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Classification and Central Extensions of a Class of Lie Algebras of Block Type

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Abstract In this paper, we determine the classification up to isomorphism and the central extensions of a class of Lie algebras $\mathcal{B}(q)$ of Block type, where q is a non-zero complex number. Our results generalize some previous results.

Keywords Block type Lie algebras; central extension; classification up to isomorphism

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1. Introduction

In 1958, motivated by the fact that many finite-dimensional nonclassical simple Lie algebras of prime characteristic have simple infinite-dimensional analogues of characteristic zero, Block [1] introduced a class of infinite dimensional simple Lie algebras as analogues of the Zassenhaus algebras. Nowadays, these Lie algebras and their generalizations are usually referred to as Lie algebras of Block type. Partially due to their close relations with the Virasoro algebra, special cases of (generalized) Cartan type Lie algebras [2] or W-infinity algebras, the Lie algebras of this type have received much attention in the last two decades [3–15].

For any $0 \neq q \in \mathbb{C}$, there is a class of Lie algebras $\mathcal{B}(q)$ of Block type with basis $\{L_{\alpha,i} \,|\, \alpha \in \mathbb{Z},\, i \in \mathbb{Z}_+\}$ over $\mathbb{C}$, and relations

$$[L_{\alpha,i}, L_{\beta,j}] = (\beta(i+q) - \alpha(j+q))L_{\alpha+\beta,i+j}. \tag{1}$$

This class of Lie algebras $\mathcal{B}(q)$ can be viewed as subalgebras of some special cases of generalized Block algebras studied in [3]. It is interesting that, as pointed in [4,5], each $\mathcal{B}(q)$ contains the (centerless) Virasoro subalgebra $\mathrm{span}\{q^{-1}L_{\alpha,0} \,|\, \alpha \in \mathbb{Z}\}$. It is recently found in [6,7] that it also has close relations with the twisted Heisenberg-Virasoro algebra and twisted Schrödinger-Virasoro algebra. Derivations, automorphisms and central extensions are three important aspects

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in the structure theory of Lie algebras. The derivations and automorphisms of $\mathcal{B}(q)$ have been completely determined in [5,8]. In [5], the authors also determined the classification up to isomorphism and central extensions of $\mathcal{B}(q)$ for positive integers q's. The central extension of $\mathcal{B}(1)$ was first considered in [9].

In this paper, we will consider the classification up to isomorphism and central extensions of $\mathcal{B}(q)$ for all non-zero complex numbers q's. Our first main result is

Theorem 1.1 *Lie algebras $\mathcal{B}(q)$ are different from each other for distinct non-zero complex numbers q's, namely, $\mathcal{B}(q_1) \cong \mathcal{B}(q_2) \Longleftrightarrow q_1 = q_2$.*

This generalizes the isomorphism theorem given in [5]. Our basic strategy for proving this result is to reduce the problem to showing that $\mathcal{B}(q) \ncong \mathcal{B}(-q)$ for any $q \neq 0$ (see Lemma 2.2), which is different from that used in [5].

It is well known that a Lie algebra has a non-trivial universal central extension if and only if it is perfect. One can easily check that Lie algebra $\mathcal{B}(q)$ is perfect if and only if q is not equal to any half of a negative integer, i.e, $q \notin \frac{1}{2}\mathbb{Z}_-$. As usual, we use the symbol $\delta_{i,j}$ to denote the Kronecker delta function. Our second main result is the following, which generalizes some results in [5,9], where the authors consider the cases with $0 \neq q \in \mathbb{Z}_+$ (in these cases, $\mathcal{B}(q)$ is, of cause, perfect).

Theorem 1.2 *The unique non-trivial universal central extension of $\mathcal{B}(q)$ is given by*

$$[L_{\alpha,i}, L_{\beta,j}] = (\beta(i+q) - \alpha(j+q))\, L_{\alpha+\beta,i+j} + \phi(L_{\alpha,i}, L_{\beta,j})c, \tag{2}$$

where $q \notin \frac{1}{2}\mathbb{Z}_-$, c is a central element and ϕ is the following non-trivial 2-cocycle:

$$\phi(L_{\alpha,i}, L_{\beta,j}) = \delta_{\alpha+\beta,0}\delta_{i+j,0}\frac{\alpha^3-\alpha}{12}. \tag{3}$$

Hence, the second cohomology group of $\mathcal{B}(q)$ is $H^2(\mathcal{B}(q), \mathbb{C}) = \mathbb{C}\phi$.

Throughout this paper, we work over the complex field $\mathbb{C}$. We will use $\mathbb{Z}, \mathbb{Z}_+$ and $\mathbb{Z}_-$ to denote the sets of integers, nonnegative and nonpositive integers, respectively.

2. Classification up to isomorphism

In this section, we will give the proof of Theorem 1.1. First, let us recall two useful definitions. An element $x \in \mathcal{B}(q)$ is called ad-locally finite if $\mathrm{span}\{\mathrm{ad}_x^m(v) \,|\, m \in \mathbb{Z}_+\}$ is a finite dimensional subspace of $\mathcal{B}(q)$ for any $v \in \mathcal{B}(q)$; ad-locally nilpotent if there exists some positive integer N such that $\mathrm{ad}_x^N(v) = 0$ for any $v \in \mathcal{B}(q)$. Denote by $\mathcal{F}_q$ the set of ad-locally finite elements of $\mathcal{B}(q)$, and by $\mathcal{N}_q$ the set of ad-locally nilpotent elements of $\mathcal{B}(q)$. We have

Lemma 2.1 *The sets $\mathcal{F}_q$ and $\mathcal{N}_q$ are as follows:*

(i) *If $q \in \mathbb{Z}_-$, then $\mathcal{F}_q = \mathbb{C}L_{0,0} + \mathbb{C}L_{0,-q}$, and $\mathcal{N}_q = \mathbb{C}L_{0,-q}$;*

(ii) *If $q \notin \mathbb{Z}_-$, then $\mathcal{F}_q = \mathbb{C}L_{0,0}$, and $\mathcal{N}_q$ is an empty set.*

Proof The conclusion (i) with $q \in \mathbb{Z}_-$ has been proved in [8, Lemma 3.2]. The conclusion (ii) with $q \notin \mathbb{Z}_-$ can be processed by similar arguments (with very minor changes) as those in [5, Lemma 2.1]. Here, we omit the details. $\square$

Let $\mathcal{B}(q_1)$ and $\mathcal{B}(q_2)$ be two Lie algebras of Block type defined as in (1). We use the notations $L_{\alpha,i}$ and $L'_{\alpha,i}$ to stand for the base elements of $\mathcal{B}(q_1)$ and $\mathcal{B}(q_2)$, respectively. Assume that

$$\tau : \mathcal{B}(q_1) \to \mathcal{B}(q_2)$$

is a Lie algebra isomorphism. We want to prove $q_1 = q_2$. Let $\mathcal{B}(q)_\alpha = \mathrm{span}\{L_{\alpha,i} | \, i \in \mathbb{Z}_+\}$ for $\alpha \in \mathbb{Z}$. We first prove the following lemma.

Lemma 2.2 *(i) We have $q_1 q_2^{-1} \in \{\pm 1\}$.*

(ii) If $q_1 = -q_2$, then there exists $s \in \{\pm 1\}$ such that $\tau(\mathcal{B}(q_1)_\alpha) \subseteq \mathcal{B}(q_2)_{s\alpha}$ for $\alpha \in \mathbb{Z}$.

Proof First, by Lemma 2.1, we must have $\tau(L_{0,0}) = aL'_{0,0} + bL'_{0,-q_2}$, where $a \neq 0$ and we treat b as zero if $q_2 \notin \mathbb{Z}_-$. Define a new Lie algebra isomorphism τ from $\mathcal{B}(q_1)$ to $\mathcal{B}(q_2)$ and an integer $s \in \{\pm 1\}$ as follows:

$$\tau = \begin{cases} -a^{-1}\tau, & \text{if } a < 0, \\ a^{-1}\tau, & \text{otherwise}, \end{cases} \qquad s = \begin{cases} -1, & \text{if } a < 0, \\ 1, & \text{otherwise}. \end{cases}$$

Then we have $\tau(L_{0,0}) = sL'_{0,0} + sa^{-1}bL'_{0,-q_2}$. For $\alpha \in \mathbb{Z}$ and $i \in \mathbb{Z}_+$, we may assume that

$$\tau(L_{\alpha,i}) = \sum_{(\beta,j)\in I_{\alpha,i}} \lambda'_{\beta,j} L'_{\beta,j},$$

where $I_{\alpha,i}$ is a finite subset of $\mathbb{Z} \times \mathbb{Z}_+$. Applying τ to $[L_{0,0}, L_{\alpha,i}] = q_1 \alpha L_{\alpha,i}$ and noticing that $L'_{0,-q_2}$ is a central element of $\mathcal{B}(q_2)$ if $q_2 \in \mathbb{Z}_-$, we obtain $\sum_{(\beta,j)\in I_{\alpha,i}} (sq_2\beta - q_1\alpha)\lambda'_{\beta,j} L'_{\beta,j} = 0$, which implies that

$$\lambda'_{\beta,j} = 0 \ \text{ if } \ \beta \neq (sq_2)^{-1} q_1 \alpha. \tag{4}$$

Since τ is a Lie algebra isomorphism, there exists at least one pair $(\beta, j) \in I_{\alpha,i}$ such that $\lambda'_{\beta,j} \neq 0$. Then (4) with $\alpha = 1$ implies that $\beta = (sq_2)^{-1} q_1$. Hence $q_1 q_2^{-1} = s\beta \in \mathbb{Z}$. Similarly, since τ^{-1} is Lie algebra isomorphism from $\mathcal{B}(q_2)$ to $\mathcal{B}(q_1)$, one can show that $q_2 q_1^{-1} \in \mathbb{Z}$. Hence $q_1 q_2^{-1} \in \{\pm 1\}$, namely, the conclusion (i) holds.

If $q_1 = -q_2$, then as above, it follows from (4) that $\beta = -s^{-1}\alpha = -s\alpha$. Hence $\tau(\mathcal{B}(q_1)_\alpha) \subseteq \mathcal{B}(q_2)_{s'\alpha}$ for $\alpha \in \mathbb{Z}$, where $s' = -s$. Since the original Lie algebra isomorphism τ is always a multiple of the new τ, the conclusion (ii) holds. $\square$

Note that Lemma 2.2(i) reduces the classification problem (up to isomorphism) to showing that $\mathcal{B}(q) \not\cong \mathcal{B}(-q)$ for any $q \neq 0$. Now, we give the proof of Theorem 1.1.

Proof of Theorem 1.1 First, note that if $q_1 = -q_2 \in \mathbb{Z}$, it follows from Lemma 2.1 that $\mathcal{B}(q_1) \not\cong \mathcal{B}(q_2)$. By Lemma 2.2(i), we only need to derive a contradiction from the assumption $q_1 = -q_2$ and $q_1 \notin \mathbb{Z}$.

Suppose $q_1 = -q_2 = q \notin \mathbb{Z}$. By Lemmas 2.1(ii) and 2.2(ii), we can assume that $\tau(L_{0,0}) =$

$aL'_{0,0}$ with $a \neq 0$, and

$$\begin{cases} \tau(L_{0,1}) = \sum_{i\in J} \nu_i L'_{0,i}, \text{where } J \text{ is a finite subset of } \mathbb{Z}_+, \\ \tau(L_{1,0}) = \sum_{i\in K_1} \mu_i^{(1)} L'_{s,i}, \text{ where } K_1 \text{ is a finite subset of } \mathbb{Z}_+, \\ \tau(L_{2,0}) = \sum_{i\in K_2} \mu_i^{(2)} L'_{2s,i}, \text{ where } K_2 \text{ is a finite subset of } \mathbb{Z}_+, \\ \tau(L_{-1,0}) = \sum_{i\in K_{-1}} \mu_i^{(-1)} L'_{-s,i}, \text{ where } K_{-1} \text{ is a finite subset of } \mathbb{Z}_+. \end{cases}$$

Applying τ to $[L_{-1,0}, L_{1,0}] = 2qL_{0,0}$, and considering the terms with the minimal second subscript on the two sides, we can derive that $[\mu_0^{(-1)} L'_{-s,0}, \mu_0^{(1)} L'_{s,0}] = 2aqL'_{0,0}$, which gives $s\mu_0^{(-1)}\mu_0^{(1)} = -a$, and in particular $\mu_0^{(1)} \neq 0$. Similarly, applying τ to $[L_{1,0}, L_{0,0}] = -qL_{1,0}$, we can obtain $[\mu_0^{(1)} L'_{s,0}, aL'_{0,0}] = -q\mu_0^{(1)} L'_{s,0}$, which gives $(1 + as)\mu_0^{(1)} = 0$. Hence $as = -1$, and so

$$\mu_0^{(-1)}\mu_0^{(1)} = -as^{-1} = -as = 1. \tag{5}$$

Furthermore, applying τ to $[L_{-1,0}, L_{2,0}] = 3qL_{1,0}$, we have $[\mu_0^{(-1)} L'_{-s,0}, \mu_0^{(2)} L'_{2s,0}] = 3q\mu_0^{(1)} L'_{s,0}$, which gives

$$\mu_0^{(1)} + s\mu_0^{(-1)}\mu_0^{(2)} = 0. \tag{6}$$

Note that (6) implies $\mu_0^{(2)} \neq 0$, since $\mu_0^{(1)} \neq 0$. At last, applying τ to $-2[L_{1,0}, [L_{1,0}, L_{0,1}]] = [L_{2,0}, L_{0,1}]$, we can derive that $-2[\mu_0^{(1)} L'_{s,0}, [\mu_0^{(1)} L'_{s,0}, \nu_{j_0} L'_{0,j_0}]] = [\mu_0^{(2)} L'_{2s,0}, \nu_{j_0} L'_{0,j_0}]$, where $j_0 = \min\{j \in J \mid \nu_j \neq 0\}$. This implies

$$(j_0 - q)(j_0(\mu_0^{(1)})^2 - s\mu_0^{(2)})\nu_{j_0} = 0. \tag{7}$$

Recall the assumption $q \notin \mathbb{Z}$ and the proved result $\mu_0^{(1)} \neq 0$, by (5)–(7), we can show that $j_0 = -1$, a contradiction. This completes the proof of Theorem 1.1. □

3. Central extensions

In this section, we will give the proof of Theorem 1.2. Since a non-perfect Lie algebra has no non-trivial universal central extensions, we only need to consider the cases with the Lie algebra $\mathcal{B}(q)$ being perfect, or equivalently, $q \notin \frac{1}{2}\mathbb{Z}_-$. Our argument is similar to that of [5] but with some differences (see the discussions after (12) and after (14)).

Let ψ be any 2-cocycle. Define a linear function on $\mathcal{B}(q)$ as follows:

$$f(L_{\alpha,i}) = \begin{cases} -(\alpha q)^{-1}\psi(L_{\alpha,i}, L_{0,0}), \text{ if } \alpha \neq 0, \\ (i + 2q)^{-1}\psi(L_{-1,i}, L_{1,0}), \text{ otherwise.} \end{cases}$$

Then $\phi = \psi - \psi_f$ is a 2-cocycle of $\mathcal{B}(q)$, which is equivalent to ψ, where ψ_f is the trivial 2-cocycle induced by f. Similar to (4.6), (4.7) and Lemma 4.2 of [5], one can show that if $q \notin \frac{1}{2}\mathbb{Z}_-$, then we still have

$$\phi(L_{-1,i}, L_{1,0}) = 0, \quad i \in \mathbb{Z}_+, \tag{8}$$

$$\phi(L_{1,i}, L_{-1,0}) = 0, \quad i \in \mathbb{Z}_+, \tag{9}$$

$$\phi(L_{-2,i},L_{2,0})=0,\quad i\in\mathbb{Z}_+\backslash\{0\}. \tag{10}$$

Lemma 3.1 *For $\alpha,\beta\in\mathbb{Z}$, $i,j\in\mathbb{Z}_+$, we have*

(i) $\phi(L_{\alpha,i},L_{\beta,j})=0$ *if* $\alpha+\beta\neq 0$;

(ii) $\phi(L_{\alpha,i},L_{-\alpha,j})=0$ *if* $i\neq j$.

Proof (i) This conclusion can be proved as in [5, Lemma 4.3(1)].

(ii) Applying ϕ to triples $(L_{1,0},L_{\alpha,i},L_{-1-\alpha,j})$ and $(L_{-1,0},L_{1+\alpha,i},L_{-\alpha,j})$, respectively, by (9) and (10), we obtain following two equations

$$\begin{aligned}0&=\phi(L_{1,0},[L_{\alpha,i},L_{-1-\alpha,j}])\\&=\phi([L_{1,0},L_{\alpha,i}],L_{-1-\alpha,j})+\phi([L_{-1-\alpha,j},L_{1,0}],L_{\alpha,i})\\&=((\alpha-1)q-i)\phi(L_{1+\alpha,i},L_{-1-\alpha,j})-((\alpha+2)q+j)\phi(L_{\alpha,i},L_{-\alpha,j}),\end{aligned}\tag{11}$$

$$\begin{aligned}0&=\phi(L_{-1,0},[L_{1+\alpha,i},L_{-\alpha,j}])\\&=\phi([L_{-1,0},L_{1+\alpha,i}],L_{-\alpha,j})+\phi([L_{-\alpha,j},L_{-1,0}],L_{1+\alpha,i})\\&=((\alpha+2)q+i)\phi(L_{\alpha,i},L_{-\alpha,j})-((\alpha-1)q-j)\phi(L_{1+\alpha,i},L_{-1-\alpha,j}).\end{aligned}\tag{12}$$

Multiplying (11) by -1, and then adding to (12), under the condition $i\neq j$, we must have that $\phi(L_{\alpha,i},L_{-\alpha,j})$ is a constant, say, a for any $\alpha\in\mathbb{Z}$. Then, fix $i=0$ in (12), by arbitrariness of j, we must have $a=0$. □

Lemma 3.2 *For $\alpha\in\mathbb{Z}$, $i\in\mathbb{Z}_+$, we have*

(i) $\phi(L_{\alpha,i},L_{-\alpha,i})=0$ *if* $i\neq 0$;

(ii) $\phi(L_{\alpha,0},L_{-\alpha,0})=\frac{\alpha^3-\alpha}{6}\phi(L_{2,0},L_{-2,0})$.

Proof It is shown in [5, Lemma 4.4] that

$$((\alpha+2)q+i)\phi(L_{\alpha,i},L_{-\alpha,i})=((\alpha-1)q-i)\phi(L_{1+\alpha,i},L_{-1-\alpha,i}),\tag{13}$$

$$(i+2q)(2i+3q)\phi(L_{\alpha,i},L_{-\alpha,i})=(\alpha+1)(\alpha q-i)((\alpha-1)q-i)\phi(L_{2,0},L_{-2,2i}).\tag{14}$$

If $i\neq 0,-\frac{3}{2}q$, then by (14) and (10) we have $\phi(L_{\alpha,i},L_{-\alpha,i})=0$. If $i=-\frac{3}{2}q$, then (13) becomes

$$q(\alpha+\frac{1}{2})\phi(L_{\alpha,i},L_{-\alpha,i})=q(\alpha+\frac{1}{2})\phi(L_{1+\alpha,i},L_{-1-\alpha,i}).$$

Since $q\neq 0$ and $\alpha\in\mathbb{Z}$, the above equality implies that $\phi(L_{\alpha,i},L_{-\alpha,i})$ is a constant, say, b for any $\alpha\in\mathbb{Z}$. By the anti-symmetry property of the 2-cocycle ϕ, we have

$$b=\phi(L_{\alpha,i},L_{-\alpha,i})=-\phi(L_{-\alpha,i},L_{\alpha,i})=-b.$$

Hence $b=0$, and so the conclusion (i) holds. If $i=0$, then (14) gives the conclusion (ii). □

Now, we give the proof of Theorem 1.2.

Proof of Theorem 1.2 By Lemmas 3.1 and 3.2, we see that the 2-cocycle ϕ must take the form (3), and it induces the non-trivial central extension of $\mathcal{B}(q)$ as (2) by taking $c=2\phi(L_{2,0},L_{-2,0})$. □

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