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On Derivations of Bounded Hyperlattices

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Abstract In this paper we introduce derivations in hyperlatices and derive some basic properties of them. Also, some properties of differential hyperideals and differential hypercongruences are studied. Further we prove that for an injective strong differential hyperlattice (L, d) and for a strongly differential hypercongruence R of (L, d), the quotient hyperlattice $(L/R, g)$ is an injective strongly differential hyperlattice, where g is an injective strong derivation on L/R induced by d.

Keywords hyperlattice; derivation; differential hyperideal; differential hypercongruence

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1. Introduction

The concept of hyperstructures was introduced by Marty [1] at 8th Congress of Scandinavian Mathematicians in 1934. Till now, the hyperstructures are studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics [2,3]. Algebraic hyperstructures are suitable generalizations of classical algebraic structures. Since then, there appeared many components of hyperalgebras such as hypergroups in [4,5] hyperrings in [6–8] etc. Konstantinidou and Mittas have introduced the concept of hyperlattices in [9] and superlattices in [10], also see [11,12]. The notion of hyperlattices is a generalization of the notion of lattices and there are some intimate connections between hyperlattices and lattices. In particular, Rasouli and Davvaz further studied the theory of hyperlattices and obtained some interesting results [13,14], which enrich the theory of hyperlattices.

The notion of derivations, introduced from the analytic theory, is helpful for the research of structures and properties in algebraic systems. Serval authors [15,16] studied derivations in rings and near rings. Also Jun and Xin [17] applied the notion of derivation in rings and near-rings theory to BCI-algebras. In [18], Xin, Li and Lu introduced the concept of derivations on lattices and characterized modular lattices and distributive lattices by isotone derivations. From the motivation of derivations, Vougiouklis [19,20] introduced a hyperoperation called theta hyperoperation and studied H_ν-structures. Jan Chvalina et al. [21] introduced a hyperoperation

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$*$ on a differential ring R so that $(R, *)$ is a hypergroup. In [21], Asokkumar introduced the notion of derivations on Krasner hyperrings. In [22], Kamali Ardekani and Davvaz defined the notion of derivation on milticative hyperring. Now, we define the notion of derivation on hyperlattice.

2. Preliminaries

In this section, we first recall some definitions and basic results which were given in [3,13,14].

Definition 2.1 ([13]) *Let L be a non empty set and $\vee : L \times L \longrightarrow P^*(L)$ be a hyperoperation, where $P(L)$ is a power set of L and $P^*(L) = P(L) - \emptyset$ and $\wedge : L \times L \longrightarrow L$ be an operation. Then $(L, \vee, \wedge)$ is a hyperlattice if for all $a, b, c \in L$:*

(1) $a \in a \vee a$, $a \wedge a = a$;

(2) $a \vee b = b \vee a$, $a \wedge b = b \wedge a$;

(3) $(a \vee b) \vee c = a \vee (b \vee c)$; $(a \wedge b) \wedge c = a \wedge (b \wedge c)$;

(4) $a \in [a \wedge (a \vee b)] \cap [a \vee (a \wedge b)]$;

(5) *if* $a \in a \vee b$, *then* $a \wedge b = b$.

For all non empty subsets A and B of L, $A \wedge B = \{a \wedge b | a \in A, b \in B\}$, $A \vee B = \{a \vee b | a \in A, b \in B\}$.

Let L be a hyperlattice. For each $x, y \in L$, we define two relations on L as follows: $(x, y) \in \leq$ if and only if $x = x \wedge y$, $(x, y) \in \preceq$ if and only if $y \in x \vee y$. For all nonempty subsets A and B of L, we define $A \leq B$ if there exist $a \in A$ and $b \in B$ such that $a \leq b$.

A zero of a hyperlattice L is an element 0 with $0 \leq x$ for all $x \in L$. A unit, 1, satisfies $x \leq 1$ for all $x \in L$, so we can conclude that there are at most one zero and at most one unit. A bounded hyperlattice is one that has both 0 and 1. In a bounded hyperlattice L, y is a complement of x if $x \wedge y = 0$ and $1 \in x \vee y$. The set of complement elements of x is denoted by x^c too. A complemented hyperlattice is a bounded hyperlattice in which every element has at least one complement.

Definition 2.2 ([3]) *An element $a \in L$ is called scalar element if for all $x \in L$ the set $a \vee x$ has only one element.*

Proposition 2.3 ([13]) *Let $(L, \vee, \wedge)$ be a hyperlattice. Then for all $x, y, z \in L$ and for all nonempty subsets X, Y, Z of L the following hold:*

(1) $\leq = \preceq$ *and $(L, \leq)$ is a poset. Also, we can replace Definition 2.1 (4) by* $x \in x \wedge (x \vee y)$;

(2) $x \wedge y \leq x, y \leq x \vee y$;

(3) $X \subseteq (X \vee X) \cap (X \wedge X)$;

(4) $X \vee (Y \vee Z) = (X \vee Y) \vee Z$ *and* $X \wedge (Y \wedge Z) = (X \wedge Y) \wedge Z$;

(5) *If* $x \leq y$, *then* $x \wedge z \leq y \wedge z$;

(6) *If* $x, y \in x \vee y$, *then* $x = y$, *so* $x \vee y = L$ *implies that* $x = y$;

(7) *If* $x \vee y = \{0\}$, *then* $x = y = 0$;

(8) *If 0 is a scalar element of L, then* $0 \vee 0 = 0$, $x \vee 0 = \{x\}$.

Definition 2.4 ([13]) *A subhyperlattice of a hyperlattice L is a nonempty subset of L which is closed under the hyperoperation $\vee$ and operation $\wedge$ as defined in L.*

Definition 2.5 ([13]) *A hyperlattice $(L, \vee, \wedge)$ is said to be a distributive if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ holds for every $a, b, c \in L$.*

Definition 2.6 ([13]) *A bounded hyperlattice $(L, \wedge, \vee, 0, 1)$ is said to be a hyperboolean algebra if L is a distributive and complemented.*

Definition 2.7 ([13]) *Let $(L, \wedge, \vee)$ be a hyperlattice and I be a nonempty subset of L. Then I is called a hyperideal of L when:*

(i) I is a subhyperlattice;

(ii) $x \in I$ and $y \in L$ imply $x \wedge y \in I$.

Definition 2.8 ([14]) *Let P be proper ideal of hyperlattice L. P is called a prime ideal of L if x, $y \in L$ and $x \wedge y \in P$ implies that $x \in P$ or $y \in P$.*

Definition 2.9 ([13]) *A map f from a hyperlattice L_1 to a hyperlattice L_2 is called a homomorphism if it satisfies the following condition:*

$$f(x \wedge y) = f(x) \wedge f(y), f(x \vee y) = f(x) \vee f(y).$$

Definition 2.10 *A hyperlattice L is said to be 2-torsion free if $0 \in x \vee x$ for $x \in L$ implies $x = 0$.*

Example 2.11 Let $L = \{0, a, b, 1\}$ and define $\wedge$ and $\vee$ by the following Cayley tables

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	a	0	b	b
1	0	a	b	1

$\vee$	0	a	b	1
0	L	$\{a, 1\}$	$\{b, 1\}$	$\{1\}$
a	$\{a, 1\}$	$\{b, 1\}$	$\{1\}$	$\{1\}$
b	$\{b, 1\}$	$\{1\}$	$\{b, 1\}$	$\{1\}$
1	$\{1\}$	$\{1\}$	$\{1\}$	1

Table 1 Definition of $\vee$ and $\wedge$ in Example 2.11

Then $(L, \wedge, \vee)$ is a hyperlattice. It is easy to check that L is a 2-torsion free hyperlattice.

Definition 2.12 ([13]) *Let R be an equivalence relation on a nonempty set L and X, $Y \subseteq L$. Then:*

(i) $X\overline{R}Y$ if and only if for all $x \in X$ there exists $y \in Y$ such that xRy and for all $y \in Y$ there exists $x \in X$ such that xRy.

(ii) $X\overline{\overline{R}}Y$ if and only if for all $x \in X$ and $y \in Y$ we have xRy.

(iii) Let L be a nonempty set and R be an equivalence relation on L. In this article, for each $x \in L$ the equivalence class of x is denoted by $[x]$ and is defined with $[x] = \{y \in L|xRy\}$. Also, the quotient set of L with respect to R is denoted by L/R and is defined with $L/R = \{[x]|x \in L\}$.

Definition 2.13 ([13]) *Let L be a hyperlattice. An equivalence relation R is said to be a hypercongruence relation on L if xRy implies $x \vee z\overline{R}y \vee z$ and $x \wedge z\overline{\overline{R}}y \wedge z$, for all $x, y, z \in L$.*

Theorem 2.14 ([13]) *Let $(L, \vee, \wedge)$ be a hyperlattice. If R is a hypercongruence on L, then L/R is a hyperlattice.*

Let S be a non-empty subset of a bounded hyperlattice L. Define $C(S) = \{x \in L | x \wedge S = 0\}$. The set $C(S)$ is similar to the annihilator of hyperring R. So we call the set $C(S)$ the annihilator of L.

Proposition 2.15 *Let L be a bounded hyperlattice and S be a non-empty subset of L. Then the following hold:*

(1) If L is a bounded distributive hyperlattice, then $C(S)$ is a hyperideal of L.

(2) If S_1 and S_2 are subsets of L such that $S_1 \subseteq S_2$, then $C(S_2) \subseteq C(S_1)$.

Proof (1) Assume that L is a bounded distribute hyperlattice. Since $0 \in C(S)$, we see that $C(S)$ is non-empty. Let x, $y \in C(S)$. Then $x \wedge s = 0$, $y \wedge s = 0$ for all $s \in S$. Now, $(x \wedge y) \wedge s = (x \wedge s) \wedge (x \wedge s) = 0 \wedge 0 = 0$. Also, $(x \vee y) \wedge s = (x \wedge s) \vee (y \wedge s) = 0 \vee 0 = 0$. Thus for any x, $y \in L$, $x \wedge y \in C(S)$, $x \vee y \subseteq C(S)$. So, $C(S)$ is a subhyperlattice of L. For any $x \in L$, $y \in C(S)$. We have $x \wedge (y \wedge s) = (x \wedge y) \wedge s = 0$. Thus $x \wedge y \in C(S)$. So, $C(S)$ is a hyperideal of L.

(2) Let $x \in C(S_2)$. Then $S_2 \wedge x = 0$. That is, $s_2 \wedge x = 0$ for all $s_2 \in S_2$. This means that x annihilates all elements of S_2. In particular, x annihilates all elements of S_1. Therefore, $x \in C(S_1)$. $\Box$

3. Derivations of bounded hyperlattices

In what follows, let L denote a bounded hyperlattice with 0 being a scalar element unless otherwise specified.

In this section we define derivation and strong derivation of hyperlattice and give examples.

Definition 3.1 *Let L be a hyperlattice. A mapping $d : L \longrightarrow L$ such that, for all $x, y \in L$, we have*

(1) $d(x \vee y) \subseteq d(x) \vee d(y)$, (2) $d(x \wedge y) \in (d(x) \wedge y) \vee (x \wedge d(y))$

is said to be a derivation on L, and the pair (L, d) is said to be a differential hyperlattice, or more precisely, a hyperlattice with a derivation. If the map d such that $d(x \vee y) = d(x) \vee d(y)$ for all $x, y \in L$ and satisfies the condition (2), then d is called a strong derivation of L. In this case, the pair (L, d) is called a strongly differential hyperlattice.

Example 3.2 (1) Let L be a hyperlattice and $d : L \longrightarrow L$ is a map defined by $d(x) = 0$ for all $x \in L$. Then d is a derivation on L. This derivation is a strong derivation, called the trivial derivation.

(2) Let L be a hyperlattice. Then the identity function, $d(x) = x$ for every $x \in L$, is a strong derivation of L and is called the identity derivation.

(3) Let $L = \{0, a, b, 1\}$ and define $\wedge$ and $\vee$ by the following Cayley tables

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\vee$	0	a	b	1
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{1\}$
a	$\{a\}$	$L-\{b\}$	$\{0,1\}$	$L-\{a\}$
b	$\{b\}$	$\{0,1\}$	$L-\{a\}$	$L-\{b\}$
1	$\{1\}$	$L-\{a\}$	$L-\{b\}$	L

Table 2 Definition of $\vee$ and $\wedge$ in Example 3.2 (3)

Then $(L, \wedge, \vee)$ is a hyperlattice. Define a map $d : L \longrightarrow L$ by $d0 = 0$, $da = db = d1 = 1$. Then we can see that d is a derivation on L.

(4) Consider the hyperlattice $L = \{0, a, b, 1\}$ with meet $\wedge$ and hyperjoin $\vee$ defined as follows.

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\vee$	0	a	b	1
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{1\}$
a	$\{a\}$	$\{0,a\}$	$\{1\}$	$\{b,1\}$
b	$\{b\}$	$\{1\}$	$\{0,b\}$	$\{a,1\}$
1	$\{1\}$	$\{b,1\}$	$\{a,1\}$	L

Table 3 Definition of $\vee$ and $\wedge$ in Example 3.2 (4)

It is clear that the map $d : L \longrightarrow L$ defined by $d0 = da = 0$, $db = d1 = b$ is a derivation of L.

Definition 3.3 *Let d be a derivation on L. If $x \leq y$ implies $dx \leq dy$ for all $x, y \in L$, d is called an isotone derivation.*

Example 3.4 Let L be an hyperlattice as in Example 3.2. It is easy to check that d is an isotone derivation of L.

Proposition 3.5 *Let d be a derivation on L. Then the following hold: for all $x, y \in L$,*

(1) $d0 = 0$;

(2) $dx \in dx \vee (x \wedge d1)$;

(3) d is an isotone derivation on L;

(4) $d(x \wedge y) = dx \wedge dy$;

(5) $d1 \in d1 \vee d1$.

Proof (1) It is clear that $d(0) = 0$.

(2) $d(x) = d(x \wedge 1) \in (d(x) \wedge 1) \vee (x \wedge d(1)) = d(x) \vee (x \wedge d(1))$.

(3) If $x \leq y$, then $y \in x \vee y$. Thus $d(y) \in d(x \vee y) \subseteq d(x) \vee d(y)$. By Proposition 3.1 (1) we get $dx \leq dy$.

(4) By (2), we have $dx \geq d1 \wedge x$ for any $x \in L$. Therefore, $d(x \wedge y) \geq d1 \wedge (x \wedge y) = d1 \wedge d1 \wedge (x \wedge y) = dx \wedge dy$. On the other hand, since d is an isotone derivation. We have

$d(x \wedge y) \leq dx$, $d(x \wedge y) \leq dy$, and so $d(x \wedge y) \leq dx \wedge dy$. Thus $d(x \wedge y) = d(x) \wedge d(y)$ for any $x, y \in L$.

(5) By Definition 3.1. □

Proposition 3.6 *Let d be a derivation on L. We define* $\ker(d) = \{x \in L | d(x) = 0\}$. *Then the following hold:*

(1) $\ker(d)$ *is a subhyperlattice of L;*

(2) $\ker(d)$ *is a hyperideal of L.*

Proof (1) Since $d0 = 0$, we see that $\ker(d)$ is non-empty. Let $x, y \in \ker(d)$. Then $d(x) = d(y) = 0$. Now, $d(x \vee y) \subseteq d(x) \vee d(y) = 0 \vee 0 = 0$. Also, $d(x \wedge y) \in (d(x) \wedge y) \vee (x \wedge d(y)) = 0 \vee 0 = 0$. Thus for any $x, y \in L$, $x \wedge y \in \ker(d)$, $x \vee y \subseteq \ker(d)$. So, $\ker(d)$ is a subhyperlattice of L.

(2) By (1) we have for any $x, y \in L$, $x \vee y \subseteq \ker(d)$. If $x \leq \ker(d)$, then there exists a $y \in \ker(d)$ such that $x \leq y$. That is, $y \in x \vee y$. Now, $0 = dy \in d(x \vee y) \subseteq d(x) \vee d(y) = d(x) \vee 0 = dx$. Thus $x \in \ker(d)$. So, $\ker(d)$ is a hyperideal of L. □

Definition 3.7 *Let d be a derivation on L. d is said to be a prime derivation if* $\ker(d)$ *is a prime hyperideal of L.*

Example 3.8 Let d be a derivation as in Example 3.2 (4). It is easy to check that d is prime derivation of L.

Proposition 3.9 *Let d be a prime derivation of a 2-torsion free hyperlattice L. If $d^2 = 0$, then $d = 0$.*

Proof Let $d^2 = 0$. Suppose $d \neq 0$, then there exists an element $a \in L$ such that $d(a) \neq 0$. Then for every $y \in L$,

$$\begin{aligned} d^2(a \wedge y) = 0 = d(d(a \wedge y)) &\in d((d(a) \wedge y) \vee (a \wedge d(y))) \subseteq d(d(a) \wedge y)) \vee d((a \wedge d(y))) \\ &\subseteq (d^2(a) \wedge y) \vee (d(a) \wedge d(y)) \vee ((d(a) \wedge d(y)) \vee (a \wedge d^2(y))) \\ &= (d(a) \wedge d(y)) \vee (d(a) \wedge d(y)). \end{aligned}$$

Since L is 2-torsion free hyperlattice, $d(a) \wedge d(y) = 0$. Since d is a prime derivation, by Definition 3.7, $d(y) = 0$ for every $y \in L$. That is we get $d = 0$, which is contradiction to the assumption. Hence $d = 0$. □

Proposition 3.10 *Let d_1, d_2 be prime derivation of a 2 torsion free hyperlattice L. If $d_1 d_2 = 0$, then $d_1 = 0$ or $d_2 = 0$.*

Proof For $x, y \in L$ we have

$$\begin{aligned} d_1 d_2(x \wedge y) = 0 = d_1(d_2(x \wedge y)) &\in d_1((d_2(x) \wedge y) \vee (x \wedge d_2(y)) \subseteq d_1(d_2(x) \wedge y) \vee d_1(x \wedge d_2(y)) \\ &\subseteq ((d_1 d_2(x) \wedge y) \vee (d_2(x) \wedge d_1(y))) \vee ((d_1(x) \wedge d_2(y)) \vee (x \wedge d_1 d_2(y))) \\ &= (d_2(x) \wedge d_1(y)) \vee (d_1(x) \wedge d_2(y)). \end{aligned}$$

Replace x by $d_2(x)$, we get $0 \in (d_2(d_2(x)) \wedge d_1(y)) \vee (d_1(d_2(x)) \wedge d_2(y)) = d_2^2(x) \wedge d_1(y)$. Now, since d_1 is a prime derivation. One can obtain $d_1 = 0$ or $d_2^2 = 0$. If $d_2^2 = 0$, then by the Proposition 3.9, we have $d_2 = 0$. □

Definition 3.11 *A derivation d is said to be contractive if $dx \leq x$ for any $x \in L$.*

Example 3.12 In Example 3.2 (4), it is easy to check that d is a contractive derivation of L.

Proposition 3.13 *Let d be a contractive derivation on L. Then the following hold:*

(1) $dx = x \wedge d1$ for any $x \in L$;

(2) If I is a hyperideal of L, then $dI \subseteq I$;

(3) $d^2 = d$;

(4) d is a co-closure operator on L;

Proof (1) Since d is isotone, $dx \leq d1$. Note that $dx \leq x$, we can get $dx \leq d1 \wedge x$. By Proposition 3.5 (2), we have $dx \geq d1 \wedge x$. Thus $dx = x \wedge d1$ for any $x \in L$.

(2) Assume that $y \in dI$. Then there exists an $x \in I$ such that $y = dx \leq x$. Since I is a hyperideal of L, we have $y \in I$.

(3) On the one hand $d^2x = d(dx) = d(x \wedge dx) \in (dx \wedge dx) \vee (x \wedge d^2x) = dx \vee d^2x$, then $dx \leq d^2x$. On the other hand d is a contractive derivation, therefore $d^2x = dx$.

(4) Clearly. □

Define $F_d(L) = \{x \in L | dx = x\}$.

Proposition 3.14 *Let d be a contractive derivation on L. If $y \leq x$ and $dx = x$, then $dy = y$.*

Proof Assume that $y \leq x$. Then $y = x \wedge y$. Thus $dy = d(x \wedge y) \in (dx \wedge y) \vee (x \wedge dy) = y \vee dy$. ie., $y \leq dy$ and hence $dy = y$. □

Theorem 3.15 *Let d be a contractive derivation on L. Then the following conditions are equivalent:*

(1) d is a strong derivation on L;

(2) F_d is a hyperideal of L.

Proof (1)⇒(2). By Propositions 3.5 (4), 3.14 and $d(x \vee y) = dx \vee dy$, we can get that $F_d(L)$ is a hyperideal of L.

(2)⇒(1). Since $F_d(L)$ is a hyperideal of L, $x \vee y \in F_d(L)$ for any x, $y \in L$. Thus $d(x \vee y) = dx \vee dy$. Therefore, d is a strong derivation on L. □

Proposition 3.16 *Let $(L, \wedge, \vee)$ be a distributive hyperlattice. For any $a \in L$, define the self mapping $d_a : L \longrightarrow L$ by $d_a(x) = t$ if and only if $t = a \wedge x$ for all $x \in L$. Then d_a is a strong derivation on L.*

Proof Let $x, y \in L$. Suppose that $d_a(x \wedge y) = t$. Then $t = a \wedge (x \wedge y) = (a \wedge x) \wedge y = (x \wedge a) \wedge y = x \wedge (a \wedge y)$. Hence $t = x \wedge c$ for $c = a \wedge y$. Now $c = a \wedge y$ implies $d_a(y) = c$. Therefore

$t = x \wedge c = x \wedge d_a(y)$. Thus $d_a(x \wedge y) = x \wedge d_a(y)$. Similarly, we have $d_a(x \wedge y) = d_a(x) \wedge y$. Hence $d_a(x \wedge y) \in (d_a(x) \wedge y) \vee (x \wedge d_a(y)$.

By $d_a(x \wedge y) = d_a(x) \wedge y$, we have $d_a(x) = x \wedge d_a(1)$. Thus $d_a(x \vee y) = (x \vee y) \wedge d_a(1) = (x \wedge d_a(1)) \vee (y \wedge d_a(y)) = d_a(x) \vee d_a(y)$ for any $x, y \in L$. By Definition 2.1, we get that d_a is a strong derivation of L. $\Box$

Let L be a hyperlattice and $a \in L$. Consider the map d_a defined by $d_a(x) = a \wedge x$ for all $x \in L$. Denote $D(L) = \{d_a | a \in L\}$. In the following theorem, we will discuss the algebraic structures of $D(L)$.

Theorem 3.17 *If L is a hyperboolean lattice, so is $(D(L), \sqcap, \sqcup, d_0, d_1)$, where for any $d_a, d_b \in D(L)$, we define $d_a \sqcap d_b$ by $(d_a \sqcap d_b)x = (d_a x) \wedge (d_b x)$, $d_a \sqcup d_b$ by $(d_a \sqcup d_b)x = (d_a x) \vee (d_b x)$ for any $x \in L$.*

Proof We have $(d_a \sqcup d_b)x = d_a(x) \vee d_b(x) = (a \wedge x) \vee (b \wedge x) = (a \vee b) \wedge x = (a \vee b) \wedge x = d_{a \vee b}(x)$, for any $d_a, d_b \in D(L), x \in L$. Since $a, b \in L$, $a \vee b \subseteq L$. Thus $(d_a \sqcup d_b)x = d_{a \vee b}x$.

For any $d_a, d_b \in D(L), x \in A$, $(d_a \sqcap d_b)x = d_a(x) \wedge d_b(x) = (a \wedge x) \wedge (b \wedge x) = (a \wedge b) \wedge x = d_{a \wedge b}(x)$. Since $a, b \in L$, $a \wedge b \in L$. Thus $(d_a \sqcap d_b)x = d_{a \wedge b}x$. So the definitions of "$\sqcup$" and "$\sqcap$" are well defined. It is not difficult to prove that $(D(L), \sqcap, \sqcup, d_0, d_1)$ is a bounded lattice. So we omit the proof of this. $\Box$

Moreover, $(d_a \sqcup (d_b \sqcap d_c))x = d_a x \vee (d_b x \wedge d_c x) = ((d_a \sqcup d_b) \sqcap (d_a \sqcup d_c))x$, for any $d_a, d_b, d_c \in D(L), x \in L$. Thus $d_a \sqcup (d_b \sqcap d_c) = (d_a \sqcup d_b) \sqcap (d_a \sqcup d_c)$.

Therefore $(D(L), \sqcup, \sqcap, d_0, d_1)$ is a bounded distributive hypelattice.

Since L is a hyperboolean lattice, for any $a \in L$, there exist a, $b \in L$ satisfying $a \wedge b = 0$ and $1 \in a \vee b$.

For any $d_a \in D(L)$, there exists a $d_b \in D(L)$ such that $(d_a \sqcap d_b)(x) = d_{a \wedge b}(x) = d_0(x)$, $d_1(x) \in d_{a \vee b}(x) = (d_a \sqcup d_b)(x)$. That is, for any $d_a \in D(L)$, there exists a d_b such that $d_0 = d_a \sqcap d_b$, $d_1 \in d_a \sqcup d_b$.

By Definition 2.6, we get that $(D(L), \sqcap, \sqcup, d_0, d_1)$ is a hyperboolean lattice.

4. Differential hyperideals and hypercongruences

In this section, we concentrate ourselves to differential hyperideals and hypercongruences on differential hyperlattice.

Definition 4.1 *Let d be a derivation of a hyperlattice L. A subset S is called d-invariant if $x \in S$ implies $dx \in S$.*

Note that $\emptyset$ and L are the d-invariant subsets of L.

Example 4.2 In the Example 3.2 (3), $\{0, 1\}$ is the d-invariant subset of L.

Definition 4.3 *Let d be a derivation of a hyperlattice L. A hyperideal I is said to be a differential hyperideal if I is the d-invariant subset of hyperlattice L.*

Let us denote by $I_d(L)$ the set of all differential hyperideals of a hyperlattice.

Example 4.4 For every differential hyperlattice L, $[0]$ is a differential hyperideal.

Theorem 4.5 *Let L be a differential hyperlattice. Then for any subset S of L, $C(S)$ is a differential hyperideal of L.*

Proof If $x \in C(S)$, then $S \wedge x = 0$. Now, for $s \in S$, $0 = d(s \wedge x) \in (d(s) \wedge x) \vee (s \wedge d(x))$. Meeting by s from the right, we get $0 \in (d(s) \wedge x \wedge s) \vee (s \wedge d(x) \wedge s)$. Since $s \wedge x = 0$, we have $x \wedge s = 0$. Therefore, $s \wedge d(x) \wedge s = 0$. Meet by $d(x)$ from the right, we get $s \wedge d(x) \wedge s \wedge d(x) = 0$. That is, $s \wedge d(x) = 0$. This means that $d(x) \in C(S)$. Thus we have $d(C(S)) \subseteq C(S))$. $\square$

By Theorem 4.5, we know that the existence of a differential hyperideal in a hyperlattice. As an illustration we have the following example.

Example 4.6 Consider the Example 3.2 (4). Now, $C(0, b) = \{0, a\}$ is a hyperideal of L. Since $d(C(0, b)) = d(\{0, a\}) = \{0\} \in (C(0, b))$, we can see that $C(0, b) = \{0, a\}$ is a differential hyperideal of L.

Theorem 4.7 *Let d be a derivation of a hyperlattice $(L, \wedge, \vee)$. Then $(I_d(L), \vee, \wedge)$ is a hyperlattice.*

Proof Let C, $D \in I_d(L)$ and $x \in C \vee D$. Then $x \in c \vee d$ for some $c \in C$ and $d \in D$. Hence $d(x) \in d(c \vee d) \subseteq d(c) \vee d(d) \subseteq C \vee D$. Therefore, $C \vee D$ is d-invariant. It is clear that $C \subseteq C \vee C$ for any $C \subseteq L$. Again, we have $x = c \vee d = d \vee c$. Hence, $C \vee D = D \vee C$. In a similar way, we can prove that $(C \vee D) \vee E = C \vee (D \vee E)$ for any C, D, $E \subseteq I_d(L)$.

Let C, $D \in I_d(L)$ and $x \in C \wedge D$. Then $x = c \wedge d$ for some $c \in C$ and $d \in D$. Hence $d(x) = d(c \wedge d) \in (d(c) \wedge d) \vee (c \wedge d(d)) \subseteq C \wedge D$. Therefore, $C \wedge D$ is d-invariant. It is clear that $C \subseteq C \wedge C$ for any $C \subseteq L$. Again, we have $x = c \wedge d = d \wedge c$. Hence, $C \wedge D = D \wedge C$. In a similar way, we can prove that $(C \wedge D) \wedge E = C \wedge (D \wedge E)$ for any C, D, $E \subseteq I_d(L)$.

For any $c \in C$, $d \in D$. It is clear that $c \in c \wedge (c \vee d)$. Hence $C \subseteq C \vee (C \wedge D)$. We can prove that if $D \in C \vee D$, then $C = C \wedge D$ for any C, $D \in I_d(L)$. $\square$

Proposition 4.8 *Let d be a strong derivation of L. Define a relation R_d on L by xR_dy if and only if $d(x) = d(y)$ for all $x, y \in L$. Then R_d is a hypercongruence on L.*

Proof Clearly, R_d is an equivalence relation on L. Let (a, b), $(c, d) \in R_d$. Then $d(a) = d(b)$ and $d(c) = d(d)$. It is clear that $d(a) \wedge d(c) R_d d(b) \wedge d(d)$ since $d(a) \wedge d(c) = d(b) \wedge d(d)$.

Let $x \in a \vee c$. Then $d(x) \in d(a \vee c) \subseteq d(a) \vee d(c) = d(b) \vee d(d) = d(b \vee d)$. Hence $d(x) = d(y)$ for some $y \in b \vee d$. Therefore, $a \wedge c\overline{\overline{R_d}}b \wedge d$, $a \vee c\overline{R_d}b \vee d$. $\square$

Definition 4.9 *Let d be a derivation of a hyperlattice L. A hypercongruence R is said to be a differential hypercongruence if it satisfies the property: xRy implies $d(x)Rd(y)$.*

Note that if d is the identity derivation on L, every hypercongruence is a differential congruence on L.

Corollary 4.10 *If R is a differential hypercongrue on L, then L/R is a hyperlattice.*

Theorem 4.11 *Let d be an injective strong derivation of a hyperlattice L. If R is a differential hypercongruence on L, then there exists an injective strong derivation on L/R.*

Proof By Corollary 4.10, L/R is a hyperlattice. Define $g : L/R \longrightarrow L/R$ by $g([x]) = [d(x)]$ for all $[x] \in L/R$. If $[x]$, $[y] \in L/R$ such that $g([x]) = g([y])$, then $[d(x)] = [d(y)]$. Hence $d(x)Rd(y)$. Since d is an injective derivation on L, we get xRy. Thus $[x] = [y]$. Therefore, g is an injective.

Let $[x]$, $[y] \in L/R$. Now, $g([x] \vee [y]) = g(\{[u]|u \in x \vee y\}) = \{d(u)|u \in x \vee y\}$. Further, $g([x]) \vee g([y]) = [d(x)] \vee [d(y)] = \{[v]|v \in d(x) \vee d(y)\} = \{[v]|v \in d(x \vee y)\} = \{[d(u)]|u \in x \vee y\}$. Since d is a strong derivation of L, we get $g([x] \vee [y]) = g([x]) \vee g([y])$.

Also, we have $g([x] \wedge [y]) = g([x \wedge y]) = [d(x \wedge y)]$. But $(g([x]) \wedge [y]) \vee ([x] \wedge g([y])) = ([d(x)]) \wedge [y] \vee [x] \wedge ([d(y)]) = \{[w]|w \in x \wedge d(y) \vee d(x) \wedge y\}$. Since $d(x \wedge y) \in d(x) \wedge y \vee x \wedge d(y)$, we get $[d(x \wedge y)] \in \{[w]|w \in d(x) \wedge y \vee x \wedge d(y)\}$. That is, $g([x] \wedge [y]) \in g([x]) \wedge [y] \vee [x] \wedge g[y]$. Thus g is a strong derivation of L/R. $\square$

Theorem 4.12 *Let (L, d) be a differential hyperlattice and R be a differential hypercongruence of L. Then there exists a one to one correspondence between the set of all differential hypercongruence containing R and the set of all differential hypercongruence of L/R.*

Proof Let $\mathbb{C}(\mathbb{L})$ be the set of all differential hypercongruences of J containing R and $\mathbb{QC}(\mathbb{L})$ be the set of all differential hypercongruences of L/R. Consider the following map:

$$u : \mathbb{C}(\mathbb{L}) \longrightarrow \mathbb{QC}(\mathbb{L}), \quad u(R) = J/R.$$

Since R is a differential hypercongruence of J containing R, so J/R is a differential hypercongruence of L/R. Hence the map u is well defined.

Let $M, N \in \mathbb{C}(\mathbb{L})$ such that $u(M) = u(N)$. So $M/R = N/R$. For any $x \in M$, we have $[x] \in M/R = N/R$, so $[x] = [y]$ for some $y \in N$. That is $x \in [y] \subseteq N$. Hence $M \subseteq N$. The proof of $N \subseteq M$ is similar to that of $M \subseteq N$. Hence $M = N$. That is, the function u is an epic function. It is clear to see that u is a monomorphic function. Therefore, u is a bijective map between $\mathbb{C}(\mathbb{A})$ and $\mathbb{CQ}(\mathbb{A})$.

5. Conclusion and future research

In this paper we introduce derivation in hyperlatices and derive some basic properties of them. Also, some properties of differential hyperideals and differential hypercongruences are studied. Further we prove that for an injective strong differential hyperlattice (L, d) and for a strongly differential hypercongruence R of (L, d), the quotient hyperlattice $(L/R, g)$ is an injective strongly differential hyperlattice, where g is an injective strong derivation on L/R induced by d.

Since hyperlattice and hyper MV-algebras, hyperresiduated lattices are closely related, we will use the results of this paper to study derivation on hyper MV-algebras, hyperresiduated lattices and related hyperalgebraic systems. Some important issues for future works are: (i) Developing the properties of a derivation; (ii) Defining new derivation which are related to given derivation on hyperlattices; (iii) Finding useful results on the other hyperalgebraic structures.

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