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Some Notices on Mina Matrix and Allied Determinant Identities

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Abstract By means of a special LU factorization of the Mina matrix with the n-th row and k-th column entry $\mathbf{D}_x^n(f^{a_k}(x))$, we obtain not only a short proof of the Mina determinant identity but also the inverse of the Mina matrix. Finally, by use of some similar factorizations built on the Lagrange interpolation formula, two new determinant identities of Mina type are established.

Keywords Mina determinant identity; Mina matrix; LU factorization; derivative operator; inverse; combinatorial identity; Lagrange interpolation; Melzak's formula

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1. Introduction

Throughout this paper, we will use $\mathbf{D}_x^n f(x)$ to denote the n-th derivative of $f(x)$ with respect to x, and $X = (a_{n,k})_{n,k=0}^{m-1}$ (resp., $\det(X)$) the square matrix of order m with the n-th row and k-th column entry $a_{n,k} = [X]_{n,k}$ (resp., the determinant of X).

An old theorem of Mina [1], rediscovered independently fifty years later by Zeitlin [2], claims that

Theorem 1.1 (Mina determinant identity) *Let $f(x)$ be an $(m-1)$-times differentiable function. Then*

$$\det\left(\mathbf{D}_x^n(f^k(x))\right)_{n,k=0}^{m-1} = f'(x)^{m(m-1)/2}\prod_{i=0}^{m-1} i!\,. \tag{1.1}$$

This determinant identity was regarded by Poorten [3] as "should be better known". Afterwards, Wilf [4] generalized this formula to the following

Theorem 1.2 (Generalization of the Mina determinant identity: Wilf [4, Eq.(7)]) *For any sequences $\{a_n\}_{n=0}^{\infty}$, there holds*

$$\det\left(\mathbf{D}_x^n(f^{a_k}(x))\right)_{n,k=0}^{m-1} = f(x)^{\sum_{i=0}^{m-1}(a_i-i)} f'(x)^{m(m-1)/2}\prod_{0\le i<j\le m-1}(a_j-a_i). \tag{1.2}$$

Up to now, many researchers have made further analysis around these two results, among them are Poorten [3], Strehl and Wilf [5], Krattenthaler [6,7], Chu [8,9], Zeilberger [10] and

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Zeitlin [2]. A full treatment on determinants of as such and their applications can be found in the book [11] by Vein and Dale, therein a fairly complete bibliography are given. Especially, we refer the reader to Krattenthaler's two remarkable surveys [6,7] on various evaluations of determinants. Now, let us turn our attention from the Mina determinant to the matrix of the form

$$\big(\mathbf{D}_x^n(f^{a_k}(x))\big)_{n,k=0}^m \quad \text{or} \quad \big(\mathbf{D}_x^n(f_k(x))\big)_{n,k=0}^m$$

which we henceforth call the Mina matrix or a matrix of Mina type respectively (known as the Wronskian of the functions $\{f_k(x)|0 \le k \le m\}$ in the literature). We find that some more problems such as the LU factorization and the inverse of the Mina matrix (subject to $f'(x) \neq 0$) arise quite naturally. These problems seem to have received little attention so far.

The aim of this paper is to attack these problems. Our paper is planned as follows. First, by the Newton binomial formula, we shall establish an LU factorization of the Mina matrix. Analogous results in the setting of formal power series, as well as applications to combinatorial identities, are also discussed. These results are presented in Section 2. Using this LU factorization, in Sections 3 and 4 we shall present not only a short elementary proof of (generalized) the Mina determinant identity but also the inverse of the Mina matrix, the latter utilizes the Lagrange inversion formula. In the last section, we shall extend such idea of matrix factorization to those build up from the Lagrange interpolation formula, thereby to computing determinants of Mina type. As main consequences, two new determinant identities for arbitrary polynomials are established.

2. LU factorizations of the Mina matrix

In this section, we shall focus on LU factorizations of the Mina matrix separately by distinguishing two settings: the space of differentiable functions and the ring of formal power series.

2.1. LU factorizations for differentiable functions

In what follows, we proceed to establish the following LU (lower and upper triangular matrix) factorization of the Mina matrix for differentiable functions.

Theorem 2.1 (LU factorization) *Let $f(x)$ be an $(m-1)$-times differentiable function and A be the $m \times m$ Mina matrix with the (n,k)-entry given by*

$$[A]_{n,k} = \mathbf{D}_x^n(f^k(x)). \tag{2.1a}$$

Then there must exist two matrices B and C with the (n,k)-entries given by

$$[B]_{n,k} = \mathbf{D}_y^n((f(y)-f(x))^k)\big|_{y=x}, \quad [C]_{n,k} = \binom{k}{n} f^{k-n}(x), \tag{2.1b}$$

such that

$$A = BC. \tag{2.1c}$$

Proof First, by the Newton binomial formula it follows that

$$f^k(y) = ((f(y) - f(x)) + f(x))^k = \sum_{i=0}^{k} \binom{k}{i} f^{k-i}(x)(f(y) - f(x))^i.$$

Consequently, we have

$$\begin{aligned} \mathbf{D}_x^n(f^k(x)) &= \mathbf{D}_y^n(((f(y) - f(x)) + f(x))^k)\big|_{y=x} \\ &= \mathbf{D}_y^n\Big(\sum_{i=0}^{k} \binom{k}{i} f^{k-i}(x)(f(y) - f(x))^i\Big)\Big|_{y=x} \\ &= \sum_{i=0}^{k} \binom{k}{i} f^{k-i}(x)\mathbf{D}_y^n((f(y) - f(x))^i)\big|_{y=x}, \end{aligned}$$

which turns out to be, after reformulated in terms of matrix algebra, (2.1c). Thus the theorem is proved. □

It is quite clear that the matrix B is lower-triangular and C is upper-triangular. For this reason, we hereafter call Theorem 2.1 by the LU factorization of the Mina matrix. Two specific forms of Theorem 2.1, which we shall use later, are worthwhile to state in details.

Corollary 2.2 *Let $f(y)$ be infinitely differentiable in a neighborhood of a real or complex number x. Then*

$$\mathbf{D}_y^n((f(y) - f(x))^k)\big|_{y=x} = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f^{k-i}(x)\mathbf{D}_x^n(f^i(x)). \tag{2.2}$$

Proof It suffices to see that the inverse of C, denoted by C^{-1} hereafter, equals

$$\Big((-1)^{k-n} \binom{k}{n} f^{k-n}(x)\Big)_{n,k=0}^{m-1}.$$

On considering (2.1c) in view of inverse relations, we immediately find that $B = AC^{-1}$, i.e., $[B]_{n,k} = [AC^{-1}]_{n,k}$, viz.,

$$\mathbf{D}_y^n((f(y) - f(x))^k)\big|_{y=x} = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f^{k-i}(x)\mathbf{D}_x^n(f^i(x)).$$

As claimed. □

Corollary 2.3 *Under the same assumption as in Corollary 2.2. Then*

$$\begin{aligned} \mathbf{D}_y^n((f(y) - f(x))^k)\big|_{y=x} &= n! \sum_{\substack{i_j \geq 1 \\ i_1+i_2+\cdots+i_k=n}} \prod_{j=1}^{k} \frac{\mathbf{D}_x^{i_j}(f(x))}{i_j!} & (2.3) \\ &= k! B_{n,k}(\mathbf{D}_x(f), \mathbf{D}_x^2(f), \ldots, \mathbf{D}_x^{n-k+1}(f)) & (2.4) \end{aligned}$$

where $B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ denotes the usual Bell polynomials [12, p.133, Definition].

Proof It follows from the Taylor series of $f(y)$ at $y = x$

$$f(y) = f(x) + \sum_{i=1}^{\infty} \mathbf{D}_x^i(f(x)) \frac{(y - x)^i}{i!}. \quad \square$$

In particular, when $n=k$, it is clear that

Corollary 2.4 *For all $n\geq 0$, there holds*

$$\mathbf{D}_y^n((f(y)-f(x))^n)\big|_{y=x}=n!(f'(x))^n. \tag{2.5}$$

As an example, we illustrate Theorem 2.1 with the LU factorization of the Mina matrix of order four.

Example 2.5 With the same notation as above, we have

$$\begin{pmatrix} 1 & f & f^2 & f^3 \\ 0 & f' & 2ff' & 3f^2f' \\ 0 & f'' & 2(f')^2+2ff'' & 6f(f')^2+3f^2f'' \\ 0 & f^{(3)} & 6f'f''+2ff^{(3)} & 6(f')^3+18ff'f''+3f^2f^{(3)} \end{pmatrix}$$
$$=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & f' & 0 & 0 \\ 0 & f'' & 2(f')^2 & 0 \\ 0 & f^{(3)} & 6f'f'' & 6(f')^3 \end{pmatrix}\begin{pmatrix} 1 & f & f^2 & f^3 \\ 0 & 1 & 2f & 3f^2 \\ 0 & 0 & 1 & 3f \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

2.2. Analogues in formal power series

Once dropping the hypothesis that $f(x)$ is $(m-1)$-times differentiable and reconsidering all forgoing conclusions in view of our specific interests in combinatorics, we have to resort to the theory of formal power series. To be precise, we need the coefficient functional $[x^n]$ acting on the ring $\mathbb{C}[[x]]$ of formal power series over the complex field $\mathbb{C}$, defined by

$$[x^n]f^k(x)=A(n,k) \tag{2.6}$$

for any given

$$f^k(x)=\sum_{n=0}^{\infty}A(n,k)x^n. \tag{2.7}$$

In view of this definition, we now reformulate Theorem 2.1 and Corollary 2.2 in the following succinct form.

Theorem 2.6 *Let $\{A(n,k)\}_{n,k\geq 0}$ be defined by (2.7). Then we have*

$$A(n,k)=\sum_{i=0}^{k}\binom{k}{i}A(0,k-i)[x^n](f(x)-A(0,1))^i, \tag{2.8}$$

or, equivalently,

$$[x^n](f(x)-A(0,1))^k=\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}A(0,k-i)A(n,i). \tag{2.9}$$

Proof According to the definitions (2.6) and (2.7), we only need to make the replacement

$$\mathbf{D}_x^n(f^k(x))\big|_{x=0}\mapsto n!A(n,k)$$

in the identities (2.1c) and (2.2). Observe that in this process, the sum on the right side of (2.3) becomes

$$n! \sum_{\substack{i_j \geq 1 \\ i_1+i_2+\cdots+i_k=n}} \prod_{j=1}^{k} A(i_j, 1) = n!\ [x^n](f(x) - A(0,1))^k.$$

This completes the proof of the theorem. □

Setting $f(x) = e^x$ and $(1+x)^a$ in (2.9), respectively, gives rise to the famous Stirling number of the second kind, i.e., $S(n, k)$, and its polynomial generalization. It means that (2.10) is the limit of (2.11) as a tends to infinity after dividing both sides of (2.11) by a^n.

Corollary 2.7 ([13]) *Let $S(n, k)$ be the Stirling numbers of the second kind. Then*

$$k!S(n,k) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n, \tag{2.10}$$

$$\sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \binom{ai}{n} = [x^n]\big((1+x)^a - 1\big)^k, \tag{2.11}$$

where, for any complex numbers x and nonnegative integers n, the generalized binomial coefficient

$$\binom{x}{n} := \frac{(x)_n}{n!}$$

and the usual falling factorial

$$(x)_n := x(x-1)\cdots(x-n+1).$$

It should be remarked that the identity (2.11) can be regarded as extensions of both (3.150) and (3.164) listed in the book [14] by Gould. For instance, by specializing $a = -1, 1/2, 2$, we obtain correspondingly that

Example 2.8 For all integers $n \geq k \geq 0$, the following identities are valid.

$$\sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \binom{-i}{n} = (-1)^n \binom{n-1}{k-1}, \tag{2.12}$$

$$\sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \binom{i/2}{n} = (-1)^{n-k} \frac{k 2^{k-2n}}{2n-k} \binom{2n-k}{n}, \tag{2.13}$$

$$\sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \binom{2i}{n} = 2^{2k-n} \binom{k}{n-k}. \tag{2.14}$$

3. Proofs of the Mina determinant identity

In the course of proving Theorem 2.1, we already come up with a simple proof of the Mina determinant identity (1.1). It is immediate from (2.1c), by taking determinants $\det(A) = \det(BC) = \det(B) \times \det(C)$. In fact, by the same argument as in Theorem 2.1, we can also show Wilf's determinant identity (1.2) generalizing the Mina result. Such a proof is sure to be new, short, and really elementary, for the reason that it is distinct from those by Mina [1], Wilf [5], Zeitlin [2], and Chu [9].

Proof of Theorem 1.2 Without loss of generality, we may assume $f(x) \neq 0$. Hence, it is easy to check that for $0 \leq n, k \leq m-1$,

$$\begin{aligned}
\mathbf{D}_x^n(f^{a_k}(x)) &= \mathbf{D}_y^n(f^{a_k}(x)(1+\frac{f(y)-f(x)}{f(x)})^{a_k})\big|_{y=x}\\
&= f^{a_k}(x)\mathbf{D}_y^n(1+\frac{f(y)-f(x)}{f(x)})^{a_k}\big|_{y=x}\\
&= f^{a_k}(x)\ \mathbf{D}_y^n\Big(\sum_{i=0}^{\infty}\binom{a_k}{i}(\frac{f(y)-f(x)}{f(x)})^i\Big)\Big|_{y=x}\\
&= \sum_{i=0}^{\infty}\binom{a_k}{i}f^{a_k-i}(x)\mathbf{D}_y^n((f(y)-f(x))^i)\big|_{y=x}\\
&= \sum_{i=0}^{m-1}\binom{a_k}{i}f^{a_k-i}(x)\mathbf{D}_y^n((f(y)-f(x))^i)\big|_{y=x}. \qquad (3.1)
\end{aligned}$$

Note that in the last third equality we have used the fact

$$|\frac{f(y)-f(x)}{f(x)}| < 1 \ \text{ when } \ y \mapsto x.$$

And the last equality results from the fact that

$$\mathbf{D}_y^n((f(y)-f(x))^i)\big|_{y=x} = 0 \ \text{ for } \ i > m-1 \geq n.$$

As previously, we rewrite (3.1) in the form

$$\widetilde{A} = \widetilde{B} \times \widetilde{C} \qquad (3.2)$$

where $\widetilde{A}, \widetilde{B}$, and $\widetilde{C}$ are three $m \times m$ matrices of Mina type whose entries are given, respectively, by

$$[\widetilde{A}]_{n,k} = \mathbf{D}_x^n(f^{a_k}(x)); \quad [\widetilde{B}]_{n,k} = \mathbf{D}_y^n((f(y)-f(x))^k)\big|_{y=x}; \quad [\widetilde{C}]_{n,k} = \binom{a_k}{n}f^{a_k-n}(x).$$

Since

$$\begin{aligned}
\det(\widetilde{C})_{n,k=0}^{m-1} &= \det\Big(\binom{a_k}{n}f^{a_k-n}(x)\Big)_{n,k=0}^{m-1} = f(x)^{\sum_{i=0}^{m-1}(a_i-i)}\det\Big(\binom{a_k}{n}\Big)_{n,k=0}^{m-1}\\
&= \frac{f(x)^{\sum_{i=0}^{m-1}(a_i-i)}}{1!2!\cdots(m-1)!}\prod_{0\leq i<j\leq m-1}(a_j-a_i),
\end{aligned}$$

which, by taking determinants on both sides of (3.2), gives rise to

$$\begin{aligned}
\det(\widetilde{A})_{n,k=0}^{m-1} &= \det(\widetilde{B})_{n,k=0}^{m-1} \times \det(\widetilde{C})_{n,k=0}^{m-1}\\
&= f(x)^{\sum_{i=0}^{m-1}(a_i-i)}\big(f'(x)\big)^{m(m-1)/2}\prod_{0\leq i<j\leq m-1}(a_j-a_i).
\end{aligned}$$

The theorem is therefore proved. □

By virtue of Theorem 2.6, we may establish an analogous determinant identity in the setting of formal power series.

Corollary 3.1 (Mina determinant identity for formal power series) *Under the same assumptions* *as in Theorem 2.6. Then we have*

$$\det\left([x^n](f^{a_k}(x))\right)_{n,k=0}^{m-1} = \frac{\lambda_0^{\sum_{i=0}^{m-1}(a_i-i)}\lambda_1^{m(m-1)/2}}{1!2!\cdots(m-1)!}\prod_{0\le i<j\le m-1}(a_j-a_i), \tag{3.3}$$

where $\lambda_0 = f(0)$, $\lambda_1 = [x]f(x)$.

Proof Observe that for arbitrary complex numbers a_k and $f(x)$ given by the definition (2.7), it holds

$$f^{a_k}(x) = \sum_{i=0}^{\infty}\binom{a_k}{i}f^{a_k-i}(0)(f(x)-f(0))^i.$$

Applying the coefficient functional $[x^n]$ to both sides of this identity, we obtain that

$$[x^n]f^{a_k}(x) = \sum_{i=0}^{\infty}\binom{a_k}{i}f^{a_k-i}(0)[x^n](f(x)-A(0,1))^i \tag{3.4}$$

where $[x^n]f^i(x) = A(n,i)$. Under the known conditions, it is easily found that

$$f^{a_k-i}(0) = \lambda_0^{a_k-i};$$

$$[x^n](f(x)-A(0,1))^i = \begin{cases} 0, & i>n; \\ \lambda_1^n, & i=n. \end{cases}$$

We are thus led to the following matrix factorization

$$([x^n](f^{a_k}(x)))_{n,k=0}^{m-1} = ([x^n](f(x)-A(0,1))^k)_{n,k=0}^{m-1}\times\left(\binom{a_k}{n}f^{a_k-n}(0)\right)_{n,k=0}^{m-1},$$

which in turn, by taking determinants on both sides simultaneously, yields

$$\det([x^n](f^{a_k}(x)))_{n,k=0}^{m-1} = \frac{\lambda_0^{\sum_{i=0}^{m-1}(a_i-i)}\lambda_1^{m(m-1)/2}}{1!2!\cdots(m-1)!}\prod_{0\le i<j\le m-1}(a_j-a_i).$$

The corollary is proved. □

4. Inverse of the Mina matrix

It is evident that the Mina matrix $(\mathbf{D}_x^n(f^k(x)))_{n,k=0}^{m-1}$ is nonsingular if and only if $f'(x)\neq 0$. The goal of this section is to derive the inverse of the Mina matrix from the LU factorization (2.1c). To do this, we also need the following

Lemma 4.1 ([15, Def.7 and Thm.6]) *For any pair of reciprocal functions* $f(x), g(x) \in \mathcal{L}_1$, *viz.,* $f(g(x)) = g(f(x)) = x$, *and finite or infinite integer* $M \ge 0$, *we have*

$$(\frac{1}{n!}\mathbf{D}_0^n(f^k(x)))_{M\ge n\ge k\ge 0}^{-1} = (\frac{1}{n!}\mathbf{D}_0^n(g^k(x)))_{M\ge n\ge k\ge 0}, \tag{4.1}$$

where the notation

$$\mathbf{D}_0^n(f(x)) = \mathbf{D}_x^n(f(x))|_{x=0};$$
$$\mathcal{L}_1 = \{f(x)|f(x) \text{ is analytic around } x=0, f(0)=0, f'(0)\neq 0\}.$$

Combining Lemma 4.1 with the LU factorization (2.1c) of the Mina matrix, we immediately obtain that

Theorem 4.2 *Let $f(x)$ be an $(m-1)$-times differentiable function with $f^{'}(x)\neq 0$. Then*

$$\begin{aligned}&(\mathbf{D}_x^n(f^k(x)))^{-1}_{m-1\geq n,k\geq 0} &(4.2)\\ &=\Big(\sum_{i=\max\{n,k\}}^{m-1}(-1)^{i-n}\binom{i}{n}\binom{i-1}{i-k}\frac{f^{i-n}(x)}{i!}\mathbf{D}_y^{i-k}(\frac{y-x}{f(y)-f(x)})^i\big|_{y=x}\Big)_{m-1\geq n,k\geq 0}.\end{aligned}$$

Proof By making use of (2.1c) and noting $f'(x)\neq 0$, we easily find that

$$\begin{aligned}&(\mathbf{D}_x^n(f^k(x)))^{-1}_{m-1\geq n,k\geq 0} &(4.3)\\ &=\Big(\binom{k}{n}f^{k-n}(x)\Big)^{-1}_{m-1\geq k\geq n\geq 0}(\mathbf{D}_y^n((f(y)-f(x))^k)\big|_{y=x})^{-1}_{m-1\geq n\geq k\geq 0}.\end{aligned}$$

It is crucial that $F(t)\in\mathcal{L}_1$, by defining $F(t)=f(t+x)-f(x)$ with $t=y-x$. So Lemma 4.1 is applicable. In conclusion, we get

$$(\mathbf{D}_y^n((f(y)-f(x))^k)|_{y=x})^{-1}_{m-1\geq n\geq k\geq 0}=(\frac{1}{n!k!}\mathbf{D}_t^n(G^k(t))|_{t=0})_{m-1\geq n\geq k\geq 0},$$

where $G(t)$ is the compositional inverse of $F(t)$. And then, thanks to the Lagrange inversion formula [12, p.148, Theorem A], we deduce that

$$\frac{1}{n!}\mathbf{D}_t^n(G^k(t))|_{t=0}=\frac{k}{n}\frac{1}{(n-k)!}\mathbf{D}_t^{n-k}(\frac{t}{f(t+x)-f(x)})^n|_{t=0}.$$

A direct substitution of these two expressions into the relation (4.3), after a bit of simplification, gives the complete proof of the theorem. □

As an illustration, we specialize Theorem 4.2 to the Mina matrix of order four.

Example 4.3 Write $a:=f',b:=f'',c:=f^{(3)}$. Then

$$\begin{aligned}&\begin{pmatrix}1 & f & f^2 & f^3\\ 0 & a & 2af & 3af^2\\ 0 & b & 2a^2+2bf & 6fa^2+3bf^2\\ 0 & c & 6ab+2cf & 6a^3+18bfa+3cf^2\end{pmatrix}^{-1}\\ &=\begin{pmatrix}1 & -\frac{f\left(6a^4+3a^2bf-acf^2+3b^2f^2\right)}{6a^5} & \frac{f^2a^2+bf^3}{2a^4} & -\frac{f^3}{6a^3}\\ 0 & \frac{2a^4+2a^2bf-acf^2+3b^2f^2}{2a^5} & \frac{-2fa^2-3bf^2}{2a^4} & \frac{f^2}{2a^3}\\ 0 & -\frac{a^2b-acf+3b^2f}{2a^5} & \frac{a^2+3bf}{2a^4} & -\frac{f}{2a^3}\\ 0 & \frac{3b^2-ac}{6a^5} & -\frac{b}{2a^4} & \frac{1}{6a^3}\end{pmatrix}.\end{aligned}$$

5. Further discussion: new determinant identities of Mina

Reconsidering all obtained in the preceding sections, we easily find that the LU factorization of the Mina matrix in Theorem 2.1, built on the Newton binomial formula, is the heart of our argument. Just as Krattenthaler pointed out in [6], LU factorizations (generally speaking, matrix factorization) is one of the most important techniques in evaluations of determinants. To the best of our knowledge, many combinatorial identities containing such matrix factorizations (but not necessarily LU) can be used to evaluations of the resulting determinants. For example, using the well–known Newton formula of finite difference operator Δ_x, i.e.,

$$\Delta_x^n f(x) = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} f(x+k),$$

Chu [9, Lemma 1] showed that for arbitrary function sequence $\{f_n(x)\}$

$$\det\left(\Delta_x^n (f_k(x))\right)_{n,k=0}^{m-1} = \det\left(f_k(x+n)\right)_{n,k=0}^{m-1}.$$

Indeed, it is the Faà di Bruno formula by which Chu in his another paper [8] successfully extended the Mina determinant identity to the case of higher derivatives of composite functions.

All these suggest that, if we restrict our attention to determinants of Mina type, such matrix factorizations constructed by combinatorial identities may provide us with great free choices for determinant evaluations, as displayed by Wimp in [16].

Keeping such idea in mind, we now turn our attention to the Lagrange interpolation formula and its special case–Melzak's formula [17]. Recall that the Lagrange interpolation formula may be stated as follows:

Lemma 5.1 (Lagrange interpolation formula) *For arbitrary positive integer m, let $b_0, b_1, \ldots, b_m$ be a sequence of distinct complex numbers and $f(x)$ be a polynomial in x of degree at most m. Then*

$$f(x) = \sum_{j=0}^{m} f(b_j) \frac{\prod_{s=0,s\neq j}^{m}(x-b_s)}{\prod_{s=0,s\neq j}^{m}(b_j-b_s)}. \tag{5.1}$$

Making the substitutions $b_i \mapsto x-i$ and $x \mapsto x+y$ in (5.1) simultaneously, then we recover Melzak's formula [17] as follows.

Lemma 5.2 (Melzak's formula) *Let $f_n(x)$ be a polynomial in x of degree at most n. Then*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{f_n(x-k)}{y+k} = n!\frac{f_n(x+y)}{(y+n)_{n+1}}. \tag{5.2}$$

As mentioned above, using these two lemmas we can set up respectively two determinant identities as follows.

Theorem 5.3 *Let $f_k(x)$ $(0 \le k \le m)$ be $m+1$ arbitrary polynomials given by*

$$f_k(x) = a_{k,m}x^m + a_{k,m-1}x^{m-1} + \cdots + a_{k,1}x + a_{k,0}. \tag{5.3}$$

Then

$$\det\left(\mathbf{D}_x^n\Big(\frac{f_k(x)}{\prod_{s=0}^{m}(x-b_s)}\Big)\right)_{n,k=0}^{m} = \det\left(a_{n,k}\right)_{n,k=0}^{m} \prod_{n=0}^{m} \frac{n!}{(x-b_n)^{m+1}}. \tag{5.4}$$

Proof As indicated above, we first divide both sides of (5.1) by $\prod_{s=0}^{m}(x-b_s)$ to get

$$\frac{f(x)}{\prod_{s=0}^{m}(x-b_s)} = \sum_{j=0}^{m} \frac{1}{x-b_j} \frac{f(b_j)}{\prod_{s=0,s\neq j}^{m}(b_j-b_s)}. \tag{5.5}$$

It is worth noting that (5.5) is valid for all polynomials which are of degree at most m. Applied to the polynomial $f_k(x)$, we obtain

$$\frac{f_k(x)}{\prod_{s=0}^{m}(x-b_s)}=\sum_{j=0}^{m}\frac{1}{x-b_j}\frac{f_k(b_j)}{\prod_{s=0,s\neq j}^{m}(b_j-b_s)}. \tag{5.6}$$

Accordingly, take the n-th derivatives on both sides of (5.6) with respect to x. It follows that

$$\mathbf{D}_x^n\Big(\frac{f_k(x)}{\prod_{s=0}^{m}(x-b_s)}\Big)=\sum_{j=0}^{m}\mathbf{D}_x^n(\frac{1}{x-b_j})\frac{f_k(b_j)}{\prod_{s=0,s\neq j}^{m}(b_j-b_s)}. \tag{5.7}$$

Rewriting this relation in terms of matrix algebra, so we are left with the matrix factorization

$$\Big(\mathbf{D}_x^n(\frac{f_k(x)}{\prod_{s=0}^{m}(x-b_s)})\Big)_{n,k=0}^{m}=(\mathbf{D}_x^n(\frac{1}{x-b_k}))_{n,k=0}^{m}\times\Big(\frac{f_k(b_n)}{\prod_{s=0,s\neq n}^{m}(b_n-b_s)}\Big)_{n,k=0}^{m}.$$

Observe that the determinant of the first matrix on the right, after some appropriate row and column manipulation, reduces to a Vandermonde determinant while the second one on the right can now be evaluated in closed form. To be precise, on referring to (5.3) and noting that

$$(f_k(b_n))_{n,k=0}^{m}=(b_n^k)_{n,k=0}^{m}\times(a_{n,k})_{n,k=0}^{m},$$

we obtain that

$$\det\Big(\frac{f_k(b_n)}{\prod_{s=0,s\neq n}^{m}(b_n-b_s)}\Big)_{n,k=0}^{m}=\frac{\det(f_k(b_n))_{n,k=0}^{m}}{\prod_{0\le i\neq j\le m}(b_j-b_i)}=(-1)^{m(m+1)/2}\frac{\det(a_{n,k})_{n,k=0}^{m}}{\prod_{0\le i<j\le m}(b_j-b_i)}.$$

Finally, we get

$$\begin{aligned}&\det\Big(\mathbf{D}_x^n(\frac{f_k(x)}{\prod_{s=0}^{m}(x-b_s)})\Big)_{n,k=0}^{m}=\det(\mathbf{D}_x^n(\frac{1}{x-b_k}))_{n,k=0}^{m}\times\det\Big(\frac{f_k(b_n)}{\prod_{s=0,s\neq n}^{m}(b_n-b_s)}\Big)_{n,k=0}^{m}\\&=\prod_{n=0}^{m}\frac{(-1)^n n!}{(x-b_n)^{m+1}}\prod_{0\le i<j\le m}(b_j-b_i)\times(-1)^{m(m+1)/2}\frac{\det(a_{n,k})_{n,k=0}^{m}}{\prod_{0\le i<j\le m}(b_j-b_i)}\\&=\det(a_{n,k})_{n,k=0}^{m}\prod_{n=0}^{m}\frac{n!}{(x-b_n)^{m+1}}.\end{aligned}$$

This completes the proof of the theorem. □

In the same line as above, we may establish another different determinant identity with the help of Lemma 5.2.

Theorem 5.4 *Let $\{f_n(x)\}_{n\ge0}$ be an arbitrary polynomial sequence and each $f_n(x)$ be of degree at most n. Then*

$$\det(\mathbf{D}_y^n(\frac{f_k(x+y)}{(y+k)_{k+1}}))_{n,k=0}^{m-1}=(-1)^{m(m-1)/2}\prod_{n=0}^{m-1}\frac{n!f_n(x-n)}{(y+n)^m}. \tag{5.8}$$

Proof It suffices to take the i-th derivatives on both sides of (5.2) with respect to y. So we see that for all $n,i\ge0$, there holds

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}f_n(x-k)\mathbf{D}_y^i(\frac{1}{y+k})=\mathbf{D}_y^i(\frac{n!f_n(x+y)}{(y+n)_{n+1}}),$$

yielding the factorization of the Mina matrix

$$\Big((-1)^k\binom{n}{k}f_n(x-k)\Big)_{n,k=0}^{m-1}(\mathbf{D}_y^k(\frac{1}{y+n}))_{n,k=0}^{m-1}=(\mathbf{D}_y^k(\frac{n!f_n(x+y)}{(y+n)_{n+1}}))_{n,k=0}^{m-1}.$$

Observe that the first matrix on the left is triangular while the determinant of the second matrix can now be evaluated by

$$\det(\mathbf{D}_y^k(\frac{1}{y+n}))_{n,k=0}^{m-1}=\det(\frac{(-1)^k k!}{(y+n)^{k+1}})_{n,k=0}^{m-1}=\det(\frac{1}{(y+n)^k})_{n,k=0}^{m-1}\times\prod_{k=0}^{m-1}\frac{(-1)^k k!}{y+k}.$$

Note that the last determinant on the right is a Vandermonde determinant. This gives the complete proof of the theorem. □

We conclude our paper with some concrete results worthy of special attention, by specializing Theorem 5.4 to

$$f_n(x)=1,x^n,\binom{\lambda\, x}{n},\binom{x+n}{n},\binom{2x+2n}{2n}\Big/\binom{x+n}{n},$$

respectively.

Example 5.5 For any nonnegative integer m and the variable $y\neq 0,-1,\ldots,-m+1$, write $\tau(m)$ for $(-1)^{m(m-1)/2}$, then the following identities are valid.

$$\det(\mathbf{D}_y^n(\frac{1}{(y+k)_{k+1}}))_{n,k=0}^{m-1}=\tau(m)\prod_{n=0}^{m-1}\frac{n!}{(y+n)^m}, \tag{5.9}$$

$$\det(\mathbf{D}_y^n(\frac{(x+y)^k}{(y+k)_{k+1}}))_{n,k=0}^{m-1}=\tau(m)\prod_{n=0}^{m-1}\frac{n!(x-n)^n}{(y+n)^m}, \tag{5.10}$$

$$\det(\mathbf{D}_y^n(\frac{(\lambda x+\lambda y)_k}{(y+k)_{k+1}}))_{n,k=0}^{m-1}=\tau(m)\prod_{n=0}^{m-1}\frac{n!(\lambda x-\lambda n)_n}{(y+n)^m}, \tag{5.11}$$

$$\det(\mathbf{D}_y^n(\frac{(x+y+k)_k}{(y+k)_{k+1}}))_{n,k=0}^{m-1}=\tau(m)\prod_{n=0}^{m-1}\frac{n!(x)_n}{(y+n)^m}, \tag{5.12}$$

$$\det(\mathbf{D}_y^n(\frac{(2x+2y+2k)_{2k}}{(x+y+k)_k(y+k)_{k+1}}))_{n,k=0}^{m-1}=\tau(m)\prod_{n=0}^{m-1}\frac{n!(2x)_{2n}}{(x)_n(y+n)^m}. \tag{5.13}$$

In particular, for the Stirling numbers of the second kind $S(n,k)$, we have

$$\det\Big(\sum_{i=k}^{\infty}(-1)^{i-k}S(i,k)\frac{(i+n)_n}{y^i}\Big)_{n,k=0}^{m-1}=\prod_{n=0}^{m-1}\frac{n!y^{n+1}}{(y+n)^m},\quad |y|>1. \tag{5.14}$$

It is of interest that the identity (5.14) results from a combination of (5.9) with the generating function of the Stirling numbers $S(n,k)$ (see [12, p.207, Theorem C]):

$$\sum_{n=k}^{\infty}S(n,k)x^n=\frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}.$$

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