Journal of Mathematical Research with Applications
Jan., 2016, Vol. 36, No. 1, pp. 109–116
DOI:10.3770/j.issn:2095-2651.2016.01.013
Http://jmre.dlut.edu.cn

Two New Relative Efficiencies of the Weighted Mixed Estimator with Respect to the Ordinary Least Squares Estimator in Linear Regression Models

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Abstract In this paper, we present two relative efficiency of the weighted mixed estimator in respect of least squares estimator. We also derive the lower and upper bounds of those relative efficiencies.

Keywords ordinary least squares estimator; weighted mixed estimator; relative efficiency; linear regression models

MR(2010) Subject Classification 62J05

1. Introduction

Let us consider the general linear regression models:

$$y = X\beta + \epsilon, \quad \epsilon \sim N(0, \sigma^2 I_n), \tag{1}$$

where y shows an $n \times 1$ vector of observation, X defines an $n \times p$ known matrix of rank p, β denotes a $p \times 1$ vector of unknown parameters, ϵ denotes an $n \times 1$ vector of disturbances with expectation $E(\epsilon) = 0$ and variance-covariance matrix $\mathrm{Cov}(\epsilon) = \sigma^2 I_n$.

The ordinary least squares estimator (OLSE) is presented as follows:

$$\hat{\beta}_{\mathrm{OLSE}} = (X'X)^{-1}X'y. \tag{2}$$

In addition to the sample model (1), we give some prior information about β in the form of a set of j independent stochastic linear restrictions as follows:

$$r = R\beta + e, \quad e \sim (0, \sigma^2 W). \tag{3}$$

where R shows a $j \times p$ known matrix with $\mathrm{rank}(R) = j$, e shows a $j \times 1$ vector of disturbances, W is supposed to be known and positive definite, the $j \times 1$ vector, r can be interpreted as a stochastic known vector. Further suppose that ε is stochastically independent of e.

Received December 15, 2014; Accepted October 12, 2015
Supported by the National Natural Science Foundation of China (Grant Nos. 11501072; 11501254), the Natural Science Foundation Project of CQ CSTC (Grant No. cstc2015jcyjA00001) and the Scientific and Technological Research Program of Chongqing Municipal Education Commission (Grant No. KJ1501114).
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Durbin [1], Theil and Goldberger [2] and Theil [3] proposed the mixed estimator which is presented as follows:

$$\hat{\beta}_{ME} = (X'X + R'W^{-1}R)^{-1}(X'y + R'W^{-1}r).$$

When the sample information presented by (1) and the prior information depicted by (3) is to be assigned not necessarily equal weights on the basis of some extraneous considerations in the estimation of regression parameters [4], Schaffrin and Toutenburg [5] presented the weighted mixed estimator (WME), which is

$$\hat{\beta}(w) = (X'X + wR'W^{-1}R)^{-1}(X'y + wR'W^{-1}r), \quad 0 \le w \le 1. \tag{4}$$

We can easily see that the ordinary least squares estimator (OLSE) and the weighted mixed estimator are unbiased estimator of β, and we can obtain:

$$\mathrm{Cov}(\hat{\beta}_{\mathrm{OLSE}}) = \sigma^2(X'X)^{-1},$$

$$\mathrm{Cov}(\hat{\beta}(w)) = \sigma^2(X'X + wR'W^{-1}R)^{-1}(X'X + w^2R'W^{-1}R)(X'X + wR'W^{-1}R)^{-1}.$$

And

$$\begin{aligned}\mathrm{Cov}(\hat{\beta}(w)) &= \sigma^2(X'X + wR'W^{-1}R)^{-1}(X'X + w^2R'W^{-1}R) \times (X'X + wR'W^{-1}R)^{-1} \\ &\le \sigma^2(X'X + wR'W^{-1}R)^{-1} \le \sigma^2(X'X)^{-1} = \mathrm{Cov}(\hat{\beta}_{\mathrm{OLSE}}).\end{aligned} \tag{5}$$

That is in the sense of "loẅner", the WME has smaller covariance matrix than the OLSE.

In practice, the matrix W may be unknown, so we use ordinary least squares estimator to replace the weighted mixed estimator. However, if we use the OLSE to replace the WME, this may lose some efficiency. In order to define the loss, the authors introduce the relative efficiency. Many authors have discussed the relative efficiency, such as, Liu [6], Wang and Yang [7], Yang and Wang [8], Wang and Yang [9], Wang and Pan [10].

Wu and Yang [11] define two relative efficiencies:

$$g_1 = \frac{|\mathrm{Cov}(\hat{\beta}(w))|}{|\mathrm{Cov}(\hat{\beta}_{\mathrm{OLSE}})|},$$

$$g_2 = \frac{\mathrm{tr}(\mathrm{Cov}(\hat{\beta}(w)))}{\mathrm{tr}(\mathrm{Cov}(\hat{\beta}_{\mathrm{OLSE}}))}.$$

They also obtained the lower and upper bounds of g_i, $i = 1, 2$. By the definition of g_i, $i = 1, 2$, we know that the two relative efficiencies only relate to diagonal element of $\mathrm{Cov}(\hat{\beta}(w))$ and $\mathrm{Cov}(\hat{\beta}_{\mathrm{OLSE}})$ and the other elements will not affect the g_i, $i = 1, 2$. In order to overcome this problem, we present two new relative efficiencies:

$$u_1 = \frac{\|\mathrm{Cov}(\hat{\beta}(w))\|_F}{\|\mathrm{Cov}(\hat{\beta}_{\mathrm{OLSE}})\|_F}, \tag{6}$$

$$u_2 = \frac{\|\mathrm{Cov}(\hat{\beta}(w))\|_2}{\|\mathrm{Cov}(\hat{\beta}_{\mathrm{OLSE}})\|_2}, \tag{7}$$

where $\|A\|_F$ and $\|A\|_2$ define the F norm and Spectrum norm of matrix A, respectively. And

$$\|A\|_F = \sqrt{\mathrm{tr}(A'A)}, \quad \|A\|_2 = (\lambda_{\max}(A'A))^{\frac{1}{2}}.$$

It is easy to see that all the elements of $\mathrm{Cov}(\hat{\beta}(w))$ and $\mathrm{Cov}(\hat{\beta}_{\mathrm{OLSE}})$ will affect the values of $u_!$.

The main purpose of the paper is to give the lower and upper bounds of the two relative efficiencies.

The rest of the paper is organized as follows. In Section 2, we give the lower and bounds of u_1 and u_2. And a numerical example is presented to show the theoretical results. Some conclusion remarks are presented in Section 4.

2. The lower and upper bounds of u_1 and u_2

In this section we will present the lower and upper bounds of u_1 and u_2. Firstly, we list some lemmas and notations which are needed in the following discussions.

Let A be an $n \times n$ nonnegative definite matrix and $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$ stand for the ordered eigenvalues of matrix A.

Lemma 2.1 ([12]) *Let A be an $n \times n$ nonnegative definite matrix, and B be an $n \times n$ nonnegative definite matrix. Then we have*

$$\lambda_n(A)\lambda_i(B) \leq \lambda_i(AB) \leq \lambda_1(A)\lambda_i(B), \quad i = 1, 2, \ldots, n.$$

Lemma 2.2 ([12]) *Let A be a nonnegative definite matrix and B be a positive definite matrix. Then we have*

$$\frac{\mathrm{tr}(A)}{\lambda_1(B)} \leq \mathrm{tr}(AB^{-1}) \leq \frac{\mathrm{tr}(A)}{\lambda_n(B)}.$$

Firstly, we state the following theorem.

Theorem 2.3 *Let u_1 and u_2 be defined in Equs. (6) and (7). Then we have $0 \leq u_i \leq 1$, $i = 1, 2$.*

Proof Let $A = \mathrm{Cov}(\hat{\beta}(w))$ and $B = \mathrm{Cov}(\hat{\beta}_{\mathrm{OLSE}})$. Then by (5), we have $0 \leq A \leq B$. Let $\zeta_1 \geq \cdots \geq \zeta_p$ be the ordered eigenvalues of matrix A. Then there exists an orthogonal matrix such that $Q'AQ = \mathrm{diag}(\zeta_1, \ldots, \zeta_p)$. And we let $\eta_1 \geq \cdots \geq \eta_p$ be the ordered eigenvalues of matrix B. Since $B \geq A \geq 0$, we obtain $\eta_i \geq \zeta_i > 0$ $(i = 1, \ldots, p)$.

By the definition of the F norm, we obtain:

$$\| A \|_F^2 = \mathrm{tr}(A'A) = \mathrm{tr}(Q'A'QQ'AQ) = \sum_{i=1}^{p} \zeta_i^2,$$

$$\| B \|_F^2 = \sum_{i=1}^{p} \eta_i^2 \geq \sum_{i=1}^{p} \zeta_i^2 = \| A \|_F^2 .$$

So

$$0 \leq u_1 = \frac{\|\mathrm{Cov}(\hat{\beta}(w))\|_F}{\|\mathrm{Cov}(\hat{\beta}_{\mathrm{OLSE}})\|_F} \leq 1.$$

Since $B \geq A > 0$, we get

$$\| B \|_2^2 = \lambda_{\max}(B'B) = \lambda_{\max}^2(B) \geq \lambda_{\max}^2(A) = \| A \|_2^2 .$$

Thus $0 \leq u_2 \leq 1$. □

By Theorem 2.3, we can conclude that if the values of u_i, $i = 1, 2$ are closer to 1, then using $\hat{\beta}_{\mathrm{OLSE}}$ to replace $\hat{\beta}(w)$ will have smaller loss, instead, if the values of u_i, $i = 1, 2$ are closer to 0, it is included that using $\hat{\beta}_{\mathrm{OLSE}}$ to replace $\hat{\beta}(w)$ will have bigger loss. In the following theorem, we will present more accurate lower and upper bounds of the two relative efficiencies.

Theorem 2.4 *Let $\hat{\beta}_{\mathrm{OLSE}}$ and $\hat{\beta}(w)$ be defined in Equs. (2) and (4), and u_1 be given in Eq. (7). Then we have*

$$\max\Big\{\sqrt{\frac{\sum_{i=1}^{p}(1+w\zeta_{p-i+1})^{-4}(1+w^2\zeta_i)^2}{\sum_{i=1}^{p}\theta_i^{-2}\theta_1^2}}, \sqrt{\frac{(\sum_{i=1}^{p}\frac{(1+w\zeta_{p-i+1})^{-2}(1+w^2\zeta_i)}{\theta_i})^2}{p\sum_{i=1}^{p}\theta_i^{-2}}}\Big\}$$
$$\leq u_1 \leq \sqrt{\frac{\sum_{i=1}^{p}(1+w\zeta_{p-i+1})^{-4}(1+w^2\zeta_i)^2}{\sum_{i=1}^{p}\theta_i^{-2}\theta_p^2}}$$

where $\theta_1 \geq \cdots \geq \theta_p > 0$ denote the ordered eigenvalues of $X'X$, $\zeta_1 \geq \cdots \geq \zeta_p$ define the ordered eigenvalues of $R'W^{-1}R(X'X)^{-1}$.

Proof First, we have

$$\|\mathrm{Cov}(\hat{\beta}_{\mathrm{OLSE}})\|_F^2 = \sigma^4 \mathrm{tr}((X'X)^{-2}), \tag{8}$$

$$\|\mathrm{Cov}(\hat{\beta}(w))\|_F^2 = \sigma^4 \mathrm{tr}\{(X'X + wR'W^{-1}R)^{-1}(X'X + w^2R'W^{-1}R)\times (X'X + wR'W^{-1}R)^{-1}\}. \tag{9}$$

Since $X'X > 0$, there exists an orthogonal matrix H_1, such that

$$X'X = H_1'\Theta H_1, \quad \Theta = \mathrm{diag}(\theta_1, \ldots, \theta_p),$$

where $\theta_1 \geq \cdots \geq \theta_p > 0$ denote the ordered eigenvalues of $X'X$. Then we can obtain

$$(X'X)^{-1} = H_1'\Theta^{-1}H_1 = H_1'\Gamma H_1, \quad \Gamma = \mathrm{diag}(\theta_p^{-1}, \ldots, \theta_1^{-1}),$$

where $\theta_p^{-1} \geq \cdots \geq \theta_1^{-1} > 0$ denote the ordered eigenvalues of $(X'X)^{-1}$.

On the other hand, it is easy to know that $Q = R'W^{-1}R \geq 0$ and $\mathrm{rank}(Q) = j$. Define $\tilde{Q} = (X'X)^{-1/2}Q(X'X)^{-1/2}$, then $\tilde{Q} \geq 0$, so there exists an orthogonal matrix H_2, such that

$$\tilde{Q} = H_2'\Omega H_2, \quad \Omega = \mathrm{diag}(\zeta_1, \ldots, \zeta_p),$$

where $\zeta_1 \geq \cdots \geq \zeta_p$ are the ordered eigenvalues of $\tilde{Q}$. Then (8) becomes

$$\| \mathrm{Cov}(\hat{\beta}_{\mathrm{OLSE}}) \|_F^2 = \sigma^4 \sum_{i=1}^{p} \theta_i^{-2},$$

$$\begin{aligned}\lambda_i(\mathrm{Cov}(\hat{\beta}(w)) &= \lambda_i((X'X + wR'W^{-1}R)^{-1}(X'X + w^2R'W^{-1}R)(X'X + wR'W^{-1}R)^{-1}) \\ &= \lambda_i((X'X)^{-1/2}(I + w\tilde{Q})^{-1}(I + w^2\tilde{Q})(I + w\tilde{Q})^{-1}(X'X)^{-1/2})\end{aligned}$$

$$=\lambda_i((I+w\tilde{Q})^{-1}(I+w^2\tilde{Q})(I+w\tilde{Q})^{-1}(X'X)^{-1})$$
$$=\lambda_i((I+w\Omega)^{-1}(I+w^2\Omega)(I+w\Omega)^{-1}H_2H_1'\Gamma H_1'H_2)$$
$$=\lambda_i(\Phi K'\Gamma K), \tag{10}$$

where $K = H_1'H_2$ is an orthogonal matrix.

$$\begin{aligned}\Phi =&(I+w\Omega)^{-1}(I+w^2\Omega)(I+w\Omega)^{-1}\\ =&\mathrm{diag}((1+w\zeta_p)^{-2}(1+w^2\zeta_1),\ldots,(1+w\zeta_1)^{-2}(1+w^2\zeta_p))\\ =&\mathrm{diag}(\varepsilon_1,\ldots,\varepsilon_p),\end{aligned}$$

where $\varepsilon_i = (1+w\zeta_{p-i+1})^{-2}(1+w^2\zeta_i)$, $\varepsilon_1 \geq \cdots \geq \varepsilon_p$.

Now by using Lemma 2.1, we obtain

$$\lambda_p(\Gamma)\lambda_i(\Phi) \leq \lambda_i(\Phi K'\Gamma K) \leq \lambda_1(\Gamma)\lambda_i(\Phi).$$

Thus

$$\sigma^2\frac{\varepsilon_i}{\theta_1} \leq \lambda_i(\mathrm{Cov}(\hat{\beta}(w)) \leq \sigma^2\frac{\varepsilon_i}{\theta_p}, \quad i=1,\ldots,p. \tag{11}$$

Then by (9) and (11), we obtain

$$\sigma^4\sum_{i=1}^{p}\frac{\varepsilon_i^2}{\theta_1^2} \leq \parallel \mathrm{Cov}(\hat{\beta}(w)) \parallel_F^2 \leq \sigma^4\sum_{i=1}^{p}\frac{\varepsilon_i^2}{\theta_p^2},$$

$$\begin{aligned}&\sigma^4\sum_{i=1}^{p}\frac{(1+w\zeta_{p-i+1})^{-4}(1+w^2\zeta_i)^2}{\theta_1^2}\\ &\quad\leq \parallel \mathrm{Cov}(\hat{\beta}(w)) \parallel_F^2 \leq \sigma^4\sum_{i=1}^{p}\frac{(1+w\zeta_{p-i+1})^{-4}(1+w^2\zeta_i)^2}{\theta_p^2}.\end{aligned}$$

So we obtain

$$\sqrt{\frac{\sum_{i=1}^{p}(1+w\zeta_{p-i+1})^{-4}(1+w^2\zeta_i)^2}{\sum_{i=1}^{p}\theta_i^{-2}\theta_1^2}} \leq u_1 \leq \sqrt{\frac{\sum_{i=1}^{p}(1+w\zeta_{p-i+1})^{-4}(1+w^2\zeta_i)^2}{\sum_{i=1}^{p}\theta_i^{-2}\theta_p^2}}.$$

On the other hand, by (10), we can get

$$\begin{aligned}&\mathrm{tr}((X'X+wR'W^{-1}R)^{-1}(X'X+w^2R'W^{-1}R)(X'X+wR'W^{-1}R)^{-1})\\ &\quad= \mathrm{tr}(\Phi K'\Gamma K).\end{aligned}$$

Since Φ and $K\Phi K'$ have same eigenvalues, using the Neumann equality [12], we may have

$$\mathrm{tr}((X'X+wR'W^{-1}R)^{-1}(X'X+w^2R'W^{-1}R)(X'X+wR'W^{-1}R)^{-1}) \geq \sum_{i=1}^{p}\frac{\varepsilon_i}{\theta_i}.$$

Then by the equality $\frac{[\mathrm{tr}((X'X+wR'W^{-1}R)^{-1}(X'X+w^2R'W^{-1}R)(X'X+wR'W^{-1}R)^{-1})]^2}{\mathrm{tr}((X'X+wR'W^{-1}R)^{-2}(X'X+w^2R'W^{-1}R)^2(X'X+wR'W^{-1}R)^{-2}} \leq p$, we can have

$$\begin{aligned}&\mathrm{tr}((X'X+wR'W^{-1}R)^{-2}(X'X+w^2R'W^{-1}R)^2(X'X+wR'W^{-1}R)^{-2})\\ &\quad\geq \frac{1}{p}[\mathrm{tr}((X'X+wR'W^{-1}R)^{-1}(X'X+w^2R'W^{-1}R)(X'X+wR'W^{-1}R)^{-1})]^2\end{aligned}$$

$$\geq \frac{1}{p}(\sum_{i=1}^{p}\frac{\varepsilon_i}{\theta_i})^2.$$

So

$$\| \operatorname{Cov}(\hat{\beta}(w)) \|_F^2 \geq \sigma^4 \frac{1}{p}(\sum_{i=1}^{p}\frac{\varepsilon_i}{\theta_i})^2 = \sigma^4 \frac{1}{p}(\sum_{i=1}^{p}\frac{(1+w\zeta_{p-i+1})^{-2}(1+w^2\zeta_i)}{\theta_i})^2.$$

Thus

$$\max\Big\{\sqrt{\frac{\sum_{i=1}^{p}(1+w\zeta_{p-i+1})^{-4}(1+w^2\zeta_i)^2}{\sum_{i=1}^{p}\theta_i^{-2}\theta_1^2}}, \sqrt{\frac{(\sum_{i=1}^{p}\frac{(1+w\zeta_{p-i+1})^{-2}(1+w^2\zeta_i)}{\theta_i})^2}{p\sum_{i=1}^{p}\theta_i^{-2}}}\Big\}$$
$$\leq u_1 \leq \sqrt{\frac{\sum_{i=1}^{p}(1+w\zeta_{p-i+1})^{-4}(1+w^2\zeta_i)^2}{\sum_{i=1}^{p}\theta_i^{-2}\theta_p^2}}. \quad \Box$$

In the following theorem, we give the lower and upper bounds of the relative efficiency u_2.

Theorem 2.5 *Let $\hat{\beta}_{\mathrm{OLSE}}$ and $\hat{\beta}(w)$ be defined in Equs. (2) and (4), and u_2 be defined in Eq. (7). Then we have*

$$\frac{(1+w^2\zeta_1)\theta_p}{(1+w\zeta_p)^2\theta_1} \leq u_2 \leq \frac{1+w^2\zeta_1}{(1+w\zeta_p)^2},$$

where $\theta_1 \geq \cdots \geq \theta_p > 0$ are the ordered eigenvalues of $X'X$, $\zeta_1 \geq \cdots \geq \zeta_p$ are the ordered eigenvalues of $R'W^{-1}R(X'X)^{-1}$.

Proof Firstly, we have

$$\| (X'X)^{-1} \|_2 = \sqrt{\lambda_{\max}((X'X)^{-2})} = \sqrt{\lambda_{\max}^2((X'X)^{-1})} = \lambda_p^{-1}(X'X) = \theta_p^{-1}$$

and

$$\| (X'X+wR'W^{-1}R)^{-2}(X'X+w^2R'W^{-1}R)^2(X'X+wR'W^{-1}R)^{-2} \|_2$$
$$= \sqrt{\lambda_{\max}((X'X+wR'W^{-1}R)^{-2}(X'X+w^2R'W^{-1}R)^2(X'X+wR'W^{-1}R)^{-2})}$$
$$= \lambda_p^{-1}((X'X+wR'W^{-1}R)^{-1}(X'X+w^2R'W^{-1}R)(X'X+wR'W^{-1}R)^{-1}).$$

Then using Eq. (11), we get

$$\frac{\varepsilon_1}{\theta_1} \leq \lambda_p^{-1}((X'X+wR'W^{-1}R)^{-1}(X'X+w^2R'W^{-1}R)(X'X+wR'W^{-1}R)^{-1}) \leq \frac{\varepsilon_1}{\theta_p}.$$

Thus $\frac{\varepsilon_1\theta_p}{\theta_1} \leq u_2 \leq \varepsilon_1$. That is

$$\frac{(1+w\zeta_p)^{-2}(1+w^2\zeta_1)\theta_p}{\theta_1} \leq u_2 \leq (1+w\zeta_p)^{-2}(1+w^2\zeta_1). \quad \Box$$

3. Numerical example

In this section, we will present a numerical example to illustrate the theoretical results. The data is given as follows, and the data was studied by Wu and Yang [11]

$$X = \begin{pmatrix} 1.9 & 2.2 & 1.9 & 3.7 \\ 1.8 & 2.2 & 2.0 & 3.8 \\ 1.8 & 2.4 & 2.1 & 3.6 \\ 1.8 & 2.4 & 2.2 & 3.8 \\ 2.0 & 2.5 & 2.3 & 3.8 \\ 2.1 & 2.6 & 2.4 & 3.7 \\ 2.1 & 2.6 & 2.6 & 3.8 \\ 2.2 & 2.6 & 2.6 & 4.0 \\ 2.3 & 2.8 & 2.8 & 3.7 \\ 2.3 & 2.7 & 2.8 & 3.8 \end{pmatrix}, \quad y = \begin{pmatrix} 2.3 \\ 2.2 \\ 2.2 \\ 2.3 \\ 2.4 \\ 2.5 \\ 2.6 \\ 2.6 \\ 2.7 \\ 2.7 \end{pmatrix}.$$

Consider the following matrix restriction $r = R\beta + e$, where $R = (1, -2, -2, -2)'$. The estimated relative efficiencies are presented in Figure 1.

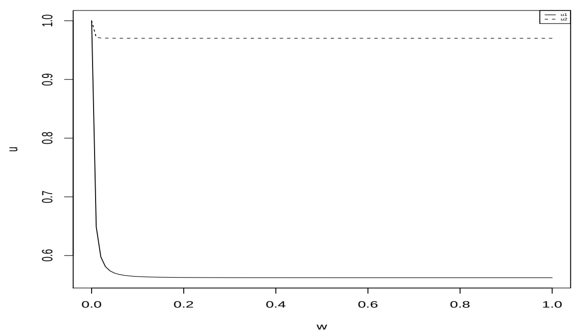

Figure 1 The estimated relative efficiencies values of u_1 and u_2 when $0 < w < 1$

4. Concluding remarks

In this paper, we define two new relative efficiencies and we also give a lower and upper bounds of the relative efficiencies.

Acknowledgments The authors wish to thank the referee and the editor for their helpful comments and suggestions.

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