Journal of Mathematical Research with Applications
Jan., 2016, Vol. 36, No. 1, pp. 117–126
DOI:10.3770/j.issn:2095-2651.2016.01.014
Http://jmre.dlut.edu.cn

L^p Solutions of BSDEs with Non-Uniformly Linear Growth Generators and General Time Interval

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Abstract In this paper, we establish the existence of the minimal L^p $(p > 1)$ solution of backward stochastic differential equations (BSDEs) where the time horizon may be finite or infinite and the generators have a non-uniformly linear growth with respect to t. The main idea is to construct a sequence of solutions $\{(Y^n, Z^n)\}$ which is a Cauchy sequence in $\mathbb{S}^p \times \mathbb{M}^p$ space, and finally we prove $\{(Y^n, Z^n)\}$ converges to the L^p $(p > 1)$ solution of BSDEs.

Keywords BSDEs; finite or infinite time interval; non-uniformly linear growth generators; Cauchy sequence; L^p $(p > 1)$ solutions

MR(2010) Subject Classification 60H10

1. Introduction

We study the following one-demensional nonlinear backward stochastic differential equations (BSDEs for short)

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \cdot \mathrm{d}B_s, \quad 0 \le t \le T, \tag{1.1}$$

where $T > 0$ is a finite or an infinite constant called the time horizon, ξ is a random variable called the terminal condition, the function $g(\omega, t, y, z) : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ is progressively measurable for each (y, z), called the generator of BSDE (1.1) ($g(t, y, z)$ for short), and $(B)_{t\in[0,T]}$ is a d-dimensional standard Brownian motion. The solution $(Y_\cdot, Z_\cdot)$ is a pair of adapted processes. The triple (g, T, ξ) is called the parameters of BSDEs (1.1).

Since the pioneering paper Pardoux and Peng [1] proved that there exists a unique L^2 solution to the multidimensional BSDEs with square integrable parameters under the Lipschitz assumption on the generator g, much effort has been done in relaxing the Lipschitz hypothesis on the generator. Here we just introduce some articles which are closely linked to our paper. Since the generators g are continuous and uniform with respect to t and $0 < T < \infty$, we have obtained a lot of results yet. For instance, while $\xi \in L^2(\Omega, \mathcal{F}_T, P)$, Lepeltier [2] first gave the linear growth condition and showed the existence of the minimal L^2 solutions to BSDEs (1.1).

Received October 18, 2014; Accepted September 2, 2015
Supported by the National Natural Science Foundation of China (Grant No. 11371362) and the Fundamental Research Funds for the Central Universities (Grant No. 2012LWB48).
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The L^p $(1 < p < 2)$ solutions for BSDEs with uniform linear-growth generator was established in [3] by constructing a Cauchy sequence which converges to the desired solution. For all $(p > 1)$, when $\xi \in L^p(\Omega, \mathcal{F}_T, P)$, Fan and Jiang [4] got the existence of L^p solution to BSDE by using the localization procedure. In particular, Izumi [5] also gave the L^p $(p > 1)$ solution by proving the sequence of solution constructed in [2] is a Cauchy one while $p > 2$.

On the other hand, as for the non-uniformly linear growth generator, only [6] established the existence of L^2 solutions to BSDEs with the time horizon T being finite or infinite and the generators g being non-uniform with t. But the L^p $(p > 1)$ solutions for BSDEs are still remaining to be solved so far. Motivated by it, the objective of this paper is to explore the existence of L^p $(p > 1)$ solutions for BSDEs with non-uniformly linear growth generators.

This paper is organized as follows. We introduce some preliminaries and lemmas in Section 2. In Section 3, we obtain the existence theorem for BSDEs with non-uniformly linear growth generators by constructing an approximation Cauchy sequence, this improves the results in [2–6].

2. Preliminaries and lemmas

Let us first introduce some notations. For what follows, we fix two positive numbers $0 < T \leqslant \infty$ and $p > 1$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space carrying a d-dimensional standard Brownian motion $(B_t)_{t\geq 0}$, and $(\mathcal{F}_t)_{t\geq 0}$ denote the natural filtration generated by $(B_t)_{t\geq 0}$, augmented by P-null sets of $\mathcal{F}$ and assume $\mathcal{F}_T = \mathcal{F}$. For any positive integer n, let $|\cdot|$ denote the norm of an Euclid space $\mathbb{R}^n$.

For $t \in [0, T]$, let $L^p(\Omega, \mathcal{F}_t, P)$ denote the set of all $\mathcal{F}_t$ -measurable random variables ξ such that $E[|\xi|^p] < +\infty$. Let $\mathbb{S}^p(0, T; R)$ be the set of all continuous and adapted processes $(Y_t)_{t\in[0,T]}$ with values in $\mathbb{R}$ such that

$$\|Y\|_{\mathbb{S}^p} := \Big(E\big[\sup_{0\leq t\leq T} |Y_t|^p\big]\Big)^{1/p} < +\infty.$$

And for each positive integer d, we denote by $\mathbb{M}^p(0, T; R^d)$ the set of all $(\mathcal{F}_t)$-progressively measurable processes $(Z_t)_{t\in[0,T]}$ with values in $\mathbb{R}^d$ such that

$$\|Z\|_{\mathbb{M}^p} := \Big(E\Big[\Big(\int_0^T |Z_t|^2 \mathrm{d}t\Big)^{p/2}\Big]\Big)^{1/p} < +\infty.$$

Moreover, let $\mathbf{S}$ be the set of all nondecreasing continuous functions $\rho(\cdot)$ from $\mathbb{R}_+$ to itself with $\rho(0) = 0$ and $\rho(u) > 0$ for $u > 0$.

Obviously, both $\mathbb{S}^p$ and $\mathbb{M}^p$ are Banach spaces.

Definition 2.1 *A pair of processes $(Y_t, Z_t)_{t\in[0,T]}$ is called an L^p solution to BSDE (1.1), if $(Y_t, Z_t)_{t\in[0,T]} \in \mathbb{S}^p \times \mathbb{M}^p$ and satisfies BSDE (1.1).*

We list some useful Lemmas, and we always assume that $0 < T \leqslant \infty$ and $p > 1$ in this article, unless otherwise specified.

Lemma 2.2 ([7]) *Let the generator g satisfy the following conditions (A1) and (A2).*

(A1) $E[(\int_0^T |g(s, 0, 0)|\mathrm{d}s)^p] < +\infty.$

(A2) g *is non-uniformly Lipschitz, i.e., there exist two deterministic functions* $u(\cdot), v(\cdot)$: $[0,T] \to \mathbb{R}_+$, *with* $\int_0^T (u(t)+v(t)^2)\mathrm{d}t < +\infty$ *such that* $\mathrm{d}P \times \mathrm{d}t$*-a.s., for each* y_1, y_2, z_1, z_2,

$$|g(t,y_1,z_1) - g(t,y_2,z_2)| \leq u(t)|y_1 - y_2| + v(t)|z_1 - z_2|.$$

Then for each $\xi \in L^p(\Omega, \mathcal{F}_T, P)$, *BSDE (1.1) has a unique* L^p *solution.*

The following priori estimates play a key role in this paper, and it was first brought forward in [5] by generalizing the conclusion in [3]. It is easy to show that the following Lemma also holds when $0 < T \leq \infty$, we omit it here.

Lemma 2.3 ([5]) *(i) If* (Y,Z) *is an* L^p *solution to BSDE (1.1), then there exists a positive constant* C_p *depending only on* p *such that*

$$\|Y\|_{\mathbb{S}^p}^p \leq C_p E\Big[|\xi|^p + \int_0^T |Y_s|^{p-1}|g(s,Y_s,Z_s)|\mathrm{d}s\Big],$$
$$\|Z\|_{\mathbb{M}^p}^p \leq C_p \Big\{E\Big[|\xi|^p + \Big(\int_0^T |Y_s||g(s,Y_s,Z_s)|\mathrm{d}s\Big)^{\frac{p}{2}}\Big] + \|Y\|_{\mathbb{S}^p}^p\Big\}.$$

(ii) If (Y^1,Z^1) *and* (Y^2,Z^2) *are, respectively, an* L^p *solution to the BSDE (1.1) with parameters* (g^1,T,ξ^1) *and* (g^2,T,ξ^2), *then there exists a positive constant* C_p *depending only on* p *such that*

$$\|\delta Y\|_{\mathbb{S}^p}^p \leq C_p E\Big[|\delta\xi|^p + \int_0^T |\delta Y_s|^{p-1}|\delta g_s|\mathrm{d}s\Big],$$
$$\|\delta Z\|_{\mathbb{M}^p}^p \leq C_p \Big\{E\Big[|\delta\xi|^p + \Big(\int_0^T |\delta Y_s||\delta g_s|\mathrm{d}s\Big)^{\frac{p}{2}}\Big] + \|\delta Y\|_{\mathbb{S}^p}^p\Big\},$$

where $\delta\xi = \xi^1 - \xi^2$, $\delta Y = Y^1 - Y^2$, $\delta Z = Z^1 - Z^2$, $\delta g_s = g^1(t,Y_s^1,Z_s^1) - g^2(t,Y_s^2,Z_s^2)$.

3. A General comparison theorem for Solutions of BSDEs

We will use the following assumptions (A3) and (A4):

(A3) g is weakly monotonic in y, i.e., there exists a deterministic function $u(\cdot) : [0,T] \to \mathbb{R}_+$ with $\int_0^T u(t)\mathrm{d}t < +\infty$ and a concave function $\phi \in \mathbf{S}$ with $\int_{0+} \frac{1}{\phi(x)}\mathrm{d}x = +\infty$ such that $\mathrm{d}P \times \mathrm{d}t$-a.s., for each y_1, y_2, z,

$$\mathrm{sgn}(y_1 - y_2) \cdot (g(t,y_1,z) - g(t,y_2,z)) \leq u(t) \cdot \phi(|y_1 - y_2|).$$

(A4) g is uniformly continuous in z uniformly with respect to y, i.e., there exists a deterministic function $v(\cdot) : [0,T] \to \mathbb{R}_+$ with $\int_0^T v(t)^2\mathrm{d}t < +\infty$ and a function $\rho \in \mathbf{S}$ of linear growth such that $\mathrm{d}P \times \mathrm{d}t$-a.s., for each y, z_1, z_2,

$$|g(t,y,z_1) - g(t,y,z_2)| \leq v(t) \cdot \rho(|z_1 - z_2|).$$

Here and henceforth, we always assume that $0 \leq \rho(x) \leq ax + b$ for all $x \in \mathbb{R}_+$. Furthermore, we also assume that $\int_0^T v(t)\mathrm{d}t < +\infty$ when $b \neq 0$.

It can be proved that the following Comparison Theorem holds true for all $p > 1$.

Theorem 3.1 *Let* $\xi^1, \xi^2 \in L^p(\Omega, \mathcal{F}_T, P)$, g^1 *and* g^2 *be two generators of BSDEs, and let* $(y_\cdot^1, z_\cdot^1)$

(resp., $(y^2_\cdot, z^2_\cdot)$) be an L^p solution to BSDE with parameters (g^1, T, ξ^1) (resp., (g^2, T, ξ^2)). If $\mathrm{d}P$-a.s., $\xi^1 \le \xi^2$, g^1 (or g^2) satisfies (A3) and (A4) and $\mathrm{d}P \times \mathrm{d}t$-a.s., $g^1(t, y^2_t, z^2_t) \le g^2(t, y^2_t, z^2_t)$ (or $g^1(t, y^1_t, z^1_t) \le g^2(t, y^1_t, z^1_t)$), then for each $t \in [0, T]$, we have $\mathrm{d}P$-a.s., $y^1_t \le y^2_t$.

Remark 3.2 It is easy to prove the above Theorem is also true while g^1 (resp., g^2) satisfies (A2) as the assumption (A2) implies (A3) and (A4).

As Theorem 3.1 is a generalized one of Theorem 2 in [6] from $p = 2$ to $p > 1$, it is easy to be proved, and we omit it here.

4. Existence theorem for BSDEs with non-uniformly linear-growth generators

In this section we obtain an existence result of the minimal L^p solutions to BSDEs with continuous and non-uniformly linear-growth generators for all $p > 1$. We will work under the following assumptions.

(H1) g is non-uniformly linear growth with t in (y, z), i.e., there exist two positive deterministic functions $u(\cdot), v(\cdot) : [0, T] \to \mathbb{R}_+$ with $\int_0^T (u(t) + v(t)^2)\mathrm{d}t < +\infty$, and an $(\mathcal{F}_t)$-progressively measurable, non-negative stochastic process $(f_t)_{t\in[0,T]}$ with $E[(\int_0^T f_s \mathrm{d}s)^p] < +\infty$ such that $\mathrm{d}P \times \mathrm{d}t$-a.s., for each y, z, $|g(t, y, z)| \le f_t + u(t)|y| + v(t)|z|$.

(H2) g is continuous in (y, z), i.e., $\mathrm{d}P \times \mathrm{d}t$-a.s., $(y, z) \to g(t, y, z)$ is continuous.

Now let us give the main result of this paper.

Theorem 4.1 *Let the assumptions (H1) and (H2) hold for g. For $\xi \in L^p(\Omega, \mathcal{F}_T, P)$, there exists a minimal L^p $(p > 1)$ solution $(y_\cdot, z_\cdot)$ to BSDE with parameters (g, T, ξ).*

Remark 4.2 Here, let us have a look of Theorem 4.1. First, [6] is an obvious conclusion of this paper by taking $p = 2$. Second, in the case of $0 < T < \infty$ and $u(t) = v(t) = K$, one can get that the above Theorem 3.1 generalizes the corresponding result in [4] and [5]. Moreover, If we let $f_t = K$, [2] and [3] are both the deduction of our result. So Theorem 4.1 improves all the previous results.

We first state some Propositions before proving the Theorem 4.1.

The following proposition was first established in [2] for uniformly linear growth generator, then [6] developed it into non-uniformly situation.

Proposition 4.3 ([6]) *Assume that the generator g satisfies (H1) and (H2). Let g_n be the function defined as follows:*

$$g_n(t, y, z) := \inf_{(a,b)\in\mathbf{R}^{1+d}} \{g(t, a, b) + nu(t)|y - a| + nv(t)|z - b|\}.$$

Then the sequence of functions g_n is well defined for each $n \ge 1$, and it satisfies $\mathrm{d}P \times \mathrm{d}t$-a.s.,

(i) Linear growth, i.e., $\forall (y, z), |g_n(t, y, z)| \le f_t(w) + u(t)|y| + v(t)|z|$.

(ii) Monotonicity in n, i.e., $\forall (y, z), g_n(t, y, z)$ increases in n.

(iii) Lipschitz condition, i.e., $\forall y_1, y_2, z_1, z_2$, *we have*

$$|g_n(t, y_1, z_1) - g_n(t, y_2, z_2)| \le nu(t)|y_1 - y_2| + nv(t)|z_1 - z_2|.$$

(iv) Convergence, i.e., if $(y_n, z_n) \to (y, z)$, *then* $g_n(t, y_n, z_n) \to g(t, y, z)$.

Moreover, inspired by [5], if we suppose the generator g satisfies (H1) in the Lemma 2.3, then a more precisely priori estimates can be obtained for z.

Proposition 4.4 *If* (Y, Z) *is an* L^p *solution to BSDE (1.1), and the assumption (H1) holds for the generator* g *too, then there exists a positive constant* C *such that*

$$\|Z\|^p_{\mathbb{M}^p} \le C\{1 + \|Y\|^{\frac{p}{2}}_{\mathbb{S}^p} + \|Y\|^p_{\mathbb{S}^p}\},$$

where C *is a positive constant and depends on* $p, E[|\xi|^p], E[(\int_0^T f_t \mathrm{d}t)^p, \int_0^T u(t)\mathrm{d}t]$ *and* $\int_0^T v^2(t)\mathrm{d}t$.

Proof By Lemma 2.3, we have

$$\|Z\|^p_{\mathbb{M}^p} \le C_p\Big\{E\Big[|\xi|^p + \Big(\int_0^T |Y_s||g(s, Y_s, Z_s)|\mathrm{d}s\Big)^{\frac{p}{2}}\Big] + \|Y\|^p_{\mathbb{S}^p}\Big\}. \tag{4.1}$$

Since the generator g satisfies the assumption (H1) and the basic inequality

$$2ab \le \varepsilon a^2 + \varepsilon^{-1} b^2, \quad \varepsilon > 0,\ a, b \ge 0,$$

we can deduce that

$$\begin{aligned} E\Big[\Big(\int_0^T |Y_s||g(s, Y_s, Z_s)|\mathrm{d}s\Big)^{\frac{p}{2}}\Big] &\le E\Big[\Big(\int_0^T |Y_s|(f_s + u(s)|Y_s| + v(s)|Z_s|)\mathrm{d}s\Big)^{\frac{p}{2}}\Big] \\ &\le C_P^1\Big\{E\Big[\Big(\int_0^T |Y_s|f_s\mathrm{d}s\Big)^{\frac{p}{2}}\Big] + E\Big[\Big(\int_0^T |Y_s|^2 u(s)\mathrm{d}s\Big)^{\frac{p}{2}}\Big] + \\ &\quad E\Big[\Big(\int_0^T |Y_s||Z_s|v(s)\mathrm{d}s\Big)^{\frac{p}{2}}\Big]\Big\}. \end{aligned} \tag{4.2}$$

Furthermore, for the second term of (4.2), we can get

$$E\Big[\Big(\int_0^T |Y_s|^2 u(s)\mathrm{d}s\Big)^{\frac{p}{2}}\Big] \le \|Y\|^p_{\mathbb{S}^p}\Big(\int_0^T u(s)\mathrm{d}s\Big)^{\frac{p}{2}}. \tag{4.3}$$

For the third term of (4.2), it follows that

$$\begin{aligned} E\Big[\Big(\int_0^T |Y_s||Z_s|v(s)\mathrm{d}s\Big)^{\frac{p}{2}}\Big] &\le E\Big[\Big(\int_0^T (\frac{1}{\varepsilon}|Y_s|^2 v^2(s) + \varepsilon|Z_s|^2)\mathrm{d}s\Big)^{\frac{p}{2}}\Big] \\ &\le C_P^2\Big\{\varepsilon^{-\frac{p}{2}}\|Y\|^p_{\mathbb{S}^p}\Big(\int_0^T v^2(s)\mathrm{d}s\Big)^{\frac{p}{2}} + \varepsilon^{\frac{p}{2}}\|Z\|^p_{\mathbb{Z}^p}\Big\}. \end{aligned} \tag{4.4}$$

By the Hölder inequality, we can get

$$E\Big[\Big(\int_0^T |Y_s|f_s\mathrm{d}s\Big)^{\frac{p}{2}}\Big] \le \|Y\|^{\frac{p}{2}}_{\mathbb{S}^P}\Big(E\Big[\Big(\int_0^T f_s\mathrm{d}s\Big)^p\Big]\Big)^{\frac{1}{2}}. \tag{4.5}$$

Thus, combining (4.2)–(4.5), we have

$$E\Big[\Big(\int_0^T |Y_s||g(s, Y_s, Z_s)|\mathrm{d}s\Big)^{\frac{p}{2}}\Big]$$

$$\le C_P^3\Big\{\|Y\|_{\mathbb{S}^p}^{\frac{p}{2}}\Big(E\Big[\Big(\int_0^T f_s\mathrm{d}s\Big)^p\Big]\Big)^{\frac{1}{2}}+\|Y\|_{\mathbb{S}^p}^p\Big(\int_0^T u(s)\mathrm{d}s\Big)^{\frac{p}{2}}+$$
$$\varepsilon^{-\frac{p}{2}}\|Y\|_{\mathbb{S}^p}^p\Big(\int_0^T v^2(s)\mathrm{d}s\Big)^{\frac{p}{2}}+\varepsilon^{\frac{p}{2}}\|Z\|_{\mathbb{M}^p}^p\Big\}. \tag{4.6}$$

By (4.6) and (4.1), taking $C_PC_P^3\varepsilon^{\frac{p}{2}}=\frac{1}{2}$, we can get the desired result

$$\|Z\|_{\mathbb{M}^p}^p\le C\{1+\|Y\|_{\mathbb{S}^p}^{\frac{p}{2}}+\|Y\|_{\mathbb{S}^p}^p\},$$

where

$$C=C_P\Big\{E[|\xi|^p]+C_P^3\Big\{\Big(E\Big[\Big(\int_0^T f_s\mathrm{d}s\Big)^p\Big]\Big)^{\frac{1}{2}}+\Big(\int_0^T u(s)\mathrm{d}s\Big)^{\frac{p}{2}}+$$
$$\varepsilon^{-\frac{p}{2}}\Big(\int_0^T v^2(s)\mathrm{d}s\Big)^{\frac{p}{2}}\Big\}+1\Big\}<\infty.$$

The proof is completed. □

Assume (H1) and (H2) hold. Then for the following BSDEs:

$$Y_t^n=\xi+\int_t^T g_n(s,Y_s^n,Z_s^n)\mathrm{d}s-\int_t^T Z_s^n\cdot\mathrm{d}B_s,\quad n\ge 1. \tag{4.7}$$

$$Y_t'=\xi+\int_t^T (f_s+u(s)|Y_s'|+v(s)|Z_s'|)\mathrm{d}s-\int_t^T Z_s'\cdot\mathrm{d}B_s. \tag{4.8}$$

The Proposition 4.3 and Lemma 2.2 deduce the above BSDE (4.7) has a unique L^p solution in $\mathbb{S}^p\times\mathbb{M}^p$ for any $n\ge 0$, denoted by (Y^n,Z^n). And BSDE (4.8) also has a unique L^p solution (Y',Z'). By Theorem 3.1, we have $\mathrm{d}P\times\mathrm{d}t$-a.s., $Y_t^1\le Y_t^n\le Y_t^{n+1}\le Y_t'$. Thus, there must exist an $(\mathcal{F}_t)$-progressively measurable process $(Y_t)_{t\in[0,T]}$ satisfying $\mathrm{d}P\times\mathrm{d}t$-a.s.,

$$\lim_{n\to+\infty}Y_t^n=Y_t.$$

Let $M=\sup_n\sup_{t\in[0,T]}|Y_t^n|$. Then we have

$$\forall n\ge 1,\ |Y_t^n|\le|Y_t|+|Y_t^1|,\ \mathrm{d}P\times\mathrm{d}t\text{-a.s.},$$
$$E\big[\sup_{t\in[0,T]}|Y_t|^p\big]\le E[|M|^p]<+\infty.$$

The following Proposition is very useful in the proof of Theorem 4.1.

Proposition 4.5 *$\{(Y^n,Z^n)\}_{n\ge1}$ is a Cauchy sequence in $\mathbb{S}^p\times\mathbb{M}^p$.*

Proof For each $m,n\ge 1$, $t\in[0,T]$, $\mathrm{d}P$-a.s.,

$$|Y_t^m-Y_t^n|^{p-1}f_t\le 2^{p-1}M^{p-1}f_t,$$
$$|Y_t^m-Y_t^n|^{p-1}u_t\le 2^{p-1}M^{p-1}u_t,$$
$$|Y_t^m-Y_t^n|^{p-1}v_t^2\le 2^{p-1}M^{p-1}v_t^2. \tag{4.9}$$

Furthermore, by the Hölder inequality, we can get

$$E\Big[\int_0^T M^{p-1}f_s\mathrm{d}s\Big]\le(E[M^p])^{\frac{p-1}{p}}\Big(E\Big[\Big(\int_0^T f_s\mathrm{d}s\Big)^p\Big]\Big)^{\frac{1}{p}}<+\infty,$$

$$E\Big[\Big(\int_0^T M^{p-1}u_s\mathrm{d}s\Big)^{\frac{p}{p-1}}\Big]\leq E[M^p]\Big(\int_0^T u_s\mathrm{d}s\Big)^{\frac{p}{p-1}}<+\infty,$$
$$E\Big[\Big(\int_0^T M^{p-1}v_s^2\mathrm{d}s\Big)^{\frac{p}{p-1}}\Big]\leq E[M^p]\Big(\int_0^T v_s^2\mathrm{d}s\Big)^{\frac{p}{p-1}}<+\infty. \tag{4.10}$$

Then by (4.9), (4.10) and the dominated convergence theorem, let $m,n\to+\infty$, it follows that

$$E\Big[\int_0^T |Y_t^m-Y_t^n|^{p-1}f_t\mathrm{d}t\Big]\to 0,$$
$$E\Big[\Big(\int_0^T |Y_t^m-Y_t^n|^{p-1}u_t\mathrm{d}t\Big)^{\frac{p}{p-1}}\Big]\to 0,$$
$$E\Big[\Big(\int_0^T |Y_t^m-Y_t^n|^{p-1}v_t^2\mathrm{d}t\Big)^{\frac{p}{p-1}}\Big]\to 0. \tag{4.11}$$

We first show $\{Y^n\}_{n\geq 1}$ is a Cauchy sequence in $\mathbb{S}^p$. By (ii) in Lemma 2.3 and (H1), we have

$$\begin{aligned}\|Y^m-Y^n\|_{\mathbb{S}^p}^p\leq & C_pE\Big[\int_0^T |Y_s^m-Y_s^n|^{p-1}(2f_s+u(s)(|Y_s^n|+|Y_s^m|)+\\ & v(s)(|Z_s^n|+|Z_s^m|))\mathrm{d}s\Big]\\ \leq & C_p\Big\{2E\Big[\int_0^T |Y_s^m-Y_s^n|^{p-1}f_s\mathrm{d}s\Big]+\\ & E\Big[\int_0^T |Y_s^m-Y_s^n|^{p-1}u(s)\cdot(|Y_s^n|+|Y_s^m|)\mathrm{d}s\Big]+\\ & E\Big[\int_0^T |Y_s^m-Y_s^n|^{p-1}v(s)(|Z_s^n|+|Z_s^m|)\mathrm{d}s\Big]\Big\}.\end{aligned} \tag{4.12}$$

Now we estimate the second term of (4.12), by the Hölder inequality and (4.11), while $m,n\to\infty$, we have

$$\begin{aligned}&E\Big[\int_0^T |Y_s^m-Y_s^n|^{p-1}u(s)(|Y_s^n|+|Y_s^m|)\mathrm{d}s\Big]\\ &\leq 2(E[M^p])^{\frac{1}{p}}\Big(E\Big[\Big(\int_0^T |Y_s^m-Y_s^n|^{p-1}u(s)\mathrm{d}s\Big)^{\frac{p}{p-1}}\Big]\Big)^{\frac{p-1}{p}}\to 0.\end{aligned} \tag{4.13}$$

The third term of (4.12) also converges to 0, while $m,n\to\infty$. Actually,

$$\begin{aligned}&E\Big[\int_0^T |Y_s^m-Y_s^n|^{p-1}v(s)(|Z_s^n|+|Z_s^m|)\mathrm{d}s\Big]\\ &\leq E\Big[\Big(\int_0^T |Y_s^m-Y_s^n|^{2p-2}v^2(s)\mathrm{d}s\Big)^{\frac{1}{2}}\Big(\int_0^T (|Z_s^n|+|Z_s^m|)^2\mathrm{d}s\Big)^{\frac{1}{2}}\Big]\\ &\leq \Big(E\Big[\Big(\int_0^T |Y_s^m-Y_s^n|^{2p-2}v^2(s)\mathrm{d}s\Big)^{\frac{p}{2p-2}}\Big]\Big)^{\frac{p-1}{p}}\cdot\Big(E\Big[\Big(\int_0^T (|Z_s^n|+|Z_s^m|)^2\mathrm{d}s\Big)^{\frac{p}{2}}\Big]\Big)^{\frac{1}{p}}.\end{aligned}$$

By (4.11) and the Hölder inequality, let $m,n\to\infty$, we have

$$\begin{aligned}&\Big(E\Big[\Big(\int_0^T |Y_s^m-Y_s^n|^{2p-2}v^2(s)\mathrm{d}s\Big)^{\frac{p}{2p-2}}\Big]\Big)^{\frac{p-1}{p}}\\ &\leq \Big(E\Big[\sup_{s\in[0,T]}|Y_s^m-Y_s^n|^{\frac{p}{2}}\Big(\int_0^T |Y_s^m-Y_s^n|^{p-1}v^2(s)\mathrm{d}s\Big)^{\frac{p}{2p-2}}\Big]\Big)^{\frac{p-1}{p}}\end{aligned}$$

$$
\begin{aligned}
&\le \Big(E\Big[\sup_{s\in[0,T]}|Y_s^m-Y_s^n|^p\Big]\Big)^{\frac{p-1}{2p}}\Big(E\Big[\Big(\int_0^T|Y_s^m-Y_s^n|^{p-1}v^2(s)\mathrm{d}s\Big)^{\frac{p}{p-1}}\Big]\Big)^{\frac{p-1}{2p}}\\
&\le C_p^1(E[M^p])^{\frac{p-1}{2p}}\Big(E\Big[\Big(\int_0^T|Y_s^m-Y_s^n|^{p-1}v^2(s)\mathrm{d}s\Big)^{\frac{p}{p-1}}\Big]\Big)^{\frac{p-1}{2p}}\\
&\to 0. \qquad (4.14)
\end{aligned}
$$

Moreover,

$$\Big(E\Big[\Big(\int_0^T(|Z_s^n|+|Z_s^m|)^2\mathrm{d}s\Big)^{\frac{p}{2}}\Big]\Big)^{\frac{1}{p}}<+\infty. \tag{4.15}$$

So, the above (4.14) and (4.15) show

$$\lim_{m,n\to\infty}E\Big[\int_0^T|Y_s^m-Y_s^n|^{p-1}v(s)(|Z_s^n|+|Z_s^m|)\mathrm{d}s\Big]\to 0. \tag{4.16}$$

Combining (4.11), (4.12), (4.13) and (4.16), we prove that $\|Y^m-Y^n\|_{\mathbb{S}^p}^p$ converges to zero as $m,n\to\infty$. Thus, $\{Y^n\}_{n\ge1}$ is a Cauchy sequence in $\mathbb{S}^p$.

By (H1) and the Hölder inequality, we have

$$
\begin{aligned}
&E\Big[\Big(\int_0^T|Y_s^m-Y_s^n||g_m(s,Y_s^m,Z_s^m)-g_n(s,Y_s^n,Z_s^n)|\mathrm{d}s\Big)^{\frac{p}{2}}\Big]\\
&\le E\Big[\Big(\int_0^T|Y_s^m-Y_s^n|(2f_s+u(s)(|Y_s^n|+|Y_s^m|)+v(s)(|Z_s^n|+|Z_s^m|))\mathrm{d}s\Big)^{\frac{p}{2}}\Big]\\
&\le E\Big[\Big(2\int_0^T|Y_s^m-Y_s^n|f_s\mathrm{d}s+2\int_0^T|Y_s^m-Y_s^n|u(s)M\mathrm{d}s+\\
&\quad\int_0^T|Y_s^m-Y_s^n|v(s)(|Z_s^n|+|Z_s^m|)\mathrm{d}s\Big)^{\frac{p}{2}}\Big]\\
&\le C_p^3\Big(E\Big[\sup_{s\in[0,T]}|Y_s^m-Y_s^n|^{\frac{p}{2}}\Big(\int_0^T f_s\mathrm{d}s\Big)^{\frac{p}{2}}\Big]+\\
&\quad E\Big[\sup_{s\in[0,T]}|Y_s^m-Y_s^n|^{\frac{p}{2}}M^{\frac{p}{2}}\Big(\int_0^T u(s)\mathrm{d}s\Big)^{\frac{p}{2}}\Big]+\\
&\quad E\Big[\sup_{s\in[0,T]}|Y_s^m-Y_s^n|^{\frac{p}{2}}\Big(\int_0^T v^2(s)\mathrm{d}s\Big)^{\frac{p}{4}}\Big(\int_0^T(|Z_s^n|^2+|Z_s^m|^2)\mathrm{d}s\Big)^{\frac{p}{4}}\Big]\Big)\\
&\le C_p^3(\|Y^m-Y^n\|_{\mathbb{S}^p}^{\frac{p}{2}}\Big(E\Big[\Big(\int_0^T f_s\mathrm{d}s\Big)^p\Big]\Big)^{\frac{1}{2}}+\\
&\quad\|Y^m-Y^n\|_{\mathbb{S}^p}^{\frac{p}{2}}(E[M^p])^{\frac{1}{2}}\Big(\int_0^T u(s)\mathrm{d}s\Big)^{\frac{p}{2}}+\\
&\quad\|Y^m-Y^n\|_{\mathbb{S}^p}^{\frac{p}{2}}\Big(E\Big[\Big(\int_0^T|Z_s^n|^2\mathrm{d}s\Big)^{\frac{p}{2}}\Big]+E\Big[\Big(\int_0^T|Z_s^m|^2\mathrm{d}s\Big)^{\frac{p}{2}}\Big]\Big)^{\frac{1}{2}}\Big(\int_0^T v^2(s)\mathrm{d}s\Big)^{\frac{p}{4}}\Big)\\
&\le K\cdot\|Y^m-Y^n\|_{\mathbb{S}^p}^{\frac{p}{2}},
\end{aligned}
$$

where K is a positive integer. Due to (ii) in Lemma 2.3 and that $\{Y^n\}_{n\ge1}$ is Cauchy sequence, we can show that $\{Z^n\}_{n\ge1}$ is also a Cauchy one.

Then $\{(Y^n,Z^n)\}_{n\ge1}$ is a Cauchy sequence in $\mathbb{S}^p\times\mathbb{M}^p$.

The proof is completed. $\square$

We denote Z as the limit of Z^n in $\mathbb{M}^p$. Now we can prove Theorem 4.1.

Proof Because $\{Z^n\}$ is a Cauchy sequence in $\mathbb{M}^p$, we select a subsequence $\{Z^{n_k}\}$ of $\{Z^n\}$ (For simplicity, it is still written as $\{Z^n\}$) satisfying

$$\mathbf{E}\Big[\Big(\int_0^T |Z_s^n - Z_s|^2 \mathrm{d}s\Big)^{\frac{p}{2}}\Big] < \frac{1}{2^n},$$

then

$$\mathbf{E}\Big[\Big(\int_0^T \sup_{n\geq 1}|Z_s^n - Z_s|^2 \mathrm{d}s\Big)^{\frac{p}{2}}\Big] \leq \mathbf{E}\Big[\sum_{n=1}^{\infty}\Big(\int_0^T |Z_s^n - Z_s|^2 \mathrm{d}s\Big)^{\frac{p}{2}}\Big] \leq \sum_{n=1}^{\infty}\frac{1}{2^n} < \infty.$$

Therefore

$$\begin{aligned}
&\mathbf{E}\Big[\Big(\int_0^T \sup_{n\geq 1}|Z_s^n|^2 \mathrm{d}s\Big)^{\frac{p}{2}}\Big] \\
&\leq C_p\Big\{\mathbf{E}\Big[\Big(\int_0^T \sup_{n\geq 1}|Z_s^n - Z_s|^2 \mathrm{d}s\Big)^{\frac{p}{2}}\Big] + \mathbf{E}\Big[\Big(\int_0^T |Z_s|^2 \mathrm{d}s\Big)^{\frac{p}{2}}\Big]\Big\} \\
&< \infty.
\end{aligned} \tag{4.17}$$

Since $\|Y^n - Y\|_{\mathbb{S}^p} \to 0, \|Z^n - Z\|_{\mathbb{M}^p} \to 0$, as $n \to +\infty$, there exist two subsequences $\{Y^n\}$ and $\{Z^n\}$ respectively, satisfying

$$\begin{aligned}
&Y^n \to Y, \quad \text{a.e., a.s.,} \\
&Z^n \to Z, \quad \text{a.e., a.s..}
\end{aligned}$$

In view of Proposition 4.3 (iv), while $n \to +\infty$, it is easy to show that

$$g_n(t, Y_t^n, Z_t^n) \to g(t, Y_t, Z_t), \quad \text{a.e., a.s.}$$

Moreover, as

$$g_n(t, Y_t^n, Z_t^n) \leq f_t(w) + u(t)|Y_t^n| + v(t)|Z_t^n| \leq f_t(w) + u(t)\sup_{n\geq 1}|Y_t^n| + v(t)\sup_{n\geq 1}|Z_t^n|,$$

and

$$E\Big[\Big(\int_0^T (f_t(w) + u(t)\sup_{n\geq 1}|Y_t^n| + v(t)\sup_{n\geq 1}|Z_t^n|)\mathrm{d}t\Big)^p\Big] < +\infty,$$

by the dominated convergence theorem, we have

$$\lim_{n\to\infty}\mathbf{E}\Big[\Big(\int_0^T |g_n(t, Y_t^n, Z_t^n) - g(t, Y_t, Z_t)|\mathrm{d}t\Big)^p\Big] = 0.$$

By taking limits on n in BSDEs (4.7), we deduce that $(Y_t, Z_t)_{t\in[0,T]}$ is a solution of the BSDE (1.1).

The minimal solution is ensured by Theorem 3.1 (Comparison Theorem).

The Proof of Theorem 4.1 is completed. □

Acknowledgements The authors thank the referees and the editor for their helpful comments and suggestions.

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