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Local Well-Posedness for the Hyperelastic Rots Equation in Besov Space

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Abstract This paper is concerned with the cauchy problem for the hyperelastic rots equation in Besov space. By virtue of the Littlewood-Paley decomposition, the local well-posedness for the equation in Besov space is established. Furthermore, the blow-up criterion for the solutions of the hyperelastic rots equation is derived.

Keywords Hyperelastic rots equations; Cauchy problem; local well-posedness; blow-up criterion; Besov space

MR(2010) Subject Classification 35G25; 35R25

1. Introduction

The hyperelastic rod (HR) equation

$$u_t - u_{txx} + 3uu_x = \gamma(2u_x u_{xx} + uu_{xxx}) \tag{1.1}$$

was first derived by Dai in [1,2] as a model for finite-length and small-amplitude axial deformation waves in thin cylindrical rods composed of a compressible isotropic hyperelastic material, where $u(t,x)$ represents the radial stretch relative to a pre-stressed state and γ is a constant determined by the material parameters.

For $\gamma = 1$, Eq. (1.1) becomes the celebrated Camassa-Holm (CH) equation [3,4], which was proposed as a model for the unidirectional propagation of the shallow water waves over a flat bottom [5], where $u(t,x)$ represents the free surface above a flat bottom.

For $\gamma = 0$, Eq. (1.1) gives the celebrated BBM equation, which was proposed by Benjamin, Bona, and Mahony [6] as a model for the unidirectional evolution of long surface waves in a channel. Recently, Bona and Tzvetkov [7] have proved that this equation is globally well-posed in Sobolev spaces H^s, if $s \geq 0$.

The case $\gamma \neq 1$ also has been extensively studied [8–14]. It is known that when the parameter fulfills $\gamma < 1$, all the solitary-wave solutions are smooth and orbitally stable in the energy space [8]. Moreover, if $\gamma > 1$, all the solitary waves have singularities cusped at the crest [9]. In [10,11], the well-posedness of the hyperelastic rod equation in Sobolev spaces $H^s, s > 3/2$ has been studied. Their approach is to rewrite the HR equation in its non-local form, and then to verify the conditions needed to apply Kato's semi-group theory [15]. Using a vanishing viscosity

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argument, Coclite, Holden, and Karlsen [16] established existence of a strongly continuous semigroup of global weak solutions of HR equation on the line for initial data in H^1. Bendahmane, Coclite, and Karlsen [17] extended this result to traveling wave solutions that are supersonic solitary shock waves. We refer to [10,18] for more information on the existence of global solutions to the hyperelastic rod equation.

Note that $p(x) = \frac{1}{2}e^{-|x|}$ and $p(x) * f = (1-\partial_x^2)^{-1}f$ for all $f \in L^2(\mathbb{R})$, we can rewrite Eq. (1.1) as follows:

$$u_t + \gamma u\partial_x u = P(D)(\frac{3-\gamma}{2}u^2 + \frac{\gamma}{2}(\partial_x u)^2), \tag{1.2}$$

where $P(D)(f) = -\partial_x(1-\partial_x^2)^{-1}f$. Supplement (1.2) with the initial data

$$u(x,0) = u_0. \tag{1.3}$$

In this work, we study the Cauchy problem of the hyperelastic rod equation. By virtue of the Littlewood-Paley decomposition and nonhomogeneous Besov space, we show that the Cauchy problem for Eqs. (1.2) and (1.3) is locally well-posed in Besov space.

Our paper is organized as follows. In Section 2, we recall some basic facts of Besov space and the transport equation theory. In Section 3, we establish the local well-posedness of the Cauchy problem of the equation. In Section 4, we give a blow-up criterion for the Eqs. (1.2) and (1.3).

2. Preliminaries

In this section, we will recall some fact on the Littlewood-Paley decomposition, the nonhomogeneous Besov space and some useful properties, and the general transport equation theory, which will be used in the sequel.

Proposition 2.1 (Littlewood-Paley decomposition [19–23]) *There exists a couple of smooth functions (χ, φ) valued in [0,1], such that χ is supported in the ball $B \triangleq \{\xi \in \mathbb{R}^n : |\xi| \le \frac{4}{3}\}$, and φ is supported in the ring $C \triangleq \{\xi \in \mathbb{R}^n : \frac{3}{4} \leqslant |\xi| \le \frac{8}{3}\}$. Moreover,*

$$\forall \xi \in \mathbb{R}^n, \chi(\xi) + \Sigma_{q\in\mathbb{N}}\varphi(2^{-q}\xi) = 1$$

and

$$\operatorname{supp}\varphi(2^{-q}\cdot) \cap \operatorname{supp}\varphi(2^{-q'}\cdot) = \emptyset, \ \text{if } |q-q'| \ge 2,$$
$$\operatorname{supp}\chi(\cdot) \cap \operatorname{supp}\varphi(2^{-q}\cdot) = \emptyset, \ \text{if } q \ge 1.$$

Then for all $u \in \mathscr{S}'(\mathbb{R}^n)$, we can define the nonhomogeneous dyadic blocks as follows. Let

$$\Delta_q u \triangleq 0, \ \text{if } q \leqslant -2, \quad \Delta_{-1}u \triangleq \chi(D)u = \mathscr{F}_x^{-1}\chi\mathscr{F}_x u,$$

$$\Delta_q u \triangleq \varphi(2^{-q}D)u = \mathscr{F}_x^{-1}\varphi(2^{-q}\xi)\mathscr{F}_x u, \quad \text{if } q \ge 0.$$

Hence,

$$u = \sum_{q\in N} \Delta_q u \ \text{in } \mathscr{S}'(\mathbb{R}),$$

where the right-hand side is called the nonhomogeneous Little-Paley decomposition of u.

Remark 2.2 The low frequency cut-off S_q is defined by

$$S_q = \sum_{p=-1}^{q-1} \triangle_p u = \chi(2^{-q}D)u = \mathscr{F}_x^{-1}\chi(2^{-q}\xi)\mathscr{F}_x u, \quad \forall q \in \mathbb{N}.$$

It is easily checked that

$$\begin{aligned} &\triangle_p \triangle_q u \equiv 0, \text{ if } |p-q| \geq 2, \\ &\triangle_q (S_{p-1}u \triangle_p v) \equiv 0, \text{ if } |p-q| \geq 5, \ \forall u, v \in \mathscr{S}'(\mathbb{R}) \end{aligned}$$

as well as

$$\|\triangle_q u\|_{L^p} \leq \|u\|_{L^p}, \ \|S_q u\|_{L^p} \leq C\|u\|_{L^p}, \quad \forall 1 \leq p \leq \infty$$

with the aid of Young's inequality, where C is a positive constant independent of q.

Definition 2.3 (Besov spaces) *Let $s \in \mathbb{R}, 1 \leq p \leq +\infty$. The nonhomogeneous Besov space $B^s_{p,r}(\mathbb{R}^n)$ is defined by:*

$$B^s_{p,r}(\mathbb{R}^n) = \{f \in \mathscr{S}'(\mathbb{R}^n) : \|f\|_{B^s_{p,r}} = \|(2^{qs}\| \triangle_q f\|_{L^p})_{q\geq -1}\|_{l^r} < \infty\}.$$

In particular, $B^\infty_{p,r} = \cap_{s\in\mathbb{R}} B^s_{p,r}$.

Definition 2.4 *Let $T > 0, s \in \mathbb{R}$ and $1 \leq p \leq \infty$. Set*

$$E^s_{p,r}(T) \triangleq C([0,T]; B^s_{p,r}) \cap C^1([0,T]; B^{s-1}_{p,r}), \quad \text{if } r < \infty,$$

$$E^s_{p,\infty}(T) \triangleq L^\infty([0,T]; B^s_{p,\infty}) \cap \mathrm{Lip}([0,T]; B^{s-1}_{p,\infty})$$

and

$$E^s_{p,r} \triangleq \bigcap_{T>0} E^s_{p,r}(T).$$

Lemma 2.5 ([20,21]) *Let $s \in \mathbb{R}, 1 \leq p, r, p_j, r_j \leq +\infty$. Then*

(1) Topological properties: $B^s_{p,r}(\mathbb{R}^n)$ is a Banach space which is continuously embedded in $\mathscr{S}'(\mathbb{R}^n)$.

(2) Density: C_c is dense in $B^s_{p,r}(\mathbb{R}^n) \Leftrightarrow 1 \leq p, r \leq \infty$.

(3) Embedding: $B^s_{p_1,r_1} \hookrightarrow B^{s-n(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$, if $p_1 \leq p_2$ and $r_1 \leq r_2$. $B^{s_2}_{p,r_2} \hookrightarrow B^{s_1}_{p,r_1}$, locally compact if $s_1 \leq s_2$.

(4) Algebraic properties: $\forall s > 0, B^s_{p,r} \bigcap L^\infty$ is an algebra. Moreover, $B^s_{p,r}$ is an algebra, provided that $s > \frac{n}{p}$ or $s \geq \frac{n}{p}$ and $r = 1$.

(5) 1-D moser-type estimates:

(i) For $s > 0$,

$$\|fg\|_{B^s_{p,r}} \leq C(\|f\|_{B^s_{p,r}}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|g\|_{B^s_{p,r}}).$$

(ii) $\forall s_1 \leq \frac{1}{p} < s_2$ ($s_2 \geq \frac{1}{p}$ if $r = 1$), and $s_1 + s_2 > 0$, we have

$$\|fg\|_{B^{s_1}_{p,r}} \leq C\|f\|_{B^{s_1}_{p,r}}\|g\|_{B^{s_2}_{p,r}}.$$

(6) *Complex interpolation:*

$$\|f\|_{B^{\theta s_1+(1-\theta)s_2}_{p,r}} \le C\|f\|^{\theta}_{B^{s_1}_{p,r}}\|f\|^{1-\theta}_{B^{s_2}_{p,r}}, \quad \forall\theta\in[0,1].$$

(7) *A logarithmic interpolation inequality:*

$$\|f\|_{B^s_{p,1}} \le C\frac{1+\varepsilon}{\varepsilon}\|f\|_{B^s_{p,\infty}}(1+\ln\frac{\|f\|_{B^{s+\varepsilon}_{p,\infty}}}{\|f\|_{B^s_{p,\infty}}}), \quad \forall\varepsilon>0.$$

(8) *Fatou lemma: if $(u_n)_{n\in\mathbb{N}}$ is bounded in $B^s_{p,r}$ and $u_n\to u$ in $\mathscr{S}'(\mathbb{R}^n)$, then $u\in B^s_{p,r}$ and*

$$\|u\|_{B^s_{p,r}} \le \liminf_{n\to\infty}\|u_n\|_{B^s_{p,r}}.$$

(9) *Let $m\in\mathbb{R}$ and f be an S^m-mutiplier (i.e., $f:\mathbb{R}^n\to\mathbb{R}$) is a smooth and satisfies that $\forall\alpha\in\mathbb{N}^n,\exists$ a constant C_α, s.t. $|\partial^\alpha f(\xi)|\le C_\alpha(1+|\xi|)^{m-|\alpha|}$ for all $\xi\in\mathbb{R}^n$. Then the operator $f(D)$ is continuous from $B^s_{p,r}$ to $B^{s-m}_{p,r}$.*

(10) *The usual product is continuous from $B^{-\frac{1}{p}}_{p,1}\times(B^{\frac{1}{p}}_{p,\infty}\bigcap L^\infty)$ to $B^{-\frac{1}{p}}_{p,\infty}$.*

Lemma 2.6 ([21]) *Let $1\le r\le+\infty$, $1\le p\le p_1\le\infty$ and $s>-N\min(\frac{1}{p_1},\frac{1}{p})$ or $s>-1-N\min(\frac{1}{p_1},\frac{1}{p})$ if $\nabla v=0$. Then there exists a constant C depending only on s,p,p_1 and r, and such that the following inequalities are ture:*

(i) *If $s<1+\frac{N}{p_1}$,*

$$\|2^{qs}\|R_q\|_{L^p}\|_{l^r} \le C\|\nabla v\|_{B^{\frac{N}{p}}_{P_1,\infty}\cap L^\infty}\|f\|_{B^s_{p,r}}. \tag{2.1}$$

(ii) *If $s>1+\frac{N}{p_1}$, or $s=1+\frac{N}{p_1}$ and $r=1$,*

$$\|2^{qs}\|R_q\|_{L^p}\|_{l^r} \le C\|\nabla v\|_{B^{s-1}_{P_1,r}\cap L^\infty}\|f\|_{B^s_{p,r}}. \tag{2.2}$$

If $f=v$, then we also have

$$\|2^{qs}\|R_q\|_{L^p}\|_{l^r} \le C\|\nabla v\|_{L^\infty}\|f\|_{B^s_{p,r}}, \tag{2.3}$$

where $R_q=v\nabla\triangle_q f-\triangle_q(v\nabla f)$.

Lemma 2.7 *Let $1\le p,r\le+\infty$ and $s>-\min(\frac{1}{p},1-\frac{1}{p})$. Assume that $f_0\in B^s_{p,r}, g\in L^1(0,T;B^s_{p,r})$, and $\partial_x v$ belongs to $L^1(0,T;B^{s-1}_{p,r})$ if $s>1+\frac{1}{p}$ or to $L^1(0,T;B^{\frac{1}{p}}_{p,r}\bigcap L^\infty)$ otherwise. If $f\in L^\infty(0,T;B^s_{p,r}\bigcap C([0,T];\mathscr{S}'(\mathbb{R}))$ solves the following general one dimension linear transport equation:*

$$\begin{cases}\partial_t f+\gamma v\partial_x f=g,\\ f|_{t=0}=f_0,\end{cases} \tag{2.4}$$

then there exists a constant C depending only on s,p,γ and r, and such that the following statements hold:

$$\|f(t)\|_{\tilde{L}^\infty_t(B^s_{p,r})} \le e^{CV(t)}\Big(\|f_0\|_{B^s_{p,r}}+\int_0^t e^{-CV(\tau)}\|g(\tau)\|_{B^s_{p,r}}\mathrm{d}\tau\Big) \tag{2.5}$$

with $V(t)=\int_0^t\|\partial_x v(\tau)\|_{B^{\frac{1}{p}}_{p,r}\cap L^\infty}\mathrm{d}\tau$ if $s<1+\frac{1}{p}$ and $V(t)=\int_0^t\|\partial_x v(\tau)\|_{B^{s-1}_{p,r}}\mathrm{d}\tau$ else.

If $f=v$, then for all $s>0$, (2.5) holds true with $V(t)=\|\partial_x v(t)\|_{L^\infty}$.

Proof Applying the operator $\triangle_q$ to Eq. (2.4) yields

$$\begin{cases} \partial_t \triangle_q f + \gamma v \partial_x \triangle_q f = \triangle_q g + \gamma R_q, \\ \triangle_q f|_{t=0} = \triangle_q f_0, \end{cases}$$

where $R_q = v\partial_x \triangle_q f - \triangle_q(v\partial_x f)$. It is easy to prove by direct caculation that

$$\partial_t |\triangle_q f|^p + \gamma v \partial_x |\triangle_q f|^p = p\triangle_q g |\triangle_q f|^{p-2} + \gamma p R_q |\triangle_q f|^{p-2}.$$

Integrating about x gives

$$\begin{aligned}\partial_t \|\triangle_q f\|_{L^p}^p &= \int_{\mathbb{R}} (\gamma |\triangle_q f|^p \partial_x v + p\triangle_q g |\triangle_q f|^{p-2} + \gamma p R_q |\triangle_q f|^{p-2}) \mathrm{d}x \\ &\leq \gamma \|\partial_x v\|_{L^p} \| \|\triangle_q f\|_{L^p}^p + p\|\triangle_q g\|_{L^p} \|\triangle_q f\|_{L^p}^{p-1} + \gamma p \|R_q\|_{L^p} \|\triangle_q f\|_{L^p}^{p-1}.\end{aligned}$$

We have

$$\partial_t \|\triangle_q f\|_{L^p} \leq \gamma \|\partial_x v\|_{L^p} \| |\triangle_q f|_{L^p} + \|\triangle_q g\|_{L^p} + \gamma \|R_q\|_{L^p}.$$

Integrating about t, we hvae

$$\|\triangle_q f\|_{L^p} \leq \int_0^t (\gamma \|\partial_x(v(\tau))\|_{L^p} \|\triangle_q f(\tau)\|_{L^p} + \|\triangle_q g(\tau)\|_{L^p} + \gamma \|R_q(\tau)\|_{L^p}) \mathrm{d}\tau + \|\triangle_q f_0\|_{L^p}. \quad (2.6)$$

Multiplying $2^{\sigma q}$ on two sides of inequality (2.6), and using the Lemma 2.6 and the Minkowshi inequality, we obtain

$$\|f\|_{\tilde{L}_t^\infty(B_{p,r}^\sigma)} \leq \|f_0\|_{B_{p,r}^\sigma} + \int_0^t \|g(\tau)\|_{B_{p,r}^\sigma} \mathrm{d}\tau + C\int_0^t \|\partial_x(v(\tau))\|_{L^p} \|f(\tau)\|_{\tilde{L}_t^\infty(B_{p,r}^\sigma)} \mathrm{d}\tau. \quad (2.7)$$

Using the Gronwall inequality, we obtain Lemma 2.7. □

Remark 2.8 With $\gamma = 1$, Lemma 2.7 includes a prior estimates in Besov space for transport equation [21].

3. Local well-posedness

In this section, we will establish the local well-posedness of Eqs. (1.2) and (1.3) in Besov spaces.

Uniqueness and continuity with respect to the initial data in some sense can be obtained by the following a priori estimates.

Lemma 3.1 *Let $1 \leq p, r \leq \infty$ and $s > \max(1 + \frac{1}{p}, \frac{3}{2})$. Suppose that we are given $u, v \in L^\infty(0,T; B_{p,r}^s) \cap C([0,T]; B_{p,r}^{s-1})$ two solutions of Eqs. (1.2) and (1.3) with the initial data $u_0, v_0 \in B_{p,r}^s$. Then for every $t \in [0,T]$, we have*

$$\|u(t) - v(t)\|_{B_{p,r}^{s-1}} \leq \|u_0 - v_0\|_{B_{p,r}^{s-1}} e^{C\int_0^t (\|u(\tau)\|_{B_{p,r}^s} + \|v(\tau)\|_{B_{p,r}^s}) \mathrm{d}\tau}, \quad (3.1)$$

where C is a positive constant depending only on s, p, γ and r.

Proof Set $w = v - u$. It is obvious that $w \in L^\infty(0,T; B_{p,r}^s) \cap C([0,T]; B_{p,r}^{s-1})$ solves the following

Cauchy problem of the transport equation:

$$\begin{cases} \partial_t w + \gamma u \partial_x w = R(t,x), \\ w|_{t=0} = w_0 \triangleq v_0 - u_0, \end{cases}$$

where $R(t,x) = -\gamma w \partial_x v + P(D)(\frac{3-\gamma}{2}(v+u)w + \frac{\gamma}{2}(\partial_x v + \partial_x u)\partial_x w)$.

For all $s > \max(1+\frac{1}{p}, \frac{3}{2})$ and $t \in [0,T]$, $B^s_{p,r}$ is a Banach algebra, we obtain

$$\|w\partial_x v\|_{B^{s-1}_{p,r}} \le C\|w\|_{B^{s-1}_{p,r}} \|\partial_x v\|_{B^{s-1}_{p,r}} \le C\|w\|_{B^{s-1}_{p,r}} \|v\|_{B^s_{p,r}}.$$

By using the S^{-1} -multiplier property of $P(D)$ and the fact that $B^{s-1}_{p,r}$ is a Banach algebra, we have

$$\begin{aligned} &\big\|P(D)(\frac{3-\gamma}{2}(v+u)w + \frac{\gamma}{2}(\partial_x v + \partial_x u)\partial_x w)\big\|_{B^s_{p,r}} \\ &\le C\big\|\frac{3-\gamma}{2}(v+u)w + \frac{\gamma}{2}(\partial_x v + \partial_x u)\partial_x w\big\|_{B^{s-1}_{p,r}} \\ &\le C(\|u\|_{B^s_{p,r}} + \|v\|_{B^s_{p,r}})\|w\|_{B^s_{p,r}}. \end{aligned}$$

By using Lemma 2.7, we have

$$\|w(t)\|_{B^{s-1}_{p,r}} \le \|w_0\|_{B^{s-1}_{p,r}} + C\int_0^t (\|u(\tau)\|_{B^s_{p,r}} + \|v(\tau)\|_{B^s_{p,r}})\|w(\tau)\|_{B^s_{p,r}} \mathrm{d}\tau.$$

Taking advantage of Gronwall inequality, we get (3.1). $\Box$

Lemma 3.2 *Let $1 \le p, r \le \infty$ and $s > \max(1+\frac{1}{p}, \frac{3}{2})$. Let $u_0 \in B^s_{p,r}$ and $u^{(0)} \equiv 0$. Then*

(1) There exists a sequence of smooth $(u^{(n)})_{n\in\mathbb{N}}$ belonging to $C([0,T], B^s_{p,r})$ and solving the following equation:

$$\begin{aligned} &\partial_t u^{(n+1)} + \gamma u^{(n)} \partial_x u^{(n+1)} = P(D)(\frac{3-\gamma}{2}(u^{(n)})^2 + \frac{\gamma}{2}(\partial_x u^{(n)})^2), \\ &u^{(n+1)}|_{t=0} \triangleq u_0^{(n+1)}(x) = S_{n+1} u_0. \end{aligned} \tag{3.2}$$

(2) There exist $T > 0$ such that the solutions $(u^{(n)})_{n\in\mathbb{N}}$ is uniformly bounded in $E^s_{p,r}(T)$ and a cauchy sequence in $C([0,T]; B^{s-1}_{p,r})$ whence it converges to some limit $u \in C([0,T]; B^{s-1}_{p,r})$.

Proof Since all $S_{n+1}u_0 \in B^\infty_{p,r}$, by using Lemma 2.7, with the aid of induction, we show that for all $n \in \mathbb{N}$, (3.2) holds.

By using Lemma 2.7, we have

$$\|u^{(n+1)}(t)\|_{B^s_{p,r}} \le e^{U^{(n)}(t)}\Big(\|u_0\|_{B^s_{p,r}} + \int_0^t e^{U^{(n)}(\tau)}\|P(D)(\frac{3-\gamma}{2}(u^{(n)})^2 + \frac{\gamma}{2}(\partial_x u^{(n)})^2)\|_{B^s_{p,r}} \mathrm{d}\tau\Big), \tag{3.3}$$

where $U^{(n)}(t) \triangleq \int_0^t \|u^{(n)}\|_{B^s_{p,r}} \mathrm{d}\tau$.

Choose $0 < T < \frac{1}{2C^2\|u_0\|_{B^s_{p,r}}}$ and suppose that

$$\|u^{(n)}(t)\|_{B^s_{p,r}} \le \frac{C\|u_0\|_{B^s_{p,r}}}{1 - 2C^2\|u_0\|_{B^s_{p,r}} t}. \tag{3.4}$$

Since $U^{(n)}(t) \triangleq \int_0^t \|u^{(n)}\|_{B^s_{p,r}} \mathrm{d}\tau$, by using (3.4), we have

$$e^{CU^{(n)}(t) - CU^{(n)}(\tau)} = e^{C\int_\tau^t \|u^{(n)}\|_{B^s_{p,r}}(t')\mathrm{d}t'}$$

$$
\begin{aligned}
&\leq e^{C\int_\tau^t \frac{\|u_0\|_{B^s_{p,r}}}{1-2C^2\|u_0\|_{B^s_{p,r}}t'}\mathrm{d}t'}\\
&\leq (\frac{1-2C^2\|u_0\|_{B^s_{p,r}}\tau}{1-2C^2\|u_0\|_{B^s_{p,r}}t})^{\frac{1}{2}}.
\end{aligned}
\tag{3.5}
$$

From (3.5), when $\tau = 0$, we have

$$
e^{CU^{(n)}(t)} \leq (1-2C^2\|u_0\|_{B^s_{p,r}}t)^{-\frac{1}{2}}.
\tag{3.6}
$$

Substituting (3.6) into (3.3) yields, for $t \in [0,T]$

$$
\begin{aligned}
\|u^{(n+1)}(t)\|_{B^s_{p,r}} &\leq e^{U^{(n)}(t)}\Big(\|u_0\|_{B^s_{p,r}} + \int_0^t e^{U^{(n)}(\tau)}\|P(D)(\frac{3-\gamma}{2}(u^{(n)})^2+\frac{\gamma}{2}(\partial_x u^{(n)})^2)\|_{B^s_{p,r}}\mathrm{d}\tau\Big)\\
&\leq \frac{2C\|u_0\|_{B^s_{p,r}}}{1-2C^2\|u_0\|_{B^s_{p,r}}t},
\end{aligned}
$$

which implies that $(u^{(n)})_{n\in\mathbb{N}}$ is uniformly bounded in $C([0,T];B^s_{p,r})$.

By using the fact $B^{s-1}_{p,r}$ is a Banach algebra and $B^s_{p,r}\hookrightarrow B^{s-1}_{p,r}$, we have

$$
\begin{aligned}
\|u^{(n)}u_x^{(n+1)}\|_{B^{s-1}_{p,r}} &\leq C\|u^{(n)}\|_{B^{s-1}_{p,r}}\|u_x^{(n+1)}\|_{B^{s-1}_{p,r}}\\
&\leq \frac{C\|u_0\|^2_{B^s_{p,r}}}{(1-2C^2\|u_0\|_{B^s_{p,r}}t)^2}.
\end{aligned}
\tag{3.7}
$$

Since $B^{s-1}_{p,r}$ is a Banach algebra, by using the $S^{-1}-$ multiplier property of $P(D)$ and (3.4), we have

$$
\begin{aligned}
\|P(D)(\frac{3-\gamma}{2}(u^{(n)})^2+\frac{\gamma}{2}(\partial_x u^{(n)})^2)\|_{B^s_{p,r}} &\leq C\|\frac{3-\gamma}{2}(u^{(n)})^2+\frac{\gamma}{2}(\partial_x u^{(n)})^2\|_{B^{s-1}_{p,r}}\\
&\leq C\|(u^{(n)})^2\|_{B^{s-1}_{p,r}}\\
&\leq \frac{C\|u_0\|^2_{B^s_{p,r}}}{(1-2C^2\|u_0\|_{B^s_{p,r}}t)^2}.
\end{aligned}
\tag{3.8}
$$

Thus, combining (3.2) with (3.7) and (3.8), we have

$$
\begin{aligned}
\|u_t^{(n+1)}\|_{B^{s-1}_{p,r}} &\leq \|u^{(n)}\partial_x u^{(n+1)}\|_{B^{s-1}_{p,r}} + \|P(D)(\frac{3-\gamma}{2}(u^{(n)})^2+\frac{\gamma}{2}(\partial_x u^{(n)})^2)\|_{B^{s-1}_{p,r}}\\
&\leq \frac{C\|u_0\|^2_{B^s_{p,r}}}{(1-2C^2\|u_0\|_{B^s_{p,r}}t)^2}.
\end{aligned}
$$

Consequently, $(u^{(n)})_n \in E^s_{p,r}$.

Now it suffices to show that $(u^{(n)})_{n\in\mathbb{N}}$ is a cauchy sequence in $C([0,T];B^s_{p,r})$. Indeed, for $m,n\in\mathbb{N}$, we have

$$
(\partial_t+\gamma u^{(n+m)}\partial_x)(u^{(n+m+1)}-u^{(n+1)}) = \gamma(u^{(n)}-u^{(n+m)})\partial_x u^{(n+1)} + P(D)(B(x,t),
$$

where $B(x,t) \triangleq \frac{3-\gamma}{2}((u^{(n+m)})^2-(u^{(n)})^2)+\frac{\gamma}{2}((\partial_x u^{(n+m)})^2-(\partial_x u^{(n)})^2)$.

By using Lemma 2.7, we have

$$
\begin{aligned}
\|u^{(n+m+1)}-u^{(n+1)}\|_{B^{s-1}_{p,r}} \leq & Ce^{U^{(n+m)}(t)}\Big(\|u_0^{(n+m+1)}-u_0^{(n+1)}\|_{B^{s-1}_{p,r}}+\\
&\int_0^t e^{-CU^{(n+m)}(\tau)}\|\gamma(u^{(n)}-u^{(n+m)})\partial_x u^{(n+1)}+
\end{aligned}
$$

$$P(D)(B(x,\tau)\|_{B^{s-1}_{p,r}}\mathrm{d}\tau\Big). \tag{3.9}$$

Since $B^{s-1}_{p,r}$ is a Banach algebra, and $u^{(n)}$ is uniformly bounded, we have

$$\begin{aligned}\|(u^{(n)}-u^{(n+m)})\partial_x u^{(n+1)}\|_{B^{s-1}_{p,r}} &\leq \|u^{(n)}-u^{(n+m)}\|_{B^{s-1}_{p,r}}\|\partial_x u^{(n+1)}\|_{B^{s-1}_{p,r}}\\ &\leq C\|u^{(n)}-u^{(n+m)}\|_{B^{s-1}_{p,r}}\|u^{(n+1)}\|_{B^{s}_{p,r}}\\ &\leq C\|u^{(n)}-u^{(n+m)}\|_{B^{s-1}_{p,r}},\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}\|P(D)(B(x,t))\|_{B^{s-1}_{p,r}} \leq& C\|\frac{3-\gamma}{2}((u^{(n+m)})^2-(u^{(n)})^2)+\frac{\gamma}{2}((\partial_x u^{(n+m)})^2-(\partial_x u^{(n)})^2)\|_{B^{s-2}_{p,r}}\\ \leq& C\|u^{(n+m)}-u^{(n)}\|_{B^{s-2}_{p,r}}\|u^{(n+m)}+u^{(n)}\|_{B^{s-2}_{p,r}}\\ \leq& C\|u^{(n)}-u^{(n+m)}\|_{B^{s-1}_{p,r}}.\end{aligned} \tag{3.11}$$

It is easy to check that

$$\begin{aligned}\|u_0^{(n+m+1)}-u_0^{(n+1)}\|_{B^{s-1}_{p,r}} &= \Big\|\sum_{q=n+1}^{n+m}\Delta_q u_0\Big\|_{B^{s-1}_{p,r}} = \Big(\sum_{k\geq -1}2^{k(s-1r)}\|\Delta_k(\sum_{q=n+1}^{n+m}\Delta_q u_0)\|^r_{L^p}\Big)^{\frac{1}{r}}\\ &\leq C\Big(\sum_{k=n}^{n+m+1}2^{-kr}2^{ksr}\|\Delta_k u_0\|^r_{L^p}\Big)^{\frac{1}{r}} \leq C2^{-n}\|u_0\|_{B^s_{p,r}}.\end{aligned} \tag{3.12}$$

From (3.9)–(3.12), arguing by induction, one can easily prove that

$$\|u^{(n+m+1)}-u^{(n+1)}(t)\|_{L^\infty_T(B^{s-1}_{p,r})} \leq C_T\Big(2^{-n}+\int_0^t \|u^{(n+m)}-u^{(n)}\|_{L^\infty_T(B^{s-1}_{p,r})}\mathrm{d}\tau\Big).$$

As $\|u^{(m)}\|_{L^\infty_T(B^s_{p,r})}$ is uniformly bounded in $E^s_{p,r}(T)$, we can conclude the existence of some new constant C'_T, such that

$$\|u^{(n+m+1)}-u^{(n+1)}\|_{L^\infty_T(B^{s-1}_{p,r})} \leq C'_T 2^{-n}. \tag{3.13}$$

From (3.13), we know that $(u^{(n)})_{n\in\mathbb{N}}$ is a cauchy sequence in $C([0,T];B^s_{p,r})$, where $(u^{(n)})_{n\in\mathbb{N}}$ converges to some limit $u\in C([0,T];B^s_{p,r})$. □

Theorem 3.3 *Let $1\leq p,r\leq\infty$ and $s>\max(1+\frac{1}{p},\frac{3}{2})$. Let $u_0\in B^s_{p,r}$. There exists a time T such that the Eqs. (1.2) and (1.3) has a unique solution u in $E^s_{p,r}(T)$.*

Proof Now we have to check that $u\in E^s_{p,r}(T)$ solves Eqs. (1.2) and (1.3).

Since $(u^{(n)})_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;B^s_{p,r})$. From (8) in Lemma 2.5, taking limit in (3.2), we can see that u solves Eqs. (1.2) and (1.3). Using the arguments similar to those in [21,22], we can obtain that $u\in E^s_{p,r}(T)$. □

4. Brow-up criterion

In this section, we will derive the blow-up criterion of the solutions to the Eqs. (1.2) and (1.3). To this end, we first state the following estimates.

Lemma 4.1 *Let $1\leq p,r\leq\infty$ and $s>1$. Let $u\in L^\infty(0,T;B^s_{p,r})$ solving Eqs. (1.2) and (1.3)* *on* $[0,T)\times\mathbb{R}$ *with* $u_0\in B^s_{p,r}\cap\mathrm{Lip}$ *as an initial datum. There exist a constant* C *depending only on* s,γ *and* p*, and a universal constant* C'*, such that for all* $t\in[0,T)$*, we have*

$$\|u(t)\|_{B^s_{p,r}}\le\|u_0\|_{B^s_{p,r}}e^{C\int_0^t(\|u(\tau)\|_{\mathrm{Lip}})\mathrm{d}\tau},\tag{4.1}$$

$$\|u(t)\|_{\mathrm{Lip}}\le\|u_0\|_{\mathrm{Lip}}e^{C'\int_0^t\|\partial_x u(\tau)\|_{L^\infty}\mathrm{d}\tau}.\tag{4.2}$$

Proof Making use of Eqs. (1.2) and (1.3) and Lemma 2.7, yields

$$\begin{aligned}&e^{-C\int_0^t\|\partial_x u(\tau)\|_{L^\infty}\mathrm{d}\tau}\|u(t)\|_{B^s_{p,r}}\\&\le\|u_0\|_{B^s_{p,r}}+C\int_0^t e^{-C\int_0^\tau\|\partial_x u(\tau')\|_{L^\infty}\mathrm{d}\tau'}\|P(D)(\frac{3-\gamma}{2}u^2+\frac{\gamma}{2}(\partial_x u)^2)(\tau)\|_{B^s_{p,r}}\mathrm{d}\tau,\end{aligned}\tag{4.3}$$

where

$$\begin{aligned}\|P(D)(\frac{3-\gamma}{2}u^2+\frac{\gamma}{2}(\partial_x u)^2)\|_{B^s_{p,r}}&\le C\|\frac{3-\gamma}{2}u^2+\frac{\gamma}{2}(\partial_x u)^2\|_{B^{s-1}_{p,r}}\\&\le C\|u\|_{\mathrm{Lip}}\|u\|_{B^s_{p,r}}.\end{aligned}$$

Hence,

$$e^{-C\int_0^t\|\partial_x u(\tau)\|_{L^\infty}\mathrm{d}\tau}\|u(t)\|_{B^s_{p,r}}\le\|u_0\|_{B^s_{p,r}}+C\int_0^t e^{-C\int_0^\tau\|\partial_x u(\tau')\|_{L^\infty}\mathrm{d}\tau'}\|u\|_{\mathrm{Lip}}\|u\|_{B^s_{p,r}}\mathrm{d}\tau,\tag{4.4}$$

which together with Gronwall inequality yields (4.1).

Differentiating Eq. (1.2) with respect to x, we have

$$\partial_t(u_x)+\gamma u\partial_x u_x=-\gamma u_x^2+\partial_x P(D)(\frac{3-\gamma}{2}u^2+\frac{\gamma}{2}(u_x)^2).$$

Using the inequality [21, p68]

$$\|P(D)(u^2+\frac{1}{2}(\partial_x u)^2)\|_{\mathrm{Lip}}\le C\|u\|_{\mathrm{Lip}}\|\partial_x u\|_{L^\infty},$$

and applying the L^∞ estimate for generally transport equation, we can easily prove that

$$\|\partial_x u\|_{L^\infty}\le C\Big(\|u_0\|_{\mathrm{Lip}}+\int_0^t\|P(D)(\frac{3-\gamma}{2}u^2+\frac{\gamma}{2}(\partial_x u)^2)(\tau)\|_{\mathrm{Lip}}\mathrm{d}\tau\Big).$$

Hence Gronwall inequality gives inequality (4.2). □

Definition 4.2 *Let* $u_0\in B^s_{p,r}$*. We define the lifespan* $T^*_{u_0}$ *of solution of (1.2) and (1.3) with initial data* u_0 *as supremum of positive time* T *such that (1.2) and (1.3) has a solution* $u\in E^s_{p,r}(T)$ *on* $[0,T]\times R$.

Theorem 4.3 *Let* u_0 *be as in Theorem 3.3, and* u *be the corresponding solution. Then*

$$T^*_{u_0}<\infty\Rightarrow\int_0^{T^*_{u_0}}\|\partial_x u(\tau)\|_{L^\infty}\mathrm{d}\tau=\infty.$$

Proof Let $u\in\cap_{T<T^*_{u_0}}E^s_{p,r}(T)$ be such that $\int_0^{T^*_{u_0}}\|\partial_x u(\tau)\|_{L^\infty}\mathrm{d}\tau<\infty$. Thanks to (4.2), we have $\int_0^{T^*_{u_0}}\|u(\tau)\|_{\mathrm{Lip}}\mathrm{d}\tau$ is also finite. Hence, (4.1) insures that

$$\|u(t)\|_{B^s_{p,r}}\le M_{T^*_{u_0}}\triangleq e^{\int_0^{T^*_{u_0}}\|u(\tau)\|_{\mathrm{Lip}}\mathrm{d}\tau},\quad\forall t\in[0,T^*_{u_0}].$$

Let ε be positive such that $\varepsilon < \frac{1}{2C^2 M_{T^*_{u_0}}}$, where C is the same constant used in Theorem 3.3. Then we have a solution $\tilde{u}(t) \in E^s_{p,r}(\varepsilon)$ to Eqs. (1.2) and (1.3) with initial datum $u(T^*_{u_0} - \frac{\varepsilon}{2})$. For the sake of uniqueness, $\tilde{u}(t) = u(t + T^*_{u_0} - \frac{\varepsilon}{2})$ on $t \in [0, \frac{\varepsilon}{2})$. So that $\tilde{u}$ extends the solution u beyond $T^*_{u_0}$. The contradiction completes the proof of the theorem. $\square$

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