]\\ \leqslant&(1-(\alpha_1+\alpha_2))\|x_n\otimes y_n-u\|+(\alpha_1+\alpha_2)[\|x_n\otimes y_n-u\|\\ =&\|x_n\otimes y_n-u\|.\end{aligned}$$

Now, proceeding as in Theorem 2.3, we can show that the iteration scheme converges to u in $D_{X\otimes_\gamma Y}$. □

Theorem 2.9 *Let $C_1=\{T_{1_i}|T_{1_i}:D_{X\otimes_\gamma Y}\to D_{X\otimes_\gamma e_2}, i=1,2,\ldots,N\}$ and $C_2=\{T_{2_i}|T_{2_i}:D_{X\otimes_\gamma Y}\to D_{e_1\otimes_\gamma Y}, i=1,2,\ldots,N\}$ be two families of mappings where (T_{1_i},T_{2_i}), $i=1,2,\ldots,N$ satisfies weak $(k,k^/)$ contractive type condition with $k+k^/\leqslant 1$. Let $T_i(u)=T_{1_i}(u)T_{2_i}(u)$, $u\in D_{X\otimes_\gamma Y}$. If each of $\{T_{1_i}^n u\}_n$ and $\{T_{2_i}^n u\}_n$ has a sub-sequence that converges to p and q, respectively, then pq is the unique fixed point for the operators T_i, $i=1,2,\ldots,N$. Also, $\{T_i^n u\}_n$ converges to pq (Where, e_1 and e_2 are the unit elements of X and Y, respectively).*

Proof Let $\lim_{j\to\infty}T_{1_i}^{n_j}u=p$ $(\in D_{X\otimes_\gamma e_2})$ and $\lim_{j\to\infty}T_{2_i}^{m_j}u=q$ $(\in D_{e_1\otimes_\gamma Y})$. We take $r_j=\max(n_j,m_j)$. Then $\lim_{j\to\infty}T_i^{r_j}u=\lim_{j\to\infty}T_{1_i}^{r_j}uT_{2_i}^{r_j}u=pq$ $(\in D_{X\otimes_\gamma Y})$.

$$\begin{aligned}\|T_i(u)-T_i(v)\| &= \|T_{1_i}uT_{2_i}u-T_{1_i}vT_{2_i}v\|\\ &\leqslant \|T_{1_i}u-T_{1_i}v\|\cdot\|T_{2_i}u\|+\|T_{1_i}v\|\cdot\|T_{2_i}u-T_{2_i}v\|\\ &\leqslant (k+k^/)\|u-v\|-\psi_1(k\|u-v\|)-\psi_2(k^/\|u-v\|)\\ &\leqslant \|u-v\|-\{\psi_1(k\|u-v\|)+\psi_2(k^/\|u-v\|)\}\leqslant\|u-v\|.\end{aligned}$$

Now,

$$\begin{aligned}\|T_i(pq)-(pq)\| \leqslant & \|T_i(pq)-T_i^{r_{j+1}}(u)\|+\|T_i^{r_{j+1}}(u)-(pq)\| \\ \leqslant & \|pq-T_i^{r_j}(u)\|-[\psi_1(k\|pq-T_i^{r_j}(u)\|)+ \\ & \psi_2(k^/\|pq-T_i^{r_j}(u)\|)]+\|T_i^{r_{j+1}}(u)-(pq)\| \\ \to & 0 \text{ as } j\to\infty.\end{aligned}$$

Thus, $T_i(pq)=pq$. The uniqueness of pq comes immediately. Also,

$$\|pq-T_i^n(u)\| \leqslant \|pq-T_i^{n-1}(u)\|-[\psi_1(k\|pq-T_i^{n-1}(u)\|)+\psi_2(k^/\|pq-T_i^{n-1}(u)\|)]$$

Then it can be easily shown that $\lim_{n\to\infty} T_i^n u = pq$. □

Now, we want to study the converse of Theorem 2.1. In the following theorem, we construct a pair of mappings satisfying weak $(k,k^/)$ contractive type condition, using a weakly contracive mapping on $D_{X\otimes_\gamma Y}$.

Theorem 2.10 *Let $T: D_{X\otimes_\gamma Y}\to D_{X\otimes_\gamma Y}$ be a weakly contractive mapping, where $D_{X\otimes_\gamma Y}$ is bounded by some M $(0<M\leqslant 1)$. Let $\sum_i a_i\otimes b_i$ be the unique fixed point of T with $\sum_i\|a_i\|\leqslant M$ and $\sum_i\|b_i\|\leqslant M$. Then we can construct $T_1: D_{X\otimes_\gamma Y}\to D_X$ and $T_2: D_{X\otimes_\gamma Y}\to D_Y$ such that (T_1,T_2) satisfies weak $(k,k^/)$ contractive type condition. Moreover using these T_1 and T_2, we obtain two self maps S_1 and S_2 having unique fixed points in D_X and D_Y, respectively.*

Proof Let $f\in X^*$ (the dual space of X) and $g\in Y^*$ (the dual space of Y) such that $\|f\|=1$ and $\|g\|=1$. We define the linear maps

$$F_1: D_{X\otimes_\gamma Y}\to D_X \text{ by } F_1(\sum_i x_i\otimes y_i)=\sum_i x_i g(y_i),$$
$$F_2: D_{X\otimes_\gamma Y}\to D_Y \text{ by } F_2(\sum_i x_i\otimes y_i)=\sum_i f(x_i)y_i.$$

Then $\|F_1\|\leqslant\|g\|=1$ and $\|F_2\|\leqslant\|f\|=1$. Let

$$T_1: D_{X\otimes_\gamma Y}\to D_X \text{ s.t. } T_1(\sum_i x_i\otimes y_i)=F_1(T(k\sum_i x_i\otimes y_i)), \text{ where } 0<k<1,$$
$$T_2: D_{X\otimes_\gamma Y}\to D_Y \text{ s.t. } T_2(\sum_i x_i\otimes y_i)=F_2(T(k^/\sum_i x_i\otimes y_i)), \text{ where } 0<k^/<1.$$

For $u\in D_{X\otimes_\gamma Y}$, $\|T_1u\|=\|F_1(T(ku))\|\leqslant\|F_1\|\|T(ku)\|\leqslant M$. Similarly, $\|T_2u\|\leqslant M$. Now, for $u,v\in D_{X\otimes_\gamma Y}$,

$$\begin{aligned}\|T_1u-T_1v\| &= \|F_1(T(ku))-F_1(T(kv))\|\leqslant\|F_1\|\cdot\|T(ku)-T(kv)\| \\ &\leqslant \|ku-kv\|-\psi(\|ku-kv\|) \quad (T \text{ is weakly contractive}) \\ &\leqslant \frac{1}{M}[k\|u-v\|-\psi(k\|u-v\|)].\end{aligned}$$

Also, $\|T_2u-T_2v\|\leqslant\frac{1}{M}[k^/\|u-v\|-\psi(k^/\|u-v\|)]$. Thus, (T_1,T_2) satisfies weak $(k,k^/)$ contractive type condition. Now we define

$$S_1: D_X\to D_X \text{ s.t. } S_1(x)=T_1(\sum_i x\otimes b_i) \text{ and } S_2: D_Y\to D_Y \text{ s.t. } S_1(y)=T_1(\sum_i a_i\otimes y)$$

$$\begin{aligned}\|S_1x - S_1x^{/}\| &= \|T_1(\sum_i x\otimes b_i) - T_1(\sum_i x^{/}\otimes b_i)\| \\ &\leqslant \frac{1}{M}[k\|\sum_i x\otimes b_i - \sum_i x^{/}\otimes b_i\| - \psi(k\|\sum_i x\otimes b_i - \sum_i x^{/}\otimes b_i\|)] \\ &\leqslant \frac{k}{M}\|\sum_i x\otimes b_i - \sum_i x^{/}\otimes b_i\| \\ &\leqslant \frac{k}{M}\|x - x^{/}\|\sum_i \|b_i\| \leqslant k\|x - x^{/}\|\end{aligned}$$

which is contraction, so it has unique fixed point. Similarly

$$\|S_2y - S_2y^{/}\| \leqslant k^{/}\|y - y^{/}\|,$$

hence S_2 has also unique fixed point. □

3. Concluding remark

In Theorem 2.1, we obtain the fixed point of T using the condition $k + k^{/} \leqslant 1$. Now, the following question arises: Can we make analogous study if the above condition is omitted?

In [4], Choudhury defined weak C-contraction on a metric space (X, d). A mapping T on a Banach space X is said to be weak C-contraction if $\forall x, y \in X$,

$$\|Tx - Ty\| \leqslant \frac{1}{2}[\|x - Ty\| + \|y - Tx\|] - \psi(\|x - Ty\|, \|y - Tx\|)$$

where $\psi : [0,\infty)^2 \longrightarrow [0,\infty)$ is a continuous mapping such that $\psi(x, y) = 0$ if and only if $x = y = 0$. With F_1 and F_2 as defined in Theorem 2.10, let T_i $(i = 1, 2)$ satisfy the condition

$$\|T_iu - T_iv\| \leqslant \frac{1}{2}[\|F_iu - T_iv\| + \|F_iv - T_iu\|] - \psi(\|F_iu - T_iv\|, \|F_iv - T_iu\|); u, v \in D_{X\otimes Y}.$$

Now, we can rise the following problem: Can we obtain some fixed point theorems for the mapping T derived by the pair (T_1, T_2) satisfying the above condition?

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