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Singular Integral Operators on New BMO and Lipschitz Spaces of Homogeneous Type

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Abstract Let (X,d,μ) be a space of homogeneous type, $\mathrm{BMO}_A(X)$ and $\mathrm{Lip}_A(\beta,X)$ be the space of BMO type, lipschitz type associated with an approximation to the identity $\{A_t\}_{t>0}$ and introduced by Duong, Yan and Tang, respectively. Assuming that T is a bounded linear operator on $L^2(X)$, we find the sufficient condition on the kernel of T so that T is bounded from $\mathrm{BMO}(X)$ to $\mathrm{BMO}_A(X)$ and from $\mathrm{Lip}(\beta,X)$ to $\mathrm{Lip}_A(\beta,X)$. As an application, the boundedness of Calderón-Zygmund operators with nonsmooth kernels on $\mathrm{BMO}(\mathbb{R}^n)$ and $\mathrm{Lip}(\beta,\mathbb{R}^n)$ are also obtained.

Keywords singular integral operators; BMO spaces; Lipschitz spaces; heat kernel; spaces of homogeneous type

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1. Introduction

Let (X,d,μ) be the space of homogeneous type, equipped with a metric d and a measure μ. Let T be a bounded linear operator on $L^2(X)$ with kernel K such that for every $f\in L^\infty(X)$ with bounded support,

$$T(f)(x)=\int_X K(x,y)f(y)\mathrm{d}\mu(y)$$

for μ-almost all $x\notin \operatorname{supp} f$.

In [1], Duong and Yan introduced a class of new spaces-$\mathrm{BMO}_A(X)$-the space of BMO type associated with an approximation to the identity $\{A_t\}_{t>0}$. And Tang [2] defined the corresponding new Lipschitz spaces-$\mathrm{Lip}_A(\beta,X)$ associated with an approximation to the identity $\{A_t\}_{t>0}$.

As it is well known, the size condition

$$|K(x,y)|\le \frac{D}{\mu(B(x,d(x,y)))}$$

and regularity condition

$$|K(x,y)-K(z,y)|+|K(y,x)-K(y,z)|\le \frac{D}{\mu(B(x,d(x,y)))}\frac{d(x,z)^\epsilon}{d(x,y)^\epsilon},$$

whenever $d(x,z)\le \frac12 d(x,y)$, are the sufficient condition for the Caldeón-Zygmund operators T to be bounded on bounded mean oscillation space $\mathrm{BMO}(X)$ and Lipschitz space $\mathrm{Lip}(\beta,X)$ (see [3]).

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Now a natural question is whether T can be extended to be a bounded operator on $\mathrm{BMO}_A(X)$ and $\mathrm{Lip}_A(\beta, X)$. In this paper, we get that if there exists a generalized approximation to the identity $\{A_t\}_{t>0}$ such that the operators A_tT have associated kernels $K_t(x,y)$ and there exist positive constants c and C such that

$$|K_t(x,y)-K(x,y)| \le C\frac{1}{\mu(B(x,d(x,y)))}\frac{t^{\epsilon/m}}{d(x,y)^{\epsilon}} \text{ for all } y\in X \text{ and } t>0 \tag{1.1}$$

when $d(x,y)>ct^{1/m}$, then when $T(1)=0$, T is bounded operator from $\mathrm{BMO}(X)$ to $\mathrm{BMO}_A(X)$ and from $\mathrm{Lip}(\beta,X)$ to $\mathrm{Lip}_A(\beta,X)$.

Let $\Delta=\Delta_x=\sum_{k=1}^{n}\partial^2/\partial_{x_k^2}$ be the classic Laplace operator in the spatial variable $x=(x_1,\ldots,x_n)\in\mathbb{R}^n$. When $A=\Delta$ or $A=\sqrt{\Delta}$, as pointed in [1,2], $\mathrm{BMO}_A(\mathbb{R}^n)$ coincides with $\mathrm{BMO}(\mathbb{R}^n)$ and $\mathrm{Lip}_A(\beta,\mathbb{R}^n)$ coincides with $\mathrm{Lip}(\beta,\mathbb{R}^n)$, with equivalent norms. By Proposition 2 in [4], we know that the following regularity condition of Calderón-Zygmund operators

$$|K(x,y-K(z,y))| \le C\frac{|x-z|^{\epsilon}}{|x-y|^{n+\epsilon}} \text{ when } |x-z|\le\frac{1}{2}|x-y| \tag{1.2}$$

is stronger than the condition (1.1). As a corollary, we get that Calderón-Zygmund operators are bounded on $\mathrm{BMO}(\mathbb{R}^n)$ and $\mathrm{Lip}(\beta,\mathbb{R}^n)$.

On the other hand, to study the mapping properties for the commutator of Calderón, Duong, Grafakos and Yan et al. [5,6] defined a class of singular integral operators via the generalized approximation to the identity, which are called singular integral operator with nonsmooth kernel. Duong, Grafakos, Yan et al. [5] replaced the condition (1.2) by weaker regularity conditions on the kernel K. The weaker regularity conditions are as follows:

Assume that there exist operators $\{B_t\}_{t>0}$ with kernels $b_t(x,y)$. Let

$$K_t^{(0)}(x,y)=\int_{\mathbb{R}^n}K(z,y)b_t(x,z)\mathrm{d}z.$$

We assume that the kernels $K_t^{(0)}(x,y)$ satisfy the following estimates, there exist a function $\phi\in C(\mathbb{R})$ with $\operatorname{supp}\phi\subset[-1,1]$ and constants $\epsilon\in(0,1/n)$ and D such that

$$|K(x,y)-K_t^{(0)}(x',y)| \le \frac{D\phi(\frac{|x-y|}{t^{1/m}})}{|x-y|^n}+\frac{At^{n\epsilon/m}}{|x-y|^{n+n\epsilon}},$$

and

$$|K_t^{(0)}(x,y)-K_t^{(0)}(x',y)| \le \frac{At^{n\epsilon/m}}{|x-y|^{n+n\epsilon}},$$

whenever $2|x-x'|\le t^{1/m}\le\frac{1}{2}|x-y|$.

As a corollary, we also get the boundedness of singular integral operators with nonsmooth kernels on $\mathrm{BMO}(\mathbb{R}^n)$ and $\mathrm{Lip}(\beta,\mathbb{R}^n)$.

The paper is organized as follows. In Section 2, we give some necessary notion and lemmas about $\mathrm{BMO}(X)$, $\mathrm{Lip}(\beta,X)$, $\mathrm{BMO}_A(X)$ and $\mathrm{Lip}_A(\beta,X)$. In Section 3, we state our main results that T is bounded from $\mathrm{BMO}(X)$ to $\mathrm{BMO}_A(X)$ and from $\mathrm{Lip}(\beta,X)$ to $\mathrm{Lip}_A(\beta,X)$ and give their proof. As an application, we get the boundedness of Calderón-Zygmund operators with nonsmooth kernels T on $\mathrm{BMO}(\mathbb{R}^n)$ and $\mathrm{Lip}(\beta,\mathbb{R}^n)$ in Section 4. And we also give a proof which is different from that in Section 3.

Throughout the article, C always denotes a positive constant that may vary from line to line but remains independent of the main variables. We use $B(x,r)=\{y\in X: d(x,y)<r\}$ to denote a ball centered at x with radius r. For a ball $B\subset X$ and $\lambda>0$, we use λB to denote the ball concentric with B whose radius is λ times of $B's$ and $f_B=\mu(B)^{-1}\int_B f(x)\mathrm{d}\mu(x)$ denotes the integral value mean of f. As usual, $\mu(E)$ denotes the $\mu-$measure of a measurable set E in X and χ_E denotes the characteristic function of E. For $p\geq 1$, we denote by $p'=p/(p-1)$ the dual exponent of p.

2. Preliminaries

Let us first recall several definitions.

2.1. $\mathrm{BMO}(X)$ and $\mathrm{Lip}(\beta,X)$ of homogeneous type

A function d defined from $X\times X$ to $[0,\infty)$ is a quasi-measure if it satisfies the following:

(i) $d(x,y)\geq 0$ for all $x,y\in X$, $d(x,y)=0$ if and only if $x=y$.

(ii) $d(x,y)=d(y,x)$ for all $x,y\in X$.

(iii) There exists a constant $C\in[1,\infty)$ such that $d(x,y)\leq C\big(d(x,z)+d(z,y)\big)$, for all $x,y,z\in X$.

Definition 2.1 ([7, Chapter 3]) *A space of homogeneous type (X,d,μ) is a set X together with a quasi-metric d and a nonnegative Borel measure μ on X for all the associated with balls satisfying the doubling property $\mu(B(x,2r))\leq C\mu(B(x,r))<\infty$ for all $x\in X$ and all $r>0$, and the constant $C\geq 1$ independent of x and r.*

Note that the doubling property implies the following strong homogeneity property,

$$\mu(B(x,\lambda r))\leq C\lambda^n\mu(B(x,r)) \tag{2.1}$$

for some $c,n>0$ uniformly for all $x\in X$, where n denotes the homogeneous dimension of homogeneous space X. There also exist constant C and $N\in[0,n]$ such that

$$\mu(B(x,r))\leq C(1+\frac{d(x,y)}{r})^N\mu(B(y,r)) \tag{2.2}$$

uniformly for all $x,y\in X$ and $r>0$. The property (2.2) with $N=n$ is a direct consequence of triangle inequality of the metric d and the strong homogeneity property. In the cases of Euclidean spaces $\mathbb{R}^n$ and Lie groups of polynomial growth, N can be chosen to be 0.

Remark 2.2 ([8]) Let $X=\mathbb{R}^n$, $d(x,y)=\big(\sum_{j=1}^n|x_j-y_j|^2\big)^{1/2}$ and μ be the Lebesgue measure, (X,d,μ) is just the classic Euclidean space.

Definition 2.3 ([9]) *We say a locally integral function f is a* BMO *function on X, if there exists* *some constant C such that for any ball B, such that*

$$\frac{1}{\mu(B)}\int_B |f(x)-f_B|\mathrm{d}\mu(x)\le C<\infty. \tag{2.3}$$

We denote $\|f\|_{\mathrm{BMO}(X)}=\inf\{C:(2.3)\text{ holds}\}$. Two equivalent norms of $\|f\|_{\mathrm{BMO}(X)}$ are given by that

$$\|f\|_{\mathrm{BMO}(X)}=\sup_B\Big(\frac{1}{\mu(B)}\int_B |f(x)-f_B|^p\mathrm{d}\mu(x)\Big)^{1/p}$$

and

$$\|f\|_{\mathrm{BMO}(X)}=\sup_B\inf_{c\in\mathbb{C}}\Big(\frac{1}{\mu(B)}\int_B |f(x)-c|^q\mathrm{d}\mu(x)\Big)^{1/q}$$

for any $p,q\in[1,\infty)$.

Definition 2.4 ([10]) *Let $\beta\in(0,1/n)$. We say a locally integral function f is a $\mathrm{Lip}(\beta,X)$ function on X, if there exists some constant C such that for any ball B, such that*

$$\frac{|f(x)-f(y)|}{d(x,y)^\beta}\le C<\infty. \tag{2.4}$$

We denote $\|f\|_{\mathrm{Lip}(\beta,X)}=\inf\{C:(2.4)\text{ holds}\}$. Two equivalent norms of $\|f\|_{\mathrm{Lip}(\beta,X)}$ are given by

$$\|f\|_{\mathrm{Lip}(\beta,X)}=\sup_B\frac{1}{\mu(B)^\beta}\Big(\frac{1}{\mu(B)}\int_B |f(x)-f_B|^p\mathrm{d}\mu(x)\Big)^{1/p}$$

and

$$\|f\|_{\mathrm{Lip}(\beta,X)}=\sup_B\inf_{c\in\mathbb{C}}\Big(\frac{1}{\mu(B)^{1+\beta}}\int_B |f(x)-c|^q\mathrm{d}\mu(x)\Big)^{1/q}$$

for any $p,q\in[1,\infty)$.

Lemma 2.5 ([11]) *If $f\in\mathrm{BMO}(X)$ and $k>1$, then there exists a positive constant C independent of f,B,k such that, $|f_B-f_{kB}|\le C(1+\log k)\|f\|_{\mathrm{BMO}(X)}$.*

2.2. $\mathrm{BMO}_A(X)$ and $\mathrm{Lip}_A(\beta,X)$ of homogeneous type

Before introducing the new spaces of homogeneous type $\mathrm{BMO}_A(X)$ and $\mathrm{Lip}_A(\beta,X)$ which are defined in [1] and [2], we first work with a class of integral operators $\{A_t\}_{t>0}$, which play the role of the approximation to the identity [4]. We always assume that the operators A_t are given by kernels $a_t(x,y)$ in the sense that

$$A_tf(x):=\int_X a_t(x,y)f(y)\mathrm{d}\mu(y),$$

for all $f\in\cup_{p\in[1,\infty]}\mathrm{L}^p(X)$ and $x\in X$, and the kernels $a_t(x,y)$ satisfy the following conditions

$$|a_t(x,y)|\le h_t(x,y), \tag{2.5}$$

for all $x,y\in X$, where $h_t(x,y)$ is a function satisfying

$$h_t(x,y):=\frac{1}{\mu(B(x,t^{1/m}))}g(\frac{d(x,y)^m}{t}), \tag{2.6}$$

in which m is a positive fixed constant and g is a positive, bounded, decreasing function satisfying

$$\lim_{r\to\infty} r^{n+\gamma+\eta}\varphi(r^m)=0, \tag{2.7}$$

for some $\epsilon > 0$, $\eta > 0$ and $\gamma \geq 0$.

In order to define the $\mathrm{BMO}_A(X)$-the spaces of BMO type associated with an "approximation to the identity" $\{A_t\}_{t>0}$, Duong and Yan [1] introduced the a class function of type (x_0, α). Let ϵ and γ be the constants in (2.6) and $\alpha \in (0, \epsilon)$. A function $f \in L^1_{\mathrm{loc}}(X)$ is said to be a function of type (x_0, α) centered at $x_0 \in X$ if f satisfies

$$\int_X \frac{|f(x)|}{(1 + d(x_0, x))^{\gamma+\alpha} \mu(B(x_0, 1 + d(x_0, x)))} \mathrm{d}\mu(x) \leq C < \infty. \tag{2.8}$$

We denote by $\mathcal{M}_(x_0, \alpha)$ the collection of all functions of type (x_0, α). The norm of f in $\mathcal{M}_(x_0, \alpha)$ is defined by

$$\|f\|_{\mathcal{M}_{(x_0,\alpha)}} = \inf\{C \geq 0 : (2.8) \text{ holds}\}.$$

One denotes

$$\mathcal{M} = \bigcup_{x_0 \in X} \bigcup_{\alpha: 0<\alpha<\epsilon} \mathcal{M}_{(x_0,\alpha)}.$$

Definition 2.6 ([1]) *Let $\gamma = 2N$ in (2.5), where N is the power appearing in property (2.2). For $f \in \mathcal{M}$, we say f is in $\mathrm{BMO}_A(X)$, the spaces of functions of bounded mean oscillation associated with a generalized approximation to the identity $\{A_t\}_{t>0}$, if there exists some constant C such that for any ball B with radius r_B,*

$$\frac{1}{\mu(B)} \int_B |f(x) - A_{t_B}(f)(x)| \mathrm{d}\mu(x) \leq C, \tag{2.9}$$

where $t_B = r_B^m$. We denote $\|f\|_{\mathrm{BMO}_A(X)} = \inf\{C : (2.9) \text{ holds}\}$.

Later, Tang [2] defined the $\mathrm{Lip}_A(\beta, X)$-the spaces of Lipschitz type associated with an "approximation to the identity" $\{A_t\}_{t>0}$ as follows.

Definition 2.7 ([2]) *Let $\gamma = 2N + (n + N)\beta$ in (2.5), where $\beta \in (0, 1/n)$ and N is the power appearing in property (2.2). For $f \in \mathcal{M}$, we say f is in $\mathrm{Lip}_A(\beta, X)$, the spaces of functions of Lipschitz type associated with a generalized approximation to the identity $\{A_t\}_{t>0}$, if there exists some constant C such that for any ball B with radius r_B,*

$$\frac{1}{\mu(B)^{1+\beta}} \int_B |f(x) - A_{t_B}(f)(x)| \mathrm{d}\mu(x) \leq C, \tag{2.10}$$

where $t_B = r_B^m$. We denote $\|f\|_{\mathrm{Lip}_A(\beta,X)} = \inf\{C : (2.10) \text{ holds}\}$.

Remark 2.8 ([1]) Let $\Delta = \Delta_x = \sum_{k=1}^n \partial^2/\partial_{x_k^2}$ be the classic Laplace operator in the spatial variable $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. When $A = \Delta$ or $A = \sqrt{\Delta}$, $\mathrm{BMO}_A(\mathbb{R}^n)$ coincides with $\mathrm{BMO}(\mathbb{R}^n)$, with equivalent norms.

With the similar methods in [1], we can also get that when $A = \Delta$ or $A = \sqrt{\Delta}$, $\mathrm{Lip}_A(\beta, \mathbb{R}^n)$ coincides with $\mathrm{Lip}(\beta, \mathbb{R}^n)$, with equivalent norms.

Lemma 2.9 ([1]) *For each $p \in [1, \infty]$ and $t > 0$, there is a constant $C > 0$ such that $A_t f(x) \leq CM(f)(x)$ for all $f \in L^p(X)$, μ-a.e., where $M(f)$ is Hardy-Littlewood maximal function.*

2.3. Singular integral operators of homogeneous type

We first recall the definition of the Calderón-Zygmund operators [7]. Let (X, d, μ) be a space of homogeneous type. Let $C_c^\tau(X)$, $\tau > 0$, be the spaces of all continuous functions on X with compact support such that $\|f\|_{C_c^\tau(X)} = \sup_{x \neq y} \frac{|f(x)-f(y)|}{|x-y|^\tau} < \infty$. Then we define the homogeneous Hölder space $\dot{C}_c^\tau(X)$ as the closure for the $C_c^\tau(X)$ norm of functions in $C_c^\theta(X)$ where $\tau < \theta$. Let $(\dot{C}_c^\tau(X))'$ be dual space of $\dot{C}_c^\tau(X)$. A continuous linear operator $T : \dot{C}_c^\tau(X) \to (\dot{C}_c^\tau(X))'$, is said to be a singular integral operator if T is associated to a kernel K such that

$$\langle T(f), g\rangle = \int_X \int_X K(x,y) f(y) g(x) \mathrm{d}\mu(y) \mathrm{d}\mu(x)$$

for all f and $g \in \dot{C}_0^\eta(X)$ with $\mathrm{supp}(f) \cap \mathrm{supp}(g) = \varnothing$.

Definition 2.10 ([7]) *We say an operator T is a Calderón-Zygmund singular operator* $\mathrm{CZO}(D, \epsilon)$ *with kernel K if*

(1) T is bounded linear operator on $L^2(X)$ with a kernel K.

(2) There exist some constant $D > 0$ and $\epsilon \in (0, 1/n]$ such that the kernel K satisfies the size condition

$$|K(x,y)| \le \frac{D}{\mu(B(x, d(x,y)))}$$

and regularity condition

$$|K(x,y) - K(z,y)| + |K(y,x) - K(y,z)| \le \frac{D}{\mu(B(x, d(x,y)))} \frac{d(x,z)^\epsilon}{d(x,y)^\epsilon}, \tag{2.11}$$

whenever $d(x,z) \le \frac{1}{2} d(x,y)$.

In this paper, we will consider following singular integral operators.

Definition 2.11 ([1,4]) *Let (X, d, μ) be a space of homogeneous type. The singular integral operators are defined in the following way:*

(i) T is bounded linear operator on $L^2(X)$ with kernel K.

(ii) There exists a generalized approximation to the identity $\{A_t\}_{t>0}$ satisfying (2.5) and (2.6) such that the operators $A_t T$ have associated kernels $K_t(x,y)$ and there exists positive constant C such that

$$|K(x,y) - K_t(x,y)| \le C \frac{1}{\mu(B(x, d(x,y)))} \frac{t_B^{\epsilon/m}}{d(x,y)^\epsilon}, \tag{2.12}$$

where $\epsilon \in (0, 1/n)$.

Following Proposition 2 of [4], we construct $a_t(x,y)$ with the following properties

$$a_t(x,y) = 0, \text{ when } |x-y| \ge C t^{1/m} \tag{2.13}$$

and

$$\int_X a_t(x,y) \mathrm{d}\mu(x) = 1 \tag{2.14}$$

for all $y \in X$ and $t > 0$. This can be achieved by choosing

$$a_t(x,y) = \frac{1}{\mu(B(y, t^{1/m}))} \chi_{B(y, t^{1/m})}(x).$$

Let B_t be approximations to the identity which are represented by kernels $b_t(x,y)$ satisfying (2.6), (2.13) and (2.14). Following Proposition 2 in [4], we know that the condition (2.12) is weaker than the condition (2.11).

3. Boundedness of singular integral operators on space of homogeneous type

Our main results are stated as follows.

Theorem 3.1 *Assume that T is an operator with kernel K as in Definition 2.11 and $T(1)=0$. Then one has T is a bounded operator from $\mathrm{BMO}(X)$ to $\mathrm{BMO}_A(X)$.*

Theorem 3.2 *Assume that T is an operator with kernel K as in Definition 2.11, $\beta \in (0,\epsilon)$ and $T(1)=0$. Then one has T is a bounded operator from $\mathrm{Lip}(\beta,X)$ to $\mathrm{Lip}_A(\beta,X)$.*

Remark 3.3 Let T^* be the adjoint operator of T which is defined by $\langle T(b),g\rangle = \langle b, T^*(g)\rangle$. Let $\dot{C}^{\tau}_{c,0}(X) = \{f \in \dot{C}^{\tau}_{c}(X) : \int_X f(x)\mathrm{d}\mu(x) = 0\}$. Then the condition $T(1)=0$ should be understood as that there exists $g \in C^{\tau}_{c,0}(X)$ such that

$$0 = \int_X \int_X K(x,y)g(y)\mathrm{d}\mu(y)\mathrm{d}\mu(x).$$

First, we give the proof of Theorem 3.1.

Proof of Theorem 3.1 It suffices to prove that for any ball $B = B(y_0, r_B)$,

$$\frac{1}{\mu(B)} \int_B |T(f)(x) - A_{t_B}(T(f))(x)|\mathrm{d}\mu(x) \leq C\|f\|_{\mathrm{BMO}(X)}, \tag{3.1}$$

where $t_B = r_B^m$.

The condition $T(1)=0$ implies that

$$0 = \lambda T(1)(x) = \int_X K(x,y)\lambda \mathrm{d}\mu(y),$$

then we have

$$\begin{aligned} T(f)(x) &= \int_X K(x,y)f(y)\mathrm{d}\mu(y) = \int_X K(x,y)f(y)\mathrm{d}y - \int_{\mathbb{R}^n} K(x,y)\lambda\mathrm{d}\mu(y) \\ &= \int_X K(x,y)\big(f(y)-\lambda\big)\mathrm{d}\mu(y). \end{aligned}$$

Let $B^* = 5B(y_0,r)$ and $f-\lambda = (f-\lambda)^0 + (f-\lambda)^\infty := (f-\lambda)\chi_{B^*} + (f-\lambda)\chi_{(B^*)^c}$. Then $T(f)(x) - A_{t_B}\big(T(f)\big)(x)$ can be divided into three parts as follows:

$$\begin{aligned} T(f)(x) - A_{t_B}(T(f))(x) =& T(f-\lambda)(x) - A_{t_B}(T(f-\lambda))(x) \\ =& T\big((f-\lambda)^0\big)(x) - A_{t_B}\big(T((f-\lambda)^0)\big)(x) + \\ & \big(T - A_{t_B}T\big)((f-\lambda)^\infty)(x) \\ :=& J_1(x) + J_2(x) + J_3(x), \end{aligned}$$

where $\lambda := f_{B*}$.

Employing the assumption that the T is bounded on $\mathrm{L}^2(\mathrm{X})$, we get

$$\begin{aligned}&\frac{1}{\mu(B)}\int_B |J_1(x)|\mathrm{d}\mu(x)\\&\le \Big(\frac{1}{\mu(B)}\int_B |T\big((f-\lambda)^0\big)(x)|^2\mathrm{d}\mu(x)\Big)^{1/2}\le C\frac{1}{\mu(B)^{1/2}}\|(f-\lambda)^0\|_{L^2(X)}\\&\le C\frac{1}{\mu(B)^{1/2}}\mu(B^*)^{1/2}\|f\|_{\mathrm{BMO}^2(X)}\le C\|f\|_{\mathrm{BMO}(X)}.\end{aligned}$$

In terms of $J_2(x)$, by the assumption that T is bounded on $L^2(X)$ and Lemma 2.9, we have

$$\begin{aligned}&\frac{1}{\mu(B)}\int_B |J_2(x)|\mathrm{d}\mu(x)\\&\le \Big(\frac{1}{\mu(B)}\int_B \big(M\big(T((f-\lambda)^0)\big)(x)\big)^2\mathrm{d}\mu(x)\Big)^{1/2}\\&\le \frac{1}{\mu(B)^{1/2}}\|(T((f-\lambda)^0)\|_{L^2(X)}\le C\|f\|_{\mathrm{BMO}(X)}.\end{aligned}$$

By $x\in B(y_0,r)$ and $y\in (B^*)^c$, we know $d(x,z)\le \frac{1}{2}d(y,x)$ and $d(y,x)\sim d(y,y_0)$. Then an application of assumption (ii) in Definition 2.11 leads to

$$\begin{aligned}&|\big(T-A_{t_B}T\big)((f-\lambda)^\infty)(x)|\\&\le \int_X |K(x,y)-K^t(x,y)||(f(y)-\lambda)^\infty|\mathrm{d}\mu(y)\\&\le \int_{X\setminus B^*}\frac{1}{\mu(B(x,d(y_0,y)))}\frac{Ct_B^{\epsilon/s}}{d(y_0,y)^\epsilon}|f(y)-\lambda|\mathrm{d}\mu(y).\end{aligned}$$

By applying Lemma 2.5, we get

$$\begin{aligned}&\int_{X\setminus B^*}\frac{1}{\mu(B(x,d(x_0,y)))}\frac{Ct_B^{\epsilon/m}}{d(y_0,y)^\epsilon}|f(y)-\lambda|\mathrm{d}\mu(y)\\&\le \sum_{j=1}^\infty\int_{5^{j+1}B\setminus 5^jB}\frac{1}{\mu(B(x,d(y_0,y)))}\frac{Ct_B^{\epsilon/m}}{d(y_0,y)^\epsilon}\big|f(y)-f_{2B}\big|\mathrm{d}\mu(y)\\&\le \sum_{j=1}^\infty\frac{1}{5^{(j-1)\epsilon}}\frac{1}{\mu(5^jB)}\int_{5^{j+1}B}(\big|f(y)-f_{5^{j+1}B}\big|+\big|f_{5^{j+1}B}-f_{5B}\big|)\mathrm{d}\mu(y)\\&\le C\sum_{j=1}^\infty\frac{1}{5^{(j-1)\epsilon}}(\|f\|_{\mathrm{BMO}(X)}+(1+j)\|f\|_{\mathrm{BMO}(X)})\\&\le C\|f\|_{\mathrm{BMO}(X)}\sum_{j=1}^\infty\frac{j+2}{5^{(j-1)\epsilon}}\le C\|f\|_{\mathrm{BMO}(X)}.\end{aligned}$$

The inequality above leads to us that

$$\frac{1}{\mu(B)}\int_B |J_3(x)|\mathrm{d}\mu(x)\le C\|f\|_{\mathrm{BMO}(X)}.$$

Combining the estimate above for $J_1(x)$, $J_2(x)$ and $J_3(x)$, we know

$$\frac{1}{\mu(B)}\int_B |T(f)(x)-A_{t_B}(T(f))(x)|\mathrm{d}\mu(x)$$

$$
\begin{aligned}
&\le \frac{1}{\mu(B)}\int_B |J_1(x)|\mathrm{d}\mu(x) + \frac{1}{\mu(B)}\int_B |J_2(x)|\mathrm{d}\mu(x) + \frac{1}{\mu(B)}\int_B |J_3(x)|\mathrm{d}\mu(x)\\
&\le C\|f\|_{\mathrm{BMO}(X)},
\end{aligned}
$$

which completes the proof of Theorem 3.1. □

Now we turn our attention to the proof of Theorem 3.2.

Proof of Theorem 3.2 By the condition $T(1) = 0$ we have

$$
T(f)(x) = \int_X K(x,y)f(y)\mathrm{d}\mu(y) = \int_X K(x,y)\big(f(y)-\lambda\big)\mathrm{d}\mu(y).
$$

Let $B = B(y_0, r)$ be any ball and $B^* = 5B(y_0, r)$. Then $T(f)(x) - A_{t_B}\big(T(f)\big)(x)$ can be divided into three parts as follows:

$$
\begin{aligned}
T(f)(x) - A_{t_B}(T(f))(x) =& T(f-\lambda)(x) - A_{t_B}(T(f-\lambda))(x)\\
=& T\big((f-\lambda)^0\big)(x) - A_{t_B}\big(T((f-\lambda)^0)\big)(x) +\\
& \big(T - A_{t_B}T\big)((f-\lambda)^\infty)(x)\\
:=& F_1(x) + F_2(x) + F_3(x),
\end{aligned}
$$

where $\lambda := f_{B*}$.

By the assumption that T is bounded on $L^2(X)$ and Lemma 2.9, we have

$$
\begin{aligned}
&\frac{1}{\mu(B)^\beta}\frac{1}{\mu(B)}\int_B |F_1(x)+F_2(x)|\mathrm{d}\mu(x)\\
&\le \frac{1}{\mu(B)^\beta}\frac{1}{\mu(B)}\int_B \big(M(T\big((f-\lambda)^0\big))(x) + M(T\big((f-\lambda)^0\big))(x)\big)\mathrm{d}\mu(x)\\
&\le \frac{1}{\mu(B)^\beta}\Big(\frac{1}{\mu(B)}\int_B \big(M(T\big((f-\lambda)^0\big))(x)\big)^2\mathrm{d}\mu(x)\Big)^{1/2}\\
&\le \frac{1}{\mu(B)^{\beta+1/2}}\|T\big((f-\lambda)^0\big)\|_{L^2(X)} \le \frac{1}{\mu(B)^{\beta+1/2}}\|(f-\lambda)^0\|_{L^2(X)}\\
&\le C\|f\|_{\mathrm{Lip}(\beta,X)}.
\end{aligned}
$$

By $x \in B(y_0, r)$, $z \in B^* = 5B$ and $y \in (B^*)^c$, we know $d(y,z) \le C(d(y,y_0) + d(y_0,z)) \le 2Cd(y,y_0)$. Then by the definition of Lischitz space $\mathrm{Lip}(\beta, X)$ we have

$$
\begin{aligned}
&|\big(T - A_{t_B}T\big)((f-\lambda)^\infty)(x)|\\
&\quad\le C\int_{X\setminus B^*}\frac{1}{\mu(B(x,d(y_0,y)))}\frac{Ct_B^{\epsilon/m}}{d(y_0,y)^\epsilon}|f(y)-\lambda|\mathrm{d}\mu(y)\\
&\quad\le C\int_{X\setminus B^*}\frac{1}{\mu(B(x,d(y_0,y)))}\frac{Ct_B^{\epsilon/m}}{d(y_0,y)^\alpha}\frac{1}{\mu(5B)}\int_{5B}\big|f(y)-f(z)\big|\mathrm{d}\mu(z)\mathrm{d}\mu(y)\\
&\quad\le C\|f\|_{\mathrm{Lip}(\beta,X)}\int_{X\setminus B^*}\frac{1}{\mu(B(x,d(x_0,y)))}\frac{Ct_B^{\epsilon/m}}{d(y_0,y)^\epsilon}\frac{1}{\mu(5B)}\int_{5B}d(y,z)^\beta\mathrm{d}\mu(z)\mathrm{d}\mu(y)\\
&\quad\le C\|f\|_{\mathrm{Lip}(\beta,X)}\int_{X\setminus B^*}\frac{1}{\mu(B(x,d(y_0,y)))}\frac{Ct_B^{\epsilon/m}}{d(y_0,y)^{\epsilon-\beta}}\mathrm{d}\mu(y)
\end{aligned}
$$

$$\leq Cr_B^\beta \sum_{j=1}^{\infty} \frac{1}{5^{j(\epsilon-\beta)}} \frac{1}{\mu(5^j B)} \int_{5^j B \setminus 5^{j+1} B} \mathrm{d}\mu(y)$$
$$\leq Cr_B^\beta \|f\|_{\mathrm{Lip}(\beta,X)}.$$

The inequality above leads to

$$\frac{1}{\mu(B)^{1+\beta}} \int_B |F_3(x)||\mathrm{d}\mu(x) \leq C\frac{1}{\mu(B)^\beta} r_B^\beta \|f\|_{\mathrm{Lip}(\beta,\mathrm{X})} \leq C\|f\|_{\mathrm{Lip}(\beta,\mathrm{X})}.$$

Combining the estimate above for $F_1(x)$, $F_2(x)$ and $F_3(x)$, we know

$$\frac{1}{\mu(B)^{1+\beta}} \int_B |T(f)(x) - A_{t_B}(T(f))(x)|\mathrm{d}\mu(x)$$
$$\leq \frac{1}{\mu(B)^{1+\beta}} \int_B |F_1(x) + F_2(x)|\mathrm{d}\mu(x) + \frac{1}{\mu(B)^{1+\beta}} \int_B |F_3(x)|\mathrm{d}\mu(x)$$
$$\leq C\|f\|_{\mathrm{Lip}(\beta,X)}.$$

This completes the proof of Theorem 3.2. $\square$

4. An application for singular operators with nonsmooth kernel

When $A = \Delta$ or $A = \sqrt{\Delta}$, by Remark 2.8, we get following classic result.

Corollary 4.1 *Assume that T is a Calderón-Zygmund operator* $\mathrm{CZO}(D,\epsilon)$.

(i) If $T(1) = 0$, then T is a bounded operator from $\mathrm{BMO}(\mathbb{R}^n)$ *to* $\mathrm{BMO}(\mathbb{R}^n)$.

(ii) If $T(1) = 0$ and $\beta \in (0,\epsilon)$, then T is a bounded operator from $\mathrm{Lip}(\beta,\mathbb{R}^n)$ *to* $\mathrm{Lip}(\beta,\mathbb{R}^n)$.

We should point out that the condition (2.11) can be replaced by weaker regularity conditions on the kernel K. To study the mapping properties for the commutator of Calderon, Duong, Grafakos and Yan et al. [5,6] introduced a class of singular integral operators via the generalized approximation to the identity, which are called singular integral operators with nonsmooth kernel. Duong, Grafakos, Yan et al. [5] replaced the condition (2.11) by weaker regularity conditions on the kernel K given by following Assumptions 4.2 and 4.3.

Assumption 4.2 *Assume that there exist operators $\{B_t\}_{t>0}$ with kernels $b_t(x,y)$ that satisfy conditions (2.5) and (2.6) with constants m, η and $\gamma = 0$. Let*

$$K_t^{(0)}(x,y) = \int_{\mathbb{R}^n} K(z,y) b_t(x,z)\mathrm{d}z. \tag{4.1}$$

We assume that the kernels $K_t^{(0)}(x,y)$ satisfy that there exist a function $\phi \in C(\mathbb{R})$ with $\operatorname{supp}\phi \subset [-1,1]$ *and constants $\epsilon \in (0,1/n)$ and D such that*

$$|K(x,y) - K_t^{(0)}(x',y)| \leq \frac{D\phi(\frac{|x-y|}{t^{1/m}})}{|x-y|^n} + \frac{Dt^{\epsilon/m}}{|x-y|^{n+n\epsilon}}, \tag{4.2}$$

whenever $2|x-x'| \leq t^{1/m} \leq \frac{1}{2}|x-y|$, and

Assumption 4.3 *Assume that there exist operators $\{B_t\}_{t>0}$ with kernels $b_t(x,y)$ that satisfy conditions (2.5) and (2.6) with constants m, η, $\gamma = 0$ and there exist kernels $K_t^{(0)}(x,y)$ such that*

(4.1) holds. Also assume that there exist positive constants A and ϵ such that

$$|K_t^{(0)}(x,y)-K_t^{(0)}(x',y)|\le\frac{Dt^{n\epsilon/m}}{|x-y|^{n+n\epsilon}},$$

whenever $2|x-x'|\le t^{1/m}\le\frac{1}{2}|x-y|$.

Our result of the singular integral operators with nonsmooth kernels is stated as follows.

Theorem 4.4 *Assume that T is bounded linear operator on $L^2(\mathbb{R}^n)$ with kernel K satisfying Assumptions (4.2) and (4.3).*

(i) If $T(1)=0$, then T is a bounded operator from $\mathrm{BMO}(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R}^n)$.

(ii) If $T(1)=0$ and $\beta\in(0,\epsilon)$, then T is a bounded operator from $\mathrm{Lip}(\beta,\mathbb{R}^n)$ to $\mathrm{Lip}(\beta,\mathbb{R}^n)$.

We will give a proof which is different from those of Theorems 3.1 and 3.2.

Proof of Theorem 4.4 Let $B=B(y_0,r_B)$, $B^*=5B(y_0,r_B)$ and $\lambda=m_{B^*}(f)$. Then the condition that $T(1)$ is a constant implies that

$$0=\lambda T(1)(x)-\lambda T(1)(z)=\int_{\mathbb{R}^n}\big(K(x,y)-K(z,y)\big)\lambda\mathrm{d}y.$$

Let $c=\int_X K(y_0,y)\big(f(y)-\lambda\big)^\infty\mathrm{d}y$. Then $T(f)(x)-c$ can be divided into two parts as follows:

$$\begin{aligned}T(f)(x)-c&=\int_X K(x,y)\big(f(y)-\lambda\big)\mathrm{d}y-\int_X K(y_0,y)\big(f(y)-\lambda\big)^\infty\mathrm{d}y\\&=T\big((f-\lambda)^0\big)(x)+\int_X\big(K(x,y)-K(y_0,y)\big)\big(f(y)-\lambda\big)^\infty\mathrm{d}y\\&=H_1(x)+H_2(x).\end{aligned}$$

By Hölder inequality and the assumption that T is bounded on $\mathrm{L}^2(\mathbb{R}^n)$, we deduce

$$\begin{aligned}\frac{1}{|B|}\int_B|H_1(x)|\mathrm{d}x&\le\Big(\frac{1}{|B|}\int_B|T\big((f-\lambda)^0\big)(x)|^2\mathrm{d}x\Big)^{1/2}\\&\le\frac{1}{|B|^{1/2}}\|(f-\lambda)^0\|_{L^2(\mathbb{R}^n)}\le C\|f\|_{\mathrm{BMO}(\mathbb{R}^n)}.\end{aligned}$$

Similarly, we can get

$$\frac{1}{|B|^{1+\beta}}\int_B|H_1(x)|\mathrm{d}x\le C\|f\|_{\mathrm{Lip}(\beta,\mathbb{R}^n)}.$$

In order to estimate the term $H_2(x)$, we claim that

$$|H_2(x)|\le\int_{\mathbb{R}^n\setminus B^*}\frac{At_B^{n\epsilon/m}}{|y_0-y|^{n+n\epsilon}}\big|\big(f(y)-\lambda\big)\big|\mathrm{d}y.\tag{4.4}$$

If (4.4) holds, then similarly to the estimate for $J_3(x)$ and $F_3(x)$ in Section 3, we get

$$\frac{1}{|B|}\int_B|H_2(x)|\mathrm{d}x\le C\|f\|_{\mathrm{BMO}(\mathbb{R}^n)},$$

and

$$\frac{1}{|B|^{n\beta+1}}\int_B|H_2(x)|\mathrm{d}x\le C\|f\|_{\mathrm{Lip}(\beta,\mathbb{R}^n)}.$$

Next, we only need to verify (4.4). $H_2(x)$ can be written as

$$\begin{aligned}|H_2(x)| &\le \int_{\mathbb{R}^n\setminus B^*} \big|K(x,y)-K(y_0,y)\big|\big|f(y)-\lambda\big|\mathrm{d}y\\ &\le \int_{\mathbb{R}^n\setminus B^*} \big|K(x,y)-K_t^0(x,y)\big|\big|f(y)-\lambda\big|\mathrm{d}y+\\ &\quad \int_{\mathbb{R}^n\setminus B^*} \big|K_t^0(x,y)-K_t^0(y_0,y)\big|\big|f(y)-\lambda\big|\mathrm{d}y+\\ &\quad \int_{\mathbb{R}^n\setminus B^*} \big|K_t^0(y_0,y)-K(y_0,y)\big|\big|f(y)-\lambda\big|\mathrm{d}y.\end{aligned}$$

Let $t=(2r_B)^m$. By $x\in B(y_0,r_B)$ and $y\in(B^*)^c$, we know $2|x-y_0|\le 2r_B=t_B^{1/m}\le\frac{1}{2}|y_0-y|$, and $|y-x|\sim|y-y_0|$. By the condition $\operatorname{supp}\phi\subset[-1,1]$, we know that $\phi(\frac{|x-y|}{t^{1/m}})=0$ and $\phi(\frac{|y_0-y|}{t^{1/m}})=0$. Then applying Assumptions 4.2 and 4.3 gives

$$\big|K(x,y)-K_t^{(0)}(x,y)\big|\le\frac{A\phi(\frac{|x-y|}{t^{1/m}})}{|x-y|^n}+\frac{At^{n\epsilon/m}}{|x-y|^{n+n\epsilon}}\le\frac{At_B^{n\epsilon/m}}{|y_0-y|^{n+n\epsilon}},$$

$$\big|K(y_0,y)-K_t^{(0)}(y_0,y)\big|\le\frac{A\phi(\frac{|y_0-y|}{t^{1/m}})}{|y_0-y|^n}+\frac{At^{n\epsilon/m}}{|y_0-y|^{n+n\epsilon}}\le\frac{At_B^{n\epsilon/m}}{|y_0-y|^{n+n\epsilon}}$$

and

$$\big|K_t^0(x,y)-K_t^0(y_0,y)\big|\le\frac{At^{n\epsilon/m}}{|y_0-y|^{n+n\epsilon}}.$$

Combining the three inequalities above, we get (4.4). □

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