The Construction of $I$-Bornological Vector Spaces

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Abstract In this paper, the concept of $I$-bornological vector spaces and two examples of the spaces are given. Two methods on constructing new $I$-bornological vector spaces are discussed, one is using a (crisp) bornological vector space to induce an $I$-bornological vector space, the other is utilizing $I$-bornological linear maps to generate an $I$-bornological vector space.

Keywords $I$-vector bornology; $I$-bornological vector space; $I$-bornological linear map

MR(2010) Subject Classification 46S40; 46A17

1. Introduction

The theory of bornological spaces was first introduced by Hogbe-Nlend [1] (It should be noticed that, the notion of bornological vector spaces is different from that in [2,3], the former [1] is a space without topology, however, the latter [2] is a space with topology). Since then, many authors have carried out various work on bornological spaces. They investigated lattice-valued bornological spaces [4], discussed topologies of strong uniform convergence on bornologies [5–8], studied bornologies for metrically generated theories [9], established Ekeland-type variational principle and Caristi fixed point theorem in bornological vector spaces [10]. Nevertheless, the work on the construction of fuzzy bornological vector spaces was rarely discussed. In this paper, we introduce the notion of fuzzy bornological vector spaces. Based on that, we investigate the method of constructing new fuzzy bornological vector spaces.

It should be mentioned here that, according to the standardized terminology in [11], fuzzy set [12] should be called an $I(= [0,1])$-valued map. For convenience, we rename the fuzzy bornological vector space as $IBVS$ in the following paper. Our purpose is three-fold. First, we introduce the concept of $IBVS$. For specifically illustrating the definition, we show two examples of $IBVS$. By using $Q$-neighborhood [13], we prove the family consisting of all $I$-bounded subset [14,15] is an $I$-vector bornology; By employing the notion of $I$-semi-norm [16], we give the concept of $I$-$\lambda$-bounded subset for the $I$-semi-norm, and obtain that the fuzzy subsets which are $I$-$\lambda$-bounded subset for the $I$-semi-norm form an $I$-vector bornology. Thus we obtain the concrete
IBVSs. Next, in consideration of the relation between \( r \)-cut sets and \( I \)-valued maps, we give special \( I \)-valued maps, which eventually induce an \( I \)-vector bornology. Further by employing an \( r \)-cut vector bornological space, we prove a characterization theorem of the induced IBVS. Finally, by using the fuzzy linear maps [17], we consider the \( I \)-bornological linear maps and prove that the primary image under inverse projective map about a family of IBVS is still an IBVS, which provides another method of constructing IBVS.

2. Preliminaries

Throughout this paper, let \( X \) be a vector space over \( K \) (\( R \) or \( C \)) and \( \theta \) denote the zero element of \( X \). Let \( I = [0, 1] \) and \( I^X \) denote a family of all fuzzy subsets of \( X \). A fuzzy subset which takes the constant value \( r \) on \( X \) (\( 0 \leq r \leq 1 \)) is denoted by \( r^* \). A fuzzy subset of \( X \) is called a fuzzy point [13], denoted by \( x_\lambda \), if it takes value 0 at \( y \in X \) \( \setminus \{x\} \) and its value at \( x \) is \( \lambda \). The set of all fuzzy points on \( X \) is denoted by \( Pt(I^X) \). A fuzzy point \( x_\lambda \) is said to be quasi-coincident with a fuzzy subset \( A \), denoted by \( x_\lambda \in A \), if \( A(x) > 1 - \lambda \). \( \mathscr{A} \) is a non-empty set.

**Definition 2.1** ([18,19]) A stratified \( I \)-topology \( \tau \) on \( X \) is said to be an \( I \)-vector topology, if the following two mappings are continuous:

\[
\begin{align*}
  f & : X \times X \to X \quad (x, y) \mapsto x + y, \\
  g & : K \times X \to X \quad (k, x) \mapsto kx,
\end{align*}
\]

where \( K \) is equipped with the \( I \)-topology induced by the usual topology, \( X \times X \) and \( K \times X \) are equipped with the corresponding product \( I \)-topologies. A vector space \( X \) with an \( I \)-vector topology, denoted by \( (X, \tau) \), is called an \( I \)-topological vector space.

**Definition 2.2** ([18,19]) Let \( A, B \in I^X \) and \( k \in K \). Then \( A + B \) and \( kA \) are defined respectively by

\[
\begin{align*}
  (A + B)(x) &= \bigvee \{A(s) \land B(t) : s + t = x\}; \\
  (kA)(x) &= A(x/k) \quad \text{whenever } k \neq 0; \\
  (0A)(x) &= \begin{cases} 
  \bigvee_{t \in X} A(t), & x = \theta, \\
  0, & x \neq \theta.
\end{cases}
\end{align*}
\]

In particular, for \( x_\lambda, y_\mu \in Pt(I^X) \), we have

\[
x_\lambda + y_\mu = (x + y)_{\lambda \land \mu}, \quad kx_\lambda = (kx)_\lambda.
\]

**Definition 2.3** ([1]) A vector bornology on \( X \) is a collection \( \mathcal{B} \) of subsets of \( X \) which satisfies the following conditions:

\begin{align*}
  (B1) & \quad X = \bigcup_{B \in \mathcal{B}} B; \\
  (B2) & \quad B_1 \subseteq B_2 \text{ and } B_2 \in \mathcal{B} \text{ implies that } B_1 \in \mathcal{B}; \\
  (B3) & \quad B_1, B_2 \in \mathcal{B} \text{ implies that } B_1 \cup B_2 \in \mathcal{B};
\end{align*}
The construction of I-bornological vector spaces

(B4) \( B_1, B_2 \in \mathfrak{B} \) implies that \( B_1 + B_2 \in \mathfrak{B} \), where

\[
B_1 + B_2 = \{ x_1 + x_2 : x_1 \in B_1, x_2 \in B_2 \};
\]

(B5) For any \( \lambda \in \mathbb{K}, B \in \mathfrak{B} \) implies that \( \lambda B \in \mathfrak{B} \) and \( \bigcup_{|\lambda| \leq 1} \lambda B \in \mathfrak{B} \).

The ordered pair \((X, \mathfrak{B})\) is called a bornological vector space (briefly BVS) and every candidate of \( \mathfrak{B} \) is called a vector bornological subset.

It is easy to prove the following conclusion:

Remark 2.4 For any \( r \in (0, 1) \) and \( A \in I^X \), \( \sigma_r(A) = \{ x : A(x) > r \} \).

(1) If \( A_0 \in I^X, \alpha \in \mathcal{A} \), where \( \mathcal{A} \) denotes indicator set, then \( \sigma_r(\bigcup_{\alpha \in \mathcal{A}} A_0) = \bigcup_{\alpha \in \mathcal{A}} \sigma_r(A_0) \);

(2) If \( A, B \in I^X \), then \( \sigma_r(A + B) = \sigma_r(A) + \sigma_r(B) \);

(3) For \( r \in (0, 1] \), \( \sigma_r(A) = r \sigma_r(A) \);

(4) For any \( \lambda \in \mathbb{K} \) and \( A \in I^X \), \( \lambda \sigma_r(A) = \sigma_r(\lambda A) \).

We just prove (4). It is not difficult to prove that when \( \lambda \neq 0 \), \( \lambda \sigma_r(A) = \sigma_r(\lambda A) \). So, we just need to prove the conditions when \( \lambda = 0 \). For each \( x \in \sigma_r(A) \), we have \( 0A(0x) = 0A(\theta) = \sup_{t \in X} A(t) \geq A(x) > r \), which means \( 0x \in \sigma_r(A) \), thus \( 0\sigma_r(A) \subset \sigma_r(A) \). Conversely, for each \( y \in \sigma_r(0A) \), we have \( 0A(y) > r \), so \( y = \theta = 0\sigma_r(A) \), which shows that \( \sigma_r(0A) \subset 0\sigma_r(A) \).

3. IBVS and examples

In this section, we introduce the notion of an IBVS and give two examples of IBVS.

Definition 3.1 An I-vector bornology on \( X \) is a family \( \mathfrak{B} \) of fuzzy subsets of \( X \) \((\mathfrak{B} \subset I^X)\) satisfying the following conditions:

(IB1) \( X = \bigcup_{B^* \in \mathfrak{B}} B^*(x) = 1, \forall x \in X \);

(IB2) \( B_1^* \subset B_2^* \) and \( B_2^* \in \mathfrak{B} \) implies that \( B_1^* \in \mathfrak{B} \);

(IB3) \( B_1^* \subset B_2^* \) and \( B_2^* \in \mathfrak{B} \) implies that \( B_1^* \cup B_2^* \in \mathfrak{B} \);

(IB4) \( B_1^* \subset B_2^* \) and \( B_2^* \in \mathfrak{B} \) implies that \( B_1^* + B_2^* \in \mathfrak{B} \), where

\[
(B_1^* + B_2^*)(x) = \sup_{s + t = x} \min\{B_1^*(s), B_2^*(t)\}, \forall x \in X;
\]

(IB5) For any \( \lambda \in \mathbb{K}, B^* \in \mathfrak{B} \) implies that \( \lambda B^* \in \mathfrak{B} \) and \( \bigcup_{|\lambda| \leq 1} \lambda B^* \in \mathfrak{B} \), where

\[
(\lambda B^*)(x) = \begin{cases} 
B^*(x/\lambda), & \lambda \neq 0 \\
(0B^*)(x) = \bigvee_{t \in X} B^*(t), & x = \theta \\
0, & x \neq \theta
\end{cases}
\]

for each \( x \in X \). Every candidate of \( \mathfrak{B} \) is called an I-vector bornological subset. If \( \mathfrak{B} \subset I^X \) is an I-vector bornology on \( X \), then the ordered pair \((X, \mathfrak{B})\) is called an I-bornological vector space (briefly IBVS).

Let \( X \) be a (crisp) topological vector space. A bounded subset of \( X \) is a subset that is absorbed by every neighborhood of zero. The collection of bounded subsets of \( X \) forms a vector bornology on \( X \) called the Vov Neumann Bornology of \( X \) (see [1]). Corresponding to the bounded subsets in (crisp) topological vector space, there is an I-bounded subset defined
by $Q$-neighborhood in $I$-topological vector space $(X, \tau)$ (A fuzzy subset $U$ of $X$ is called $Q$-neighborhood of $x_0$ if there exists $G \in \tau$ such that $x_0 \in G \subseteq U$ (see [13]). Now, as a reference we mention the definition of the $I$-bounded subset. Then we prove the $I$-bounded subset defined by $Q$-neighborhood is an $I$-vector bornological subset. We refer to [14,19,20] for other symbols or definitions which are not mentioned here.

**Definition 3.2** ([14,15]) A fuzzy set $B$ in $(X, \tau)$ is said to be $\lambda$-bounded $(\lambda \in (0,1])$, if for each $Q$-neighborhood $U$ of $\theta_\lambda$ in $X$, there exist $t > 0$ and $r \in (1 - \lambda, 1]$ such that $B \cap r \subseteq tU$. $B$ is said to be $I$-bounded, if it is $\lambda$-bounded for each $\lambda \in (0,1]$. The family consisting of all $I$-bounded subsets is denoted by $\mathcal{V}$.

**Example 3.3** Let $(X, \tau)$ be an $I$-topological vector space. Then $\mathcal{V}$ is an $I$-vector bornology.

**Proof** We need to prove that $\mathcal{V}$ satisfies the conditions (IB1)–(IB5) in Definition 3.1. 

(IB1) For any $x \in X$, we have $\{x\} \in \mathcal{V}$, which means $\{x\}$ is an $I$-bounded subset in the sense of Definition 3.2. In fact, for each $\lambda \in (0,1]$ and $Q$-neighborhood $U$ of $\theta_\lambda$, by [21], there exists a balanced and $Q - \lambda$ absorbing subset $V$ such that $V \subseteq U$, where $V$ is a $Q - \lambda$ absorbing subset, that is, there exists $t > 0$ such that $V(tx) > 1 - \lambda$. Obviously, there exists $\varepsilon > 0$ such that $V(tx) > 1 - \lambda + \varepsilon$, therefore, $tx_1 - \lambda + \varepsilon \in V$. Notice that $\{x\} \cap (1 - \lambda + \varepsilon)^* = x_1 - \lambda + \varepsilon \in 1/tV \cap 1/U$, so $\{x\} \in \mathcal{V}$. At the same time, it is easy to see that $\{x\}(x) = 1$ for each $x \in X$. Combining it with $\{x\} \in \mathcal{V}$, we obtain that $\sup_{B \in \mathcal{V}} B(x) = 1$. Thus $X = \bigcup_{B \in \mathcal{V}} B$.

(IB2) Let $B_1 \subset B_2$ and $B_2 \in \mathcal{V}$. Then for each $\lambda \in (0,1]$ and $Q$-neighborhood $U$ of $\theta_\lambda$, there exist $t > 0$ and $r \in (1 - \lambda, 1]$ such that $B_2 \cap r \subseteq tU$. Since $B_1 \subset B_2$, $B_1 \cap r \subseteq tU$, which means $B_1 \in \mathcal{V}$.

(IB3) Let $B_1, B_2 \in \mathcal{V}$. Then for each $\lambda \in (0,1]$ and $Q$-neighborhood $U$ of $\theta_\lambda$, there exists a balanced $Q - \lambda$ subset $V$ such that $V \subseteq U$. Since $B_1, B_2 \in \mathcal{V}$, there exist $t_1, t_2 > 0$ and $r_1, r_2 \in (1 - \lambda, 1]$ such that $B_1 \cap r_1 \subseteq t_1 U$, $B_2 \cap r_2 \subseteq t_2 V$. Taking $r = \min\{r_1, r_2\}$, $t = \max\{t_1, t_2\}$, then we have $(B_1 \cup B_2) \cap r = (B_1 \cap r_1) \cup (B_2 \cap r_2) \subseteq (B_1 \cap r_1) \cup (B_2 \cap r_2) \subseteq t_1 V \cap t_2 V \subseteq tV \subseteq tU$, thus $B_1 \cup B_2 \in \mathcal{V}$.

(IB4) Let $B_1, B_2 \in \mathcal{V}$. Then for each $\lambda \in (0,1]$ and $Q$-neighborhood $U$ of $\theta_\lambda$, there exists a balanced $Q - \lambda$ subset $V$ such that $V \cap V \subseteq U$ (see [19]). Considering that $B_1, B_2 \in \mathcal{V}$, there exist $t_1, t_2 > 0$ and $r_1, r_2 \in (1 - \lambda, 1]$ such that $B_1 \cap r_1 \subseteq t_1 V$, $B_2 \cap r_2 \subseteq t_2 V$. Let $r = \min\{r_1, r_2\}$, $t = \max\{t_1, t_2\}$. We have $(B_1 + B_2) \cap r = (B_1 \cap r) + (B_2 \cap r) \subseteq (B_1 \cap r_1) + (B_2 \cap r_2) \subseteq t_1 V + t_2 V \subseteq tV + tV \subseteq tU$, thus $B_1 + B_2 \in \mathcal{V}$.

(IB5) Let $B \in \mathcal{V}$. For any $\alpha \in \mathbb{K}$, firstly, we need to prove $\alpha B \in \mathcal{V}$. In fact, for each $\lambda \in (0,1]$ and $Q$-neighborhood $U$ of $\theta_\lambda$, there exists a balanced $Q - \lambda$ subset $V$ such that $V \subseteq U$. Note that $B \in \mathcal{V}$, so there exist $t > 0$ and $r \in (1 - \lambda, 1]$ such that $B \cap r \subseteq tU$. Thus, we have $(\alpha B) \cap r^* = \alpha(B \cap r^*) \in \alpha V \subseteq ((\alpha t) + 1)V \subseteq ((\alpha t) + 1)U$. Next, we prove for each $B \in \mathcal{V}$, $\bigcup_{|\alpha| \leq 1} \alpha B \in \mathcal{V}$. In fact, for each $\lambda \in (0,1]$ and $Q$-neighborhood $U$ of $\theta_\lambda$, there exists a balanced $Q - \lambda$ subset $V$ such that $V \subseteq U$. Note that $B \in \mathcal{V}$, so there exist $t > 0$ and $r \in (1 - \lambda, 1]$ such that $B \cap r \subseteq tU$. Therefore, we have $(\bigcup_{|\alpha| \leq 1} \alpha B) \cap r^* = \bigcup_{|\alpha| \leq 1} \alpha(B \cap r^*) \subseteq \bigcup_{|\alpha| \leq 1} \alpha V \subseteq tV \subseteq tU$. 

In [22], Wu and Fang introduced the definition of fuzzy norms. Then Wu and Ma gave the notion of fuzzy semi-norms [20], and indicate that the fuzzy norm is a special fuzzy semi-norm.

Furthermore, in [16], Wu and Ma gave several examples of fuzzy norm on concrete vector spaces such as the vector space $L^w[0, 1] = \cap_{p \geq 1} L_p[0, 1]$ (where $L_p[0, 1] = \{f(t) : f(t)$ is measurable function on $[0, 1]$, and satisfies $\int_0^1 |f(t)|^p dt < \infty\}$). In which, letting $p(f) = (\int_0^1 |f(t)|^p dt)^{\lambda}$, for each $\lambda \in (0, 1]$ and for all $f \in L^w[0, 1]$, they proved $p(f_{\lambda})$ is a fuzzy norm on $L^w[0, 1]$, so it is a fuzzy semi-norm naturally. The examples show that the definition of fuzzy semi-norm in the sense of Wu and Ma is practical. Now we use the definition of I-semi-norm in [16] to introduce another example of I-vector bornology.

**Definition 3.4** ([16]) A mapping $p : P(I^X) \to [0, +\infty)$ is called an I-semi-norm on $X$, if it satisfies the following conditions:

1. $p(kx) = |k|p(x)$ for all $x \in P(I^X)$ and all $k \in \mathbb{K}$;
2. $p(x + y) \leq p(x) + p(y)$;
3. $p(x)$ is nonincreasing and left continuous for $\lambda$.

**Definition 3.5** Let $X$ be a vector space over $\mathbb{K}$, $p$ be an I-semi-norm on $X$. A fuzzy subset $A$ of $X$ is said to be an $I$-$\lambda$-bounded subset for the I-semi norm $p$ if $p(A) = \sup\{p(x) | x \in A\} < \infty$, $\lambda \in (0, 1]$. (Appoint $\sup \emptyset = 0$).

**Example 3.6** The fuzzy subsets of $X$ which are $I$-$\lambda$-bounded subsets for the I-semi-norm $p$ form an I-vector bornology on $X$, $\lambda \in (0, 1]$.

**Proof** Let $\mathcal{W}^* = \{A | p(A) = \sup\{p(x) | x \in A\} < \infty\}$. We need to prove it satisfies the conditions (IB1)–(IB5) in Definition 3.1.

(1) For any $x \in X$, it is easy to know that $\sup\{p(x) | x \in A\} = p(x) < \infty$, which means that $\{x\} \in \mathcal{W}^*$, so $X = \bigcup_{A \in \mathcal{W}^*} A$.

(2) Let $A_1 \subset A_2$ and $A_2 \in \mathcal{W}^*$. Consider the fact that $\{p(x) | x \in A_1\} \subset \{p(x) | x \in A_2\}$, hence $\sup\{p(x) | x \in A_1\} \leq \sup\{p(x) | x \in A_2\} < \infty$, which shows $A_1 \in \mathcal{W}^*$.

(3) Let $A_1, A_2 \in \mathcal{W}^*$. Notice that $\{p(x) | x \in (A_1 \cup A_2)\} = \{p(x) | x \in A_1\} \cup \{p(x) | x \in A_2\}$, hence $\sup\{p(x) | x \in (A_1 \cup A_2)\} = \max\{\sup\{p(x) | x \in A_1\}, \sup\{p(x) | x \in A_2\}\} < \infty$, which implies that $A_1 \cup A_2 \in \mathcal{W}^*$.

(4) Let $A_1, A_2 \in \mathcal{W}^*$. Notice that $\{p(x) | x \in (A_1 + A_2)\} \subset \{p(s) | s \in A_1\} + \{p(t) | t \in A_2\}$, where $s + t = x$. So

$$\sup\{p(x) | x \in (A_1 + A_2)\} \leq \sup\{p(s) | s \in A_1\} + \sup\{p(t) | t \in A_2\} < \infty,$$

thus $A_1 + A_2 \in \mathcal{W}^*$.

(5) Let $A \in \mathcal{W}^*$. For any $\alpha \in \mathbb{K}$, if $\alpha = 0$, then $\sup\{p(x) | x \in \alpha A\} = \sup\{p(\theta)\} < \infty$; if $\alpha \neq 0$, then $\{p(x) | x \in \alpha A\} = \{p(x)(1/\alpha x) | x \in \alpha A\} = \{p(\alpha \cdot 1/\alpha x) | x \in \alpha A\} = \{p(\alpha) | x \in \alpha A\}$.
Remark 4.2

From Theorem 4.1, we know that a vector bornology can induce an $I$-vector bornology $\omega(\mathcal{B})$, where $\omega(\mathcal{B})$ is called an induced $I$-vector bornology, and $(X, \omega(\mathcal{B}))$ is called an induced IBVS.

4. Induced IBVS

In this section, we introduce one method of constructing an IBVS and prove a characterization theorem of the induced IBVS.

At the beginning, we construct a family $\omega(\mathcal{B})$ of fuzzy subsets on $X$, which satisfies that a fuzzy subset $A \in \omega(\mathcal{B})$ iff $\sigma_r(A) \in \mathcal{B}$ for each $r \in [0, 1]$. Then, we prove the following theorem:

Theorem 4.1 Let $(X, \mathcal{B})$ be a BVS. Then $(X, \omega(\mathcal{B}))$ is an IBVS.

Proof We just need to prove that $\omega(\mathcal{B})$ satisfies the conditions (IB1)–(IB5) in Definition 3.1.

(IB1) By (B1), we have $X = \bigcup_{A \in \mathcal{B}} A$, which implies $\sup_{A \in \mathcal{B}} A(x) = 1, \forall x \in X$. From $A = \sigma_r(A) \in \mathcal{B}$ for any $r \in [0, 1]$, we know that $A \in \omega(\mathcal{B})$. Thus $\sup_{A \in \omega(\mathcal{B})} A(x) = 1$, hence $X = \bigcup_{A \in \omega(\mathcal{B})} A$.

(IB2) Let $A_1 \subseteq A_2$ and $A_2 \in \omega(\mathcal{B})$. Thus $\sigma_r(A_1) \subseteq \sigma_r(A_2)$ for each $r \in [0, 1]$. Meanwhile, note that $A_1 = \sigma_r(A_1)$, we have $\sigma_r(A_1) \in \mathcal{B}$ for every $r \in [0, 1]$, which means that $A_1 \in \omega(\mathcal{B})$.

(IB3) Let $A_1, A_2 \in \omega(\mathcal{B})$. Then $\sigma_r(A_1), \sigma_r(A_2) \in \mathcal{B}$ for each $r \in [0, 1]$. From (B3) we obtain that $\sigma_r(A_1) \cup \sigma_r(A_2) \in \mathcal{B}$. Note that $\sigma_r(A_1 \cup A_2) = \sigma_r(A_1) \cup \sigma_r(A_2)$. Thus $\sigma_r(A_1 \cup A_2) \in \mathcal{B}$ for each $r \in [0, 1]$, which implies that $A_1 \cup A_2 \in \omega(\mathcal{B})$.

(IB4) Let $A_1, A_2 \in \omega(\mathcal{B})$. Then $\sigma_r(A_1), \sigma_r(A_2) \in \mathcal{B}$ for each $r \in [0, 1]$. From (B4) we have $\sigma_r(A_1) + \sigma_r(A_2) \in \mathcal{B}$. Note that $\sigma_r(A_1 + A_2) = \sigma_r(A_1) + \sigma_r(A_2)$. Thus $\sigma_r(A_1 + A_2) \in \mathcal{B}$ for each $r \in [0, 1]$, which implies that $A_1 + A_2 \in \omega(\mathcal{B})$.

(IB5) Let $A \in \omega(\mathcal{B})$. Then $\sigma_r(A) \in \mathcal{B}$ for each $r \in [0, 1]$. For any $\lambda \in \mathbb{K}$, by Remark 2.4 and (B5), we have $\lambda \sigma_r(A) = \sigma_r(\lambda A)$ and $\sigma_r(\lambda A) \in \mathcal{B}$ for any $r \in [0, 1]$, which implies that $\lambda A \in \omega(\mathcal{B})$. Simultaneously, by Remark 2.4 and (B5), we have

$$\sigma_r\left(\bigcup_{|\lambda| \leq 1} \lambda A\right) = \bigcup_{|\lambda| \leq 1} \sigma_r(\lambda A) = \bigcup_{|\lambda| \leq 1} \lambda \sigma_r(A),$$

and $\sigma_r(\bigcup_{|\lambda| \leq 1} \lambda A) \in \mathcal{B}$ for any $r \in [0, 1]$, which implies that $\bigcup_{|\lambda| \leq 1} \lambda A \in \omega(\mathcal{B})$.

Thus from Definition 4.1, we know that $(X, \omega(\mathcal{B}))$ is an IBVS. □
Lemma 4.3 Let \((X, \mathfrak{B})\) be an IBVS. Then \((X, \mathfrak{B}^v)\) is a BV for each \(r \in [0, 1]\), where \(\mathfrak{B}^v = \{\sigma_r(B^*) : B^* \in \mathfrak{B}\}\).

Proof We need to prove that for each \(r \in [0, 1]\), \(\mathfrak{B}^v\) satisfies the conditions (B1)–(B5) in Definition 2.3.

(B1) Obviously, when \((X, \mathfrak{B})\) is an IBVS, from (IB1), we have for each \(x \in X\) and \(r \in [0, 1]\), there exists \(B^* \in \mathfrak{B}\), such that \(B^*(x) > r\), hence \(x \in \sigma_r(B^*)\), which implies that \(X \subset \bigcup_{B^* \in \mathfrak{B}} \sigma_r(B^*)\). Thus \(r \in [0, 1]\), and so \(X = \bigcup_{B^* \in \mathfrak{B}} \sigma_r(B^*)\).

(B2) Let \(B_1 \subset B_2\) and \(B_2 \in \mathfrak{B}^v\). Then there exists \(B^*_1 \in \mathfrak{B}\), such that \(B_2 = \sigma_r(B^*_2)\). When \(r \in (0, 1]\), we put

\[
B^*_1(x) = \begin{cases}
(B^*_2(x) + r)/2, & x \in B_1; \\
(r - \varepsilon_0), & x \in B_2 \setminus B_1; \\
B^*_2(x), & x \in X \setminus B_2,
\end{cases}
\]

where \(\varepsilon_0 \in (0, r]\). If \(x \in B_1\), it is easy to know that \(B^*_2(x) > (B^*_2(x) + r)/2 = B^*_1(x)\). If \(x \in B_2 \setminus B_1\), \(B^*_2(x) > r > r - \varepsilon_0\). If \(x \in X \setminus B_2\), \(B^*_2(x) = B^*_1(x)\). Therefore, \(B^*_1(x) = B^*_2(x)\) for any \(x \in X\), so \(B^*_1 \subset B^*_2\). By (IB2), we obtain that \(B^*_1 \in \mathfrak{B}^v\). Meanwhile, it is not difficult to prove that \(B_1 = \sigma_r(B^*_1)\). In fact, for each \(x \in B_1\), we have \(B^*_1(x) = (B^*_2(x) + r)/2 > r\), which means \(x \in \sigma_r(B^*_1)\), thus \(B_1 \subset \sigma_r(B^*_1)\); Conversely, for each \(x \in \sigma_r(B^*_1)\), we have \(B^*_1(x) \geq r\), which implies that \(x \in B_1\), or if \(x \in \mathfrak{B}B_1\), by the definition of \(B^*_1\), \(B^*_1(x) < r\). So \(\sigma_r(B^*_1) \subset B_1\).

When \(r = 0\), we put

\[
B^*_1(x) = \begin{cases}
B^*_2(x)/2, & x \in B_1 \\
0, & x \in X \setminus B_1.
\end{cases}
\]

Repeat the above proof, we could obtain same conclusion, that is \(B_1 \in \mathfrak{B}^v\) for \(r \in (0, 1]\).

(B3) Let \(B_1, B_2 \in \mathfrak{B}^v\). Then there exist \(B^*_1, B^*_2 \in \mathfrak{B}\), such that \(B_1 = \sigma_r(B^*_1), B_2 = \sigma_r(B^*_2)\). By (IB3), we have \(B^*_1 \cup B^*_2 \in \mathfrak{B}\). Meanwhile, by Remark 2.4, we have \(B_1 \cup B_2 = \sigma_r(B^*_1) \cup \sigma_r(B^*_2) = \sigma_r(B^*_1 \cup B^*_2)\), so we have \(B_1 \cup B_2 \in \mathfrak{B}^v\).

(B4) Let \(B_1, B_2 \in \mathfrak{B}^v\). Then there exist \(B^*_1, B^*_2 \in \mathfrak{B}\), such that \(B_1 = \sigma_r(B^*_1), B_2 = \sigma_r(B^*_2)\). By Remark 2.4, we have \(\sigma_r(B^*_1) + \sigma_r(B^*_2) = \sigma_r(B^*_1 + B^*_2)\). Considering \(B^*_1 + B^*_2 \in \mathfrak{B}\), thus we obtain that \(B^*_1 + B^*_2 \in \mathfrak{B}^v\).

(B5) Let \(B \in \mathfrak{B}^v\). Then there exists \(B^* \in \mathfrak{B}\), such that \(B = \sigma_r(B^*)\). For any \(\lambda \in \mathbb{K}\), by Remark 2.4 and (IB5), we obtain that \(\lambda B = \lambda \sigma_r(B^*) = \sigma_r(\lambda B^*)\), and \(\lambda B^* \in \mathfrak{B}\), which implies that \(\lambda B \in \mathfrak{B}^v\).

Likewise, by Remark 2.4, we have

\[
\bigcup_{|\lambda| \leq 1} \lambda B = \bigcup_{|\lambda| \leq 1} \lambda \sigma_r(B^*) = \bigcup_{|\lambda| \leq 1} \sigma_r(\lambda B^*) = \sigma_r\left( \bigcup_{|\lambda| \leq 1} \lambda B^* \right).
\]

Note that \(\bigcup_{|\lambda| \leq 1} \lambda B^* \in \mathfrak{B}\), so \(\bigcup_{|\lambda| \leq 1} \lambda B \in \mathfrak{B}^v\).

Therefore, for each \(r \in [0, 1]\), \((X, \mathfrak{B}^v)\) is a BV of \(X\). □

Then, we will show the characterization theorem of the induced IBVS.

Theorem 4.4 Let \((X, \mathfrak{B})\) be an IBVS, \(\mathfrak{B}\) be a vector bornology, and \(\omega(\mathfrak{B})\) be an induced
An IBVS generated by I-bornological linear map

In this section, we will give the other method for constructing new IBVS by using I-bornological linear maps. For this, we will introduce the definition of fuzzy linear map due to Katsaras and Liu [17] firstly.

Definition 5.1 Let $X, Y$ be two vector spaces over $K$. Then $f$ is called a fuzzy linear map from $X$ into $Y$ (where $f$ is a fuzzy map extended by the general map $\tilde{f}$, see [20]), if for any $A, B \in I^X$, $f(\alpha A + \beta B) = \alpha f(A) + \beta f(B)$, $\alpha, \beta \in K$.

By using the definition of fuzzy linear map, we give:

Definition 5.2 Let $(X, \mathcal{I}B)$ and $(Y, \mathcal{I}D)$ be two I-bornological spaces. If $f$ is a fuzzy linear map of $X$ into $Y$ and for each $B^* \in \mathcal{I}B$, $f(B^*) \in \mathcal{I}D$. Then $f$ is called an I-bornological linear map.

Obviously, the identity map of any IBVS is an I-bornological linear map. As another example of the I-bornological linear map, and from [1] and Theorem 4.1, we have:

Example 5.3 The linear map from IBVS $(X, \mathcal{I}B)$ into the scalar field $K$ is an I-bornological linear map, where the scalar field $K$ is endowed with the I-vector bornology defined by $\mathcal{B}$,

$$\mathcal{B} = \{A : K \rightarrow [0, 1]|\sigma_r(A) \subset K \text{ and } \sup\{|b| : b \in \sigma_r(A)\} < \infty\}.$$
linear map.

An $I$-vector bornology $\mathcal{B}_1$ on $X$ is a Finer Bornology than an $I$-vector bornology $\mathcal{B}_2$ on $X$ (or $\mathcal{B}_2$ is a Coarser Bornology than $\mathcal{B}_1$) if $\mathcal{B}_1 \subset \mathcal{B}_2$. It is not difficult to prove the following result:

**Theorem 5.5** The identity map $(X, \mathcal{B}_1) \longrightarrow (X, \mathcal{B}_2)$ is an $I$-bornological linear map if and only if $\mathcal{B}_1 \subset \mathcal{B}_2$.

It is easy to prove or obtain from [20] the following result:

**Lemma 5.6** Let $X, Y$ be two non-empty sets, $f : X \to Y$ and $A, B \in I_X$. Then

1. $A \subset B \Rightarrow f(A) \subset f(B)$;
2. $f(\bigcup_{\alpha \in \mathcal{A}} A_\alpha) = \bigcup_{\alpha \in \mathcal{A}} f(A_\alpha)$.

Then we show the method for obtaining a new IBVS:

**Theorem 5.7** Let $(X_\alpha, \mathcal{U}_\alpha)(\alpha \in \mathcal{A})$ be a family of $I$-vector bornological spaces, and $X$ be a vector space. Suppose that, for every $\alpha \in \mathcal{A}$, $f_\alpha : X \to X_\alpha$ is a linear map. If the family $\mathcal{U}$ of all fuzzy subsets $A$ on $X$ has the following property: Every $A \in \mathcal{U}$ iff $f_\alpha(A) \in \mathcal{U}_\alpha$ for each $\alpha \in \mathcal{A}$. Then $\mathcal{U}$ is an $I$-vector bornology on $X$.

**Proof** Using the conditions (IB1)–(IB5) in Definition 3.1 and Lemma 5.6, we can easily obtain the result, so we just prove the conditions (IB1) and (IB3).

(IB1) For each $x \in X$ and $\alpha \in \mathcal{A}$, $f_\alpha(x) \in X_\alpha$. Since $\mathcal{U}_\alpha$ is an $I$-vector bornology, by (IB1) in Definition 3.1, we know that $f_\alpha(x) \in \mathcal{U}_\alpha$, thus $\{x\} \in \mathcal{U}$, which means $X = \bigcup_{A \in \mathcal{U}} A$.

(IB3) Let $A_1, A_2 \in \mathcal{U}$. By Lemma 5.6, we know $f_\alpha(A_1 \cup A_2) = f_\alpha(A_1) \cup f_\alpha(A_2)$. Since $f_\alpha(A_1), f_\alpha(A_2) \in \mathcal{U}_\alpha, f_\alpha(A_1) \cup f_\alpha(A_2) \in \mathcal{U}_\alpha$, which implies $A_1 \cup A_2 \in \mathcal{U}$.

It follows that the theorem is proved.

**Acknowledgements** We thank the referees for their time and comments.

**References**

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