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Limiting Property of the Distribution Function of L^p Function at Endpoints

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Abstract We consider the limiting property of the distribution function of L^p function at endpoints 0 and ∞ and prove that for $\lambda > 0$ the following two equations

$$\lim_{\lambda\to+\infty} \lambda^p m(\{x : |f(x)| > \lambda\}) = 0, \quad \lim_{\lambda\to 0^+} \lambda^p m(\{x : |f(x)| > \lambda\}) = 0$$

hold for $f \in L^p(\mathbb{R}^n)$ with $1 \le p < \infty$. This result is naturally applied to many operators of type (p, q) as well.

Keywords Hardy-Littlewood maximal function; limiting behavior; distribution function

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1. Introduction

The Hardy-Littlewood maximal operator M acting on $f \in L^1_{\rm loc}(\mathbb{R}^n)$ is defined by

$$Mf(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\mathrm{d}y, \tag{1.1}$$

where $B(x, r)$ denotes a ball centered at x with radius r, and $|B(x, r)|$ represents the Lebesgue measure of the ball $B(x, r)$. As a sublinear operator, M maps $L^p(\mathbb{R}^n)$ to itself whenever $1 < p \le \infty$. That is, there exists a constant $C_{n,p} > 0$ such that the following inequality

$$\|Mf\|_p \le C_{n,p}\|f\|_p$$

holds for all $f \in L^p(\mathbb{R}^n)$.

A weak-type result related to the Hardy-Littlewood maximal operator is that M maps $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$. That is, there exists a constant C_n such that the inequality

$$\lambda m(\{x : Mf(x) > \lambda\}) \le C_n\|f\|_1 \tag{1.2}$$

holds for all $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$, where m denotes the Lebesgue measure on $\mathbb{R}^n$. As we have shown in (1.2), the supremum of the left side of the inequality over λ is finite. In [1], Janakiraman investigated the limiting properties of $\lambda m(\{x : Mf(x) > \lambda\})$ as λ tends to 0 or ∞ and obtained the following Theorem 1.1.

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Theorem 1.1 *Let M be the Hardy-Littlewood maximal operator and $f \in L^1(\mathbb{R}^n)$. For $\lambda > 0$, the following two equations*

$$\lim_{\lambda \to 0^+} \lambda m(\{x \in \mathbb{R}^n : Mf(x) > \lambda\}) = \|f\|_1 \tag{1.3}$$

and

$$\lim_{\lambda \to +\infty} \lambda m(\{x \in \mathbb{R}^n : Mf(x) > \lambda\}) = 0 \tag{1.4}$$

hold.

In Theorem 1.1, the authors studied the limiting property of the product of λ and the distribution function of the Hardy-Littlewood maximal function of L^1 function at the two endpoints 0 and ∞. In fact, if we take the supremum for λ over $\mathbb{R}^+ := (0, \infty)$ in the left side of the Eq. (1.3), then the supremum is just the weak L^1 norm of Mf. For $f \in L^p(\mathbb{R}^n)$ with $1 \leq p \leq \infty$, we want to know what properties the limiting behavior of $\lambda^p m(\{x : |f(x)| > \lambda\})$ has, as λ tends to 0 or ∞. In this paper, motivated by the result in Theorem 1.1, we will investigate the limiting question. Now let us formulate our main theorems.

Theorem 1.2 *Suppose that $f \in L^p(\mathbb{R}^n)$ with $1 \leq p < \infty$. Then the following two equations*

$$\lim_{\lambda \to +\infty} \lambda^p m(\{x : |f(x)| > \lambda\}) = 0 \tag{1.5}$$

and

$$\lim_{\lambda \to 0^+} \lambda^p m(\{x : |f(x)| > \lambda\}) = 0 \tag{1.6}$$

hold.

As an application of Theorem 1.2, we can obtain the following results.

Theorem 1.3 *Suppose that the operator T is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ with $1 \leq p \leq \infty, 1 \leq q < \infty$. If $f \in L^p(\mathbb{R}^n)$, then the following two equations*

$$\lim_{\lambda \to +\infty} \lambda^q m(\{x : |Tf(x)| > \lambda\}) = 0 \tag{1.7}$$

and

$$\lim_{\lambda \to 0^+} \lambda^q m(\{x : |Tf(x)| > \lambda\}) = 0 \tag{1.8}$$

hold.

In fact, since the operator T is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, we immediately have $Tf \in L^q(\mathbb{R}^n)$, provided that $f \in L^p(\mathbb{R}^n)$. Thus Theorem 1.3 can be directly obtained by Theorem 1.2. It should be pointed out that, when the case $1 < p = q < \infty$ holds, the operator T covers many famous operators such as the Hardy-Littlewood maximal function, Hardy operator and Calderon-Zygmund singular integral operator etc. More details can be found in [2] and [3]. For the general p, q, the operator T also covers many operators such as the convolution operator, the Fourier transform and some fractional singular integral operators, etc.

2. Proofs of Theorems

Next we will give the proofs of Theorems 1.2 and 1.3.

Proof of Theorem 1.2 We first prove the Eq. (1.5). Consider the case $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$. Actually if $f \in L^1(\mathbb{R}^n)$, the Eq. (1.5) is just the corollary of the Eq. (1.4) in Theorem 1.1.

For a fixed function $f \in L^p(\mathbb{R}^n)$ with $1 \leq p < \infty$, there exists a sequence of functions $\{f_k\}_{k\in\mathbb{N}}$ which belong to $C_c(\mathbb{R}^n)$ such that $\|f - f_k\|_p < \frac{1}{k}$. For any positive real number $\lambda > 0$, we have

$$\{x : |f(x)| > \lambda\} \subset \{x : |f(x) - f_k(x)| > \frac{\lambda}{2}\} \bigcup \{x : |f_k(x)| > \frac{\lambda}{2}\}.$$

Since $f \in L^p(\mathbb{R}^n)$ and $f_k \in L^p(\mathbb{R}^n)$, we have $f - f_k \in L^p(\mathbb{R}^n)$. Thus it follows that

$$m(\{x : |f(x) - f_k(x)| > \frac{\lambda}{2}\}) \leq \frac{2^p\|f - f_k\|_p^p}{\lambda^p}.$$

Since $C_c(\mathbb{R}^n)$ is a subspace of $L^\infty(\mathbb{R}^n)$, we have

$$m(\{x : |f_k(x)| > \frac{\lambda}{2}\}) = 0, \tag{2.1}$$

whenever $\lambda \geq 2\|f_k\|_\infty$. Consequently, when $\lambda \geq 2\|f_k\|_\infty$, we conclude that

$$\begin{aligned}\lambda^p m(\{x : |f(x)| > \lambda\}) \leq& \lambda^p m(\{x : |f(x) - f_k(x)| > \frac{\lambda}{2}\} \bigcup \{x : |f_k(x)| > \frac{\lambda}{2}\}) \\ \leq& 2^p\|f - f_k\|_p^p + \lambda^p m(\{x : |f_k(x)| > \frac{\lambda}{2}\}) \\ \leq& \frac{2^p}{k^p}.\end{aligned} \tag{2.2}$$

Therefore, the inequality (2.2) implies that

$$\limsup_{\lambda\to+\infty} \lambda^p m(\{x : |f(x)| > \lambda\}) \leq \frac{2^p}{k^p}. \tag{2.3}$$

Note that k may be arbitrarily large, then we can easily obtain that the Eq. (1.5) holds for every $f \in L^p(\mathbb{R}^n)$.

Next we will prove the Eq. (1.6).

This proof is similar to the proof of Eq. (1.5). In the same way we also need a sequence of functions $\{f_k\}_{k\in\mathbb{N}}$ belonging to $C_c(\mathbb{R}^n)$ to approximate a function $f(x)$ in $L^p(\mathbb{R}^n)$. And the argument is similar, but one point should be noted that in this part the estimate of $m(\{x : |f_k(x)| > \frac{\lambda}{2}\})$ is different from (2.1).

Since $f_k \in C_c(\mathbb{R}^n)$, we have $m(\{x : |f_k(x)| > \frac{\lambda}{2}\}) \leq m(S_k) < \infty$, where $S_k = \overline{\{x : f_k(x) \neq 0\}}$ is a compact set. Thus it follows that

$$\lim_{\lambda\to 0^+} \lambda^p m(\{x : |f_k(x)| > \frac{\lambda}{2}\}) = 0. \tag{2.4}$$

Noting the inequality (2.2) and the Eq. (2.4), we conclude that

$$\begin{aligned}\limsup_{\lambda\to 0^+} \lambda^p m(\{x : |f(x)| > \lambda\}) \leq& \limsup_{\lambda\to 0^+} \lambda^p m(\{x : |f(x) - f_k(x)| > \frac{\lambda}{2}\} \bigcup \{x : |f_k(x)| > \frac{\lambda}{2}\}) \\ \leq& 2^p\|f - f_k\|_p^p + \limsup_{\lambda\to 0^+} \lambda^p m(\{x : |f_k(x)| > \frac{\lambda}{2}\}) \\ \leq& \frac{2^p}{k^p}.\end{aligned} \tag{2.5}$$

Consequently, the Eq. (1.6) holds for all $f \in L^p(\mathbb{R}^n)$ with $1 \leq p < \infty$. $\square$

We remark that if a function $f \in L^p(\mathbb{R}^n)$ and $\|f\|_p > 0$ with $1 \leq p < \infty$, then we can easily obtain $\|f\|_{L^{p,\infty}} = \sup_{\lambda>0} \lambda(m(\{x : |f(x)| > \lambda\}))^{\frac{1}{p}} > 0$. However, the both Eqs. (1.5) and (1.6) in Theorem 1.2 show us that the weak L^1 norm of f must not be reached at the endpoints $\lambda = 0$ and $\lambda = \infty$.

Furthermore, except the two endpoints $\lambda = 0$ and $\lambda = \infty$, for every positive number $\alpha > 0$ and $c < 1$, there must exist a function $f \in L^p(\mathbb{R}^n)$ and $\|f\|_p > 0$ with $1 \leq p < \infty$ such that

$$\lim_{\lambda \to \alpha^-} \lambda(m(\{x : |f(x)| > \lambda\}))^{\frac{1}{p}} > c\|f\|_p \tag{2.6}$$

and

$$\lim_{\lambda \to \alpha^+} \lambda(m(\{x : |f(x)| > \lambda\}))^{\frac{1}{p}} > c\|f\|_p \tag{2.7}$$

hold.

3. Further results

Next we will consider the problem of the critical value. When $f \in L^p(\mathbb{R}^n)$ with $1 \leq p < \infty$, the exponential p in (1.5) and (1.6) is the critical value. In fact, we have the following theorem.

Theroem 3.1 *For every $\varepsilon > 0$, there must exist a function $f \in L^p(\mathbb{R}^n)$ such that the following equation*

$$\lim_{\lambda \to +\infty} \lambda^{p+\varepsilon} m(\{x : |f(x)| > \lambda\}) = \infty \tag{3.1}$$

holds.

In the same way, for every $\epsilon > 0$, there must exist a function $f \in L^p(\mathbb{R}^n)$ such that the following equation

$$\lim_{\lambda \to 0^+} \lambda^{p-\epsilon} m(\{x : |f(x)| > \lambda\}) = \infty \tag{3.2}$$

holds.

We remark that we immediately deduce from the Eq. (1.5) that

$$\lim_{\lambda \to +\infty} \lambda^{p-\alpha} m(\{x : |f(x)| > \lambda\}) = 0 \tag{3.3}$$

holds for $f \in L^p(\mathbb{R}^n)$ and any $\alpha \geq 0$. Consequently, the both Eqs. (3.1) and (3.3) show that the exponential p is the critical value in the sense of weak type norm of the L^p function f at the endpoint ∞.

The proof of Theorem 3.1 is constructive. That is, for any fixed $\varepsilon > 0$, it suffices to find a L^p function which satisfies the Eq. (3.1).

Proof of Theorem 3.1 We first prove the Eq. (3.1). For some p with $1 \leq p < \infty$, set

$$f(x) := \sum_{k=3}^{\infty} k^{\frac{1}{p}} \chi_{[k \leq |x| \leq k + \frac{1}{k^{n+1}\log^2 k}]}(x), \tag{3.4}$$

for $x \in \mathbb{R}^n$. Now we prove $f \in L^p(\mathbb{R}^n)$.

It follows from (3.4) that

$$\|f\|_p^p = \sum_{k=3}^{\infty} k \int_{k \leq |x| \leq k + \frac{1}{k^{n+1}\log^2 k}} 1 \mathrm{d}x = \sum_{k=3}^{\infty} k\omega_{n-1} \int_k^{k + \frac{1}{k^{n+1}\log^2 k}} r^{n-1} \mathrm{d}r$$

$$= \sum_{k=3}^{\infty} k\omega_{n-1} \frac{r^n}{n}\Big|_k^{k+\frac{1}{k^{n+1}\log^2 k}} = \sum_{k=3}^{\infty} k\frac{\omega_{n-1}}{n}[(k+\frac{1}{k^{n+1}\log^2 k})^n - k^n],$$

where ω_{n-1} denotes the surface area of n dimensional unit sphere.

By the following inequality

$$nk^{n-1}\frac{1}{k^{n+1}\log^2 k} \le [(k+\frac{1}{k^{n+1}\log^2 k})^n - k^n] \le 2^n k^{n-1}\frac{1}{k^{n+1}\log^2 k}, \tag{3.5}$$

we have

$$\|f\|_p^p \le \frac{2^n\omega_{n-1}}{n}\sum_{k=3}^{\infty}\frac{1}{k\log^2 k} < \infty.$$

For any positive integer $N > 3$, we have

$$\begin{aligned} m(\{x : |f(x)| > N\}) = &\sum_{k=N^p+1}^{\infty}\int_{k\le|x|\le k+\frac{1}{k^{n+1}\log^2 k}} 1\mathrm{d}x \\ = &\sum_{k=N^p+1}^{\infty}\omega_{n-1}\int_k^{k+\frac{1}{k^{n+1}\log^2 k}} r^{n-1}\mathrm{d}r \\ \ge &\omega_{n-1}\sum_{k=N^p+1}^{\infty}\frac{1}{k^2\log^2 k} \\ \ge &\frac{\omega_{n-1}}{4(N^p+1)\log^2[2(N^p+1)]}. \end{aligned} \tag{3.6}$$

The elementary property of limit implies that

$$\lim_{N\to\infty}\frac{\omega_{n-1}N^{p+\varepsilon}}{4(N^p+1)\log^2[2(N^p+1)]} = \infty \tag{3.7}$$

holds for any $\varepsilon > 0$.

Thus combining the inequality (3.6) with the Eq. (3.7) yields that

$$\lim_{\lambda\to+\infty}\lambda^{p+\varepsilon}m(\{x : f(x) > \lambda\}) = \infty$$

holds for any $\varepsilon > 0$, that is, the Eq. (3.1) holds.

We now prove the Eq. (3.2). The proof of Eq. (3.2) has a lot of similars as that in the Eq. (3.1), so we give the different parts.

In the same way, set

$$f(x) := \sum_{k=3}^{\infty} k^{-\frac{1}{p}}\chi_{[k\le|x|\le k+\frac{1}{k^{n-1}\log^2 k}]}(x) \tag{3.8}$$

for $x \in \mathbb{R}^n$. By employing the fundamental knowledge of the mathematical analysis, we can easily show that $f \in L^p(\mathbb{R}^n)$ holds for $1 \le p < \infty$.

For any positive integer $N > 3$, the inequality $Mg(x) \ge |g(x)|$ holds for almost every $x \in \mathbb{R}^n$, then we have

$$m(\{x : |f(x)| > \frac{1}{N}\}) = \sum_{k=3}^{N^p}\int_{k\le|x|\le k+\frac{1}{k^{n-1}\log^2 k}} 1\mathrm{d}x$$

$$
\begin{aligned}
&=\sum_{k=3}^{N^p}\omega_{n-1}\int_k^{k+\frac{1}{k^{n-1}\log^2 k}} r^{n-1}\mathrm{d}r\\
&=\frac{\omega_{n-1}}{n}\sum_{k=3}^{N^p}[(k+\frac{1}{k^{n-1}\log^2 k})^n-k^n]\\
&\geq\frac{\omega_{n-1}}{n}\frac{N^p-3}{p^2\log^2 N}. \qquad (3.9)
\end{aligned}
$$

Obviously we conclude from (3.9) that

$$\lim_{\lambda\to 0^+}\lambda^{p-\epsilon}m(\{x:|f(x)|>\lambda\})=\infty$$

holds for any $\epsilon>0$. This is our desired result. $\square$

We do not know whether or not the Eqs. (3.1) and (3.2) in Theorem 3.1 still hold, if we replace $|f|$ by $|Tf|$, provided that T is bounded on L^p. However, when Tf is the Hardy-Littlewood maximal function of f, the corresponding results hold. In fact, we have the following theorem.

Theroem 3.2 *Suppose that M is the Hardy-Littlewood maximal function operator and $1<p<\infty$. For every $\varepsilon>0$, there must exist a function $f\in L^p(\mathbb{R}^n)$ such that the following equation*

$$\lim_{\lambda\to+\infty}\lambda^{p+\varepsilon}m(\{x:Mf(x)>\lambda\})=\infty \qquad (3.10)$$

holds. In the same way, for every $\epsilon>0$, there must exist a function $f\in L^p(\mathbb{R}^n)$ such that the following equation

$$\lim_{\lambda\to 0^+}\lambda^{p-\epsilon}m(\{x:Mf(x)>\lambda\})=\infty \qquad (3.11)$$

holds.

Noting that $Mf(x)\geq|f(x)|$ for almost every $x\in\mathbb{R}^n$ and using the almost similar method as in proving Theorem 3.1, we can easily obtain the proof of Theorem 3.2, so we omit the proof here.

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