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Expanding Integrable Models and Their Some Reductions as Well as Darboux Transformations

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Abstract In this paper we first present a 3-dimensional Lie algebra H and enlarge it into a 6-dimensional Lie algebra T with corresponding loop algebras $\tilde{H}$ and $\tilde{T}$, respectively. By using the loop algebra $\tilde{H}$ and the Tu scheme, we obtain an integrable hierarchy from which we derive a new Darboux transformation to produce a set of exact periodic solutions. With the loop algebra $\tilde{T}$, a new integrable-coupling hierarchy is obtained and reduced to some variable-coefficient nonlinear equations, whose Hamiltonian structure is derived by using the variational identity. Furthermore, we construct a higher-dimensional loop algebra $\bar{H}$ of the Lie algebra H from which a new Liouville-integrable hierarchy with 5-potential functions is produced and reduced to a complex mKdV equation, whose 3-Hamiltonian structure can be obtained by using the trace identity. A new approach is then given for deriving multi-Hamiltonian structures of integrable hierarchies. Finally, we extend the loop algebra $\tilde{H}$ to obtain an integrable hierarchy with variable coefficients.

Keywords Lie algebra; Hamiltonian structure; integrable hierarchy

MR(2010) Subject Classification 47E05; 37K10; 34L40

1. Introduction

The development of new integrable hierarchies of evolution equations is important in the study of soliton theory. These developments include the recent Lax pair method for generating integrable hierarchies of evolution equations [1] among which Tu [2] employed the matrix Lie algebras to introduce Lax pairs to derive some well-known integrable hierarchies of evolution type, such as the AKNS hierarchy, the KN hierarchy, and the WKI hierarchy. In this paper Tu proposed a milestone formula to deduce Hamiltonian structures of the integrable hierarchies, called the trace identity. The scheme for generating integrable hierarchies and the corresponding Hamiltonian structures was now known as the Tu scheme [3]. By adopting the Tu scheme, some interesting integrable hierarchies and their some properties were obtained [4–9]. Guo and Yu [10] introduced a loop algebra to obtain a multi-potential integrable hierarchy with 3-Hamiltonian

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structure. In [11,12] the authors proved that the self-dual Yang-Mills equations can be presented as a compatibility condition of the general Lax pair. The Lax pair method has been shown in [13] that it provides an effective approach to generate integrable hierarchies of evolution equations. To the knowledge of the authors, there is still very few work on applying the Lax pair method for generating variable-coefficient integrable hierarchies. Li [14] has proposed an approach to produce variable-coefficient integrable hierarchies under isospectral and non-isospectral problems by Lax pairs. In this paper, we first present a 3-dimensional Lie algebra H and enlarge it into a 6-dimensional Lie algebra T with corresponding loop algebras $\tilde{H}$ and $\tilde{T}$, respectively. Starting from the loop algebra $\tilde{H}$, we obtain two various integrable hierarchies under the isospectral problems by using the Tu scheme, one is an isospectral integrable hierarchy and the another one is a non-isospectral integrable hierarchy with variable coefficients. Based on the work given in [15], we derive a new Darboux transformation of the obtained integrable hierarchy which can be used to produce exact periodic solutions. Furthermore, we provide a new approach for generating variable-coefficient integrable hierarchies by producing an integrable couplings which can be reduced to a series of variable-coefficient nonlinear integrable equations. As a result, the Hamiltonian structure can be derived by the variational identity. Finally, we establish a higher dimensional loop algebra $\bar{H}$ of the Lie algebra H to produce a multi-potential Liouville-integrable hierarchy along with 3-Hamiltonian structure which can be reduced to an mKdV equation with complex coefficients.

2. A Lie algebra and its loop algebra $\tilde{H}$ as well as applications

The most simplest Lie algebra [16] presents:

$$A_1 = \left\{ e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}, \tag{1}$$

which possesses the following commutative relations

$$[h, e] = 2e, [h, f] = -2f, [e, f] = h.$$

We introduce the following Lie algebra

$$H = \mathrm{span}\{t_1, t_2, t_3\}, \tag{2}$$

where $t_1 = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}, t_2 = 2\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, t_3 = h$ along with the commutative relations:

$$[t_1, t_2] = 2t_3, [t_1, t_3] = -2t_1 - 4t_2, [t_2, t_3] = 2t_3.$$

A corresponding loop algebra is defined as

$$\tilde{H} = \mathrm{span}\{t_1(n), t_2(n), t_3(n)\}, \tag{3}$$

where $t_i(n) = t_i\lambda^n, i = 1, 2, 3; n \in \mathbf{Z}$. Eq. (3) will be used to generate two integrable hierarchies.

Firstly, we introduce the following isospectral problems:

$$\begin{cases} \varphi_x = U\varphi, U = t_1(1) - vt_2(0) + ut_3(0), \\ \varphi_t = V\varphi, V = \sum\limits_{m\geq 0} (a_m t_1(-m) + b_m t_2(-m) + c_m t_3(-m)). \end{cases} \tag{4}$$

Denote

$$V_+^{(n)} = \sum_{m=0}^{n} (a_m t_1(-m) + b_m t_2(-m) + c_m t_3(-m))\lambda^n, V^{(n)} = V_+^{(n)} - a_n t_1(0),$$

we have

$$-V_x^{(n)} + [U, V^{(n)}] = (4c_{n+1} - 4ua_n)t_2(0) - (2b_{n+1} + 2va_n)t_3(0).$$

Based on the Tu scheme, the zero curvature equation $U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0$ admits that

$$\begin{cases} v_{t_n} = 4c_{n+1} - 4ua_n, \\ u_{t_n} = c_{n,x}, \end{cases} \tag{5}$$

where a_n, b_n, c_n satisfy:

$$\begin{cases} c_{n+1} = -\frac{1}{2}a_{n,x} + ua_n, \\ b_{n+1} = \frac{1}{2}c_{n,x} - va_n, \\ a_{n,x} = \frac{1}{2}b_{n,x} + vc_n + ub_n. \end{cases} \tag{6}$$

Setting $n = 2, a_0 = -2, b_0 = c_0 = 0, t_2 = t$, we obtain from Eqs. (5) and (6) the following coupled nonlinear integrable equation:

$$\begin{cases} v_t = u_{xx} + 3vv_x + 2uu_x, \\ u_t = -\frac{1}{2}v_{xx} + (uv)_x, \end{cases} \tag{7}$$

which is similar to the long water wave equation. From Eq.(6), it is easy to get

$$L = \begin{pmatrix} -\frac{1}{2}\partial^{-1}v\partial - \frac{1}{2}v & -\frac{\partial}{4} - \frac{1}{2}\partial^{-1}u\partial \\ \frac{\partial}{2} - u & 0 \end{pmatrix}$$

which satisfies the following recurrence relation:

$$\begin{pmatrix} -2a_{n+1} \\ 2c_{n+1} \end{pmatrix} = L \begin{pmatrix} -2a_n \\ 2c_n \end{pmatrix}. \tag{8}$$

Combined with Eq. (6), Eq. (5) can be rewritten as:

$$\tilde{u}_{t_n} = \begin{pmatrix} v \\ u \end{pmatrix}_{t_n} = \begin{pmatrix} \partial & 0 \\ 0 & \frac{\partial}{2} \end{pmatrix} \begin{pmatrix} -2a_n \\ 2c_n \end{pmatrix} = J \begin{pmatrix} -2a_n \\ 2c_n \end{pmatrix}. \tag{9}$$

3. A Darboux transformation of the integrable hierarchy (5)

Ma [17] once investigated some Darboux transformations of a Lax integrable system in 2n-dimensions. Based on this and [15], we shall discuss a Darboux transformation of equation hierarchy (5) in what follows.

Denote $\Phi = JLJ^{-1}$, where $J = \begin{pmatrix} \partial & 0 \\ 0 & \frac{\partial}{2} \end{pmatrix}$, and L is presented in Eq. (8). We have

$$\Phi = \begin{pmatrix} -\frac{1}{2}(\partial v \partial^{-1} + v) & -\frac{\partial}{2} - u \\ \frac{\partial}{4} - \frac{1}{2}\partial u \partial^{-1} & 0 \end{pmatrix}. \tag{10}$$

Thus, the isospectral integrable hierarchy (9) can be written as

$$\tilde{u}_t = \begin{pmatrix} v \\ u \end{pmatrix}_t = \Phi^{n-1} \begin{pmatrix} \alpha v_x \\ 2\alpha u_x \end{pmatrix}. \tag{11}$$

The Lax pair of Eq. (20) is given by

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}_x = \begin{pmatrix} u & \lambda \\ -2\lambda - 2v & -u \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \tag{12}$$

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}_t = \begin{pmatrix} C & A + a_n \\ -2A + 2B - 2a_n & -C \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \tag{13}$$

where

$$A = \sum_{m=0}^{n} a_m \lambda^{n-m},\ B = \sum_{m=0}^{n} b_m \lambda^{n-m},\ C = \sum_{m=0}^{n} c_m \lambda^{n-m}.$$

The compatibility condition of Eqs. (12) and (13) leads to

$$\begin{cases} A_x = -2\lambda C + 2uA, \\ B_x = -4\lambda C - 2vC + 4uA - 2uB, \\ C_x = 2\lambda B + 2vA, \end{cases} \tag{14}$$

from which we get

$$A = \frac{1}{2}B + \partial^{-1}(vC + uB). \tag{15}$$

Suppose that the matrix $\begin{pmatrix} \varphi_{11}(x,t,\lambda) & \varphi_{12}(x,t,\lambda) \\ \varphi_{21}(x,t,\lambda) & \varphi_{22}(x,t,\lambda) \end{pmatrix}$ is a soliton matrix of Eq. (12). Denote

$$\xi_j = \frac{\mu_j \varphi_{21}(x,t,\lambda_j) + \nu_j \varphi_{22}(x,t,\lambda_j)}{\mu_j \varphi_{11}(x,t,\lambda_j) + \nu_j \varphi_{12}(x,t,\lambda_j)}, \quad j = 1, 2, \tag{16}$$

where $i \neq j \Rightarrow \mu_i \neq \mu_j, \nu_i \neq \nu_j$, and μ_j, ν_j are arbitrary constants with $|\mu_j| + |\nu_j|$ not identical zero. In terms of Eqs. (12) and (13), ξ_j satisfy the following Riccati equations

$$\xi_{j,x} = -2\lambda_j - 2u\xi_j - 2v - \xi_j^2 \lambda_j, \quad j = 1, 2, \tag{17}$$

$$\xi_{j,t} = -2A + 2B - 2a_n - 2C\xi_j - (A + a_n)\xi_j^2, \quad j = 1, 2. \tag{18}$$

Eq. (17) gives

$$\begin{cases} u = \frac{(\xi_1 - \xi_2)_x - 2(\lambda_2 - \lambda_1) + \lambda_1 \xi_1^2 - \lambda_2 \xi_2^2}{2(\xi_2 - \xi_1)}, \\ v = \frac{\xi_1 \xi_{2,x} - \xi_{1,x}\xi_2 + 2\xi_1 \lambda_2 - 2\xi_2 \lambda_1 + \lambda_1 \xi_1^2 \xi_2 - \lambda_2 \xi_2^2 \xi_1}{2(\xi_2 - \xi_1)}. \end{cases} \tag{19}$$

Hence, we obtain from Eq. (27) that

$$v_x = \frac{P}{2(\xi_2 - \xi_1)^2}, u_x = \frac{Q}{2(\xi_2 - \xi_1)^2}, \tag{20}$$

where

$$\begin{aligned}P =&(2\lambda_2\xi_1\xi_2 - 2\lambda_1\xi_1^2 - 2u\xi_1 + 2u\xi_2)\xi_1\xi_{2,x} + \xi_2\xi_{1,x}(2\lambda_1\xi_1\xi_2 - 2u\xi_2 - 2\lambda_2\xi_2^2 + 2u\xi_1)+\\ &(\xi_1\xi_2 - \xi_1^2)\xi_{2,xx} + (\xi_1 - \xi_2)\xi_2\xi_{1,xx},\\ Q =&(\xi_1 - \xi_2)(\xi_2 - \xi_1)_{xx} + [2u(\xi_2 - \xi_1) + 4(\lambda_2 - \lambda_1) - 3\lambda_1\xi_1^2 + 2\lambda_2\xi_2^2 + 2\lambda_1\xi_1\xi_2 - \lambda_1\xi_1]\xi_{1,x}+\\ &[4(\lambda_1 - \lambda_2) + 2u(\xi_1 - \xi_2) + 2\lambda_2\xi_1\xi_2 - 4\lambda_2\xi_2^2 + \lambda_1\xi_1^2 + \lambda_1\xi_1]\xi_{2,x}.\end{aligned}$$

Assume

$$T = \frac{\alpha}{2(\xi_2 - \xi_1)^2}\begin{pmatrix} a_1\partial^2 + a_2\partial & b_1\partial^2 + b_2\partial \\ c_1\partial^2 + c_2\partial & d_1\partial^2 + d_2\partial \end{pmatrix},$$

where

$$\begin{aligned}&a_1 = \xi_1\xi_2 - {\xi_2}^2, a_2 = 2\lambda_1\xi_1{\xi_2}^2 - 2u{\xi_2}^2 - 2\lambda_2{\xi_2}^3 + 2u\xi_1\xi_2, b_1 = \xi_1\xi_2 - {\xi_1}^2,\\ &b_2 = 2\lambda_2{\xi_1}^2\xi_2 - 2\lambda_1{\xi_1}^3 - 2u{\xi_1}^2 + 2u\xi_1\xi_2, c_1 = \xi_2 - \xi_1,\\ &c_2 = 2u(\xi_2 - \xi_1) + 4(\lambda_2 - \lambda_1) - 3\lambda_1{\xi_1}^2 + 2\lambda_2{\xi_2}^2 + 2\lambda_1\xi_1\xi_2 - \lambda_1\xi_1, d_1 = \xi_1 - \xi_2,\\ &d_2 = 4(\lambda_1 - \lambda_2) + 2u(\xi_1 - \xi_2) + 2\lambda_2\xi_1\xi_2 - 4\lambda_2\xi_2^2 + \lambda_1\xi_1^2 + \lambda_1\xi_1.\end{aligned}$$

The inverse operator of T presents that

$$T^{-1} = \begin{pmatrix} \partial^{-2}a^{-1} + \partial^{-1}a_2^{-1} & \partial^{-2}c_1^{-1} + \partial^{-1}c_2^{-1} \\ \partial^{-2}b_1^{-1} + \partial^{-1}b_2^{-1} & \partial^{-2}d_1^{-1} + \partial^{-1}d_2^{-1} \end{pmatrix}.$$

It is easy to obtain

$$\begin{pmatrix} v \\ u \end{pmatrix}_t = T\begin{pmatrix} \xi_{1,t} \\ \xi_{2,t} \end{pmatrix}, \ \lambda_{i,t} = 0, \quad i = 1, 2.$$

Set

$$\tilde{\Phi} = \begin{pmatrix} \tilde{\Phi}_{11}(\lambda_1, \lambda_2, \xi_1, \xi_2) & \tilde{\Phi}_{12}(\lambda_1, \lambda_2, \xi_1, \xi_2) \\ \tilde{\Phi}_{21}(\lambda_1, \lambda_2, \xi_1, \xi_2) & \tilde{\Phi}_{22}(\lambda_1, \lambda_2, \xi_1, \xi_2) \end{pmatrix},$$

where

$$\begin{aligned}\tilde{\Phi}_{11}(\lambda_1, \lambda_2, \xi_1\xi_2) =&(\partial^{-1}a_1^{-1} + \partial^{-1}a_2^{-1})(-\frac{1}{2}v_x\partial^{-1}a_1\partial^2 - va_1\partial^2 - \frac{1}{2}\partial^{-1}v\partial a_1\partial^2-\\ &\frac{1}{2}v_x\partial^{-1}a_2\partial - va_2\partial - \frac{1}{2}\partial^{-1}v\partial a_2\partial - uc_1\partial^2-\\ &uc_2\partial + \partial^{-1}u\partial c_1\partial^2 + \partial^{-1}u\partial c_2\partial),\\ \tilde{\Phi}_{21}(\lambda_1, \lambda_2, \xi_1, \xi_2) =&(\partial^{-2}b_1^{-1} + \partial^{-1}b_2^{-1})(-\frac{1}{2}v_x\partial^{-1}a_1\partial^2 - va_1\partial^2 - \frac{1}{2}\partial^{-1}v\partial a_1\partial^2-\\ &\frac{1}{2}v_x\partial^{-1}a_2\partial - va_2\partial - \frac{1}{2}\partial^{-1}v\partial a_2\partial - uc_1\partial^2-\\ &uc_2\partial + \partial^{-1}u\partial c_1\partial^2 + \partial^{-1}u\partial c_2\partial),\\ \tilde{\Phi}_{12}(\lambda_1, \lambda_2, \xi_1, \xi_2) =&\tilde{\Phi}_{21}(\lambda_2, \lambda_1, \xi_2, \xi_1), \ \tilde{\Phi}_{22}(\lambda_1, \lambda_2, \xi_1, \xi_2) = \tilde{\Phi}_{11}(\lambda_2, \lambda_1, \xi_2, \xi_1).\end{aligned}$$

After simplification and computation, we obtain

$$\Phi T = T\tilde{\Phi}, \tag{21}$$

which indicates that

$$\Phi^n T = T\tilde{\Phi}^n. \tag{22}$$

Therefore, we have

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}_t = \begin{pmatrix} -2A+2B-2a_n-2C\xi_1-(A+a_n)\xi_1^2 \\ -2A+2B-2a_n-2C\xi_2-(A+a_n)\xi_2^2 \end{pmatrix} = \tilde{\Phi}^{m-1}\begin{pmatrix} \xi_{1,x} \\ \xi_{2,x} \end{pmatrix}, \tag{23}$$

$$\begin{pmatrix} v \\ u \end{pmatrix}_t - \Phi^{m-1}\begin{pmatrix} \alpha v_x \\ 2\alpha u_x \end{pmatrix} = T\left\{\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}_t - \tilde{\Phi}^{m-1}\begin{pmatrix} \xi_{1,x} \\ \xi_{2,x} \end{pmatrix}\right\}. \tag{24}$$

Choosing an appropriate ξ_i $(i=1,2)$ and employing Eqs. (12)and (13), we obtain

$$\begin{cases} \bar{v} = v + \frac{(\lambda_1-\lambda_2)(\xi_1+\xi_2)(2+\xi_1\xi_2)}{2(\xi_2-\xi_1)}, \\ \bar{u} = u + \frac{(\lambda_2-\lambda_1)(4+\xi_1^2+\xi_2^2)}{2(\xi_2-\xi_1)}. \end{cases} \tag{25}$$

It can be verified that Eq. (25) satisfies Eq. (11). Therefore, Eq. (25) is a Darboux transformation of the isospectral integrable hierarchy (5).

In the following, we consider some exact solutions of the nonlinear integrable equations (7) by using the Darboux transformation (25). Eq. (12) can be written as

$$\begin{cases} \varphi_{11,x} = u\varphi_{11} + \lambda\varphi_{21}, \\ \varphi_{12,x} = u\varphi_{12} + \lambda\varphi_{22}, \\ \varphi_{21,x} = (-2\lambda-2v)\varphi_{11} - u\varphi_{21}, \\ \varphi_{22,x} = (-2\lambda-2v)\varphi_{12} - u\varphi_{22}. \end{cases} \tag{26}$$

We take a special solution of Eq.(7) to be $u=v=0$. Eq. (26) reduces to

$$\varphi_{11,x} = \lambda\varphi_{21},\ \varphi_{12,x} = \lambda\varphi_{22},\ \varphi_{21,x} = -2\lambda\varphi_{11},\ \varphi_{22,x} = -2\lambda\varphi_{12}, \tag{27}$$

which gives rise to

$$\varphi_{11,xx} = -2\lambda^2\varphi_{11},\ \varphi_{12,xx} = -2\lambda^2\varphi_{12}, \tag{28}$$

whose solutions are given by

$$\varphi_{11} = \sin(c_1(t)+\sqrt{2}\lambda x),\ \varphi_{12} = \sin(c_2(t)+\sqrt{2}\lambda x). \tag{29}$$

Substituting (29) into (27) yields that

$$\varphi_{21} = \sqrt{2}\cos(c_1(t)+\sqrt{2}\lambda x),\ \varphi_{22} = \sqrt{2}\cos(c_2(t)+\sqrt{2}\lambda x). \tag{30}$$

Eq. (13) can be written as

$$\begin{cases} \varphi_{11,t} = C\varphi_{11} + (A+a_n)\varphi_{21}, \\ \varphi_{12,t} = C\varphi_{12} + (A+a_n)\varphi_{22}, \\ \varphi_{21,t} = (-2A+2B-2a_n)\varphi_{11} - C\varphi_{21}, \\ \varphi_{22,t} = (-2A+2B-2a_n)\varphi_{12} - C\varphi_{22}. \end{cases} \tag{31}$$

When $n=2$, we have from Eq. (6) that

$$b_1 = 2v, c_1 = -2u, a_1 = v, b_2 = -u_x - v^2, c_2 = -\frac{1}{2}v_x + uv, a_2 = -\frac{1}{2}u_x - \frac{3}{4}v^2 - \frac{1}{2}u^2, \ldots. \tag{32}$$

Thus, we obtain

$$A = -2\lambda^2 + v\lambda - \frac{1}{2}u_x - \frac{3}{4}v^2 - \frac{1}{2}u^2,\ B = 2v\lambda - u_x - v^2,\ C = -2u\lambda - \frac{1}{2}v_x + uv. \tag{33}$$

When $n = 2, u = v = 0$, Eq. (31) reduces to

$$\varphi_{11,t} = -2\lambda^2\varphi_{21},\ \varphi_{12,t} = -2\lambda^2\varphi_{22},\ \varphi_{21,t} = 4\lambda^2\varphi_{11},\ \varphi_{22,t} = 4\lambda^2\varphi_{12}, \tag{34}$$

which gives

$$\varphi_{11,xx} = -8\lambda^4\varphi_{11},\ \varphi_{12,tt} = -8\lambda^4\varphi_{12}. \tag{35}$$

When $c_1(t) = c_2(t) = -2\sqrt{2}\lambda^2 t$, the solutions to Eq. (27) satisfy Eq. (34). That is, the following equations

$$\varphi_{11} = \varphi_{12} = \sin(\sqrt{2}\lambda x - 2\sqrt{2}\lambda^2 t),\ \varphi_{21} = \varphi_{22} = \sqrt{2}\cos(\sqrt{2}\lambda x - 2\sqrt{2}\lambda^2 t) \tag{36}$$

satisfy Eq. (34). Substituting (36) into Eq. (16) yields that

$$\xi_i = \sqrt{2}\cot(\sqrt{2}\lambda_i x - 2\sqrt{2}\lambda_i^2 t),\quad i = 1, 2. \tag{37}$$

When $\lambda_1 \neq \lambda_2$, Eq.(37) represents two formulas

$$\xi_1 = \sqrt{2}\cot(\sqrt{2}\lambda_1 x - 2\sqrt{2}\lambda_1^2 t), \tag{38}$$

$$\xi_2 = \sqrt{2}\cot(\sqrt{2}\lambda_2 x - 2\sqrt{2}\lambda_2^2 t). \tag{39}$$

When set $u = v = 0$, Eq. (25) reduces to

$$\begin{cases} \bar{v} = \frac{(\lambda_1-\lambda_2)(\xi_1+\xi_2)(2+\xi_1\xi_2)}{2(\xi_2-\xi_1)}, \\ \bar{u} = \frac{(\lambda_2-\lambda_1)(4+\xi_1^2+\xi_2^2)}{2(\xi_2-\xi_1)}. \end{cases} \tag{40}$$

Substituting (38), (39) into (40), we obtain the following set of exact periodic solutions:

$$\begin{cases} \bar{v} = (\lambda_1 - \lambda_2)\frac{\cot(\sqrt{2}\lambda_1 x - 2\sqrt{2}\lambda_1^2 t) + \cot(\sqrt{2}\lambda_2 x - 2\sqrt{2}\lambda_2^2 t)}{\cot(\sqrt{2}\lambda_2 x - 2\sqrt{2}\lambda_2^2 t) - \cot(\sqrt{2}\lambda_1 x - 2\sqrt{2}\lambda_1^2 t)}(1 + \cot(\sqrt{2}\lambda_1 x - 2\sqrt{2}\lambda_1^2 t) \\ \quad \cot(\sqrt{2}\lambda_2 x - 2\sqrt{2}\lambda_2^2 t)), \\ \bar{u} = \frac{\lambda_2-\lambda_1}{\sqrt{2}}\frac{2+\cot^2(\sqrt{2}\lambda_1 x - 2\sqrt{2}\lambda_1^2 t)+\cot^2(\sqrt{2}\lambda_2 x - 2\sqrt{2}\lambda_2^2 t)}{\cot(\sqrt{2}\lambda_2 x - 2\sqrt{2}\lambda_2^2 t) - \cot(\sqrt{2}\lambda_1 x - 2\sqrt{2}\lambda_1^2 t)}. \end{cases} \tag{41}$$

4. Enlarged Lie algebra of the Lie algebra H and its applications

In the section we further enlarge the Lie algebra H into another one with the help of the known Lie algebra proposed in [18]. If we denote

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

which is the same as the one given in Eq.(1), then one Lie algebra was presented in [16] as:

$$G = \text{span}\{f_1, f_2, \ldots, f_6\}, \tag{42}$$

where

$$f_i = \begin{pmatrix} e_i & 0 \\ 0 & e_i \end{pmatrix},\ f_j = \begin{pmatrix} 0 & e_i \\ 0 & e_i \end{pmatrix},\quad j = 4, 5, 6; i = 1, 2, 3$$

equipped with the following commutative relations

$$[f_1, f_2] = 2f_2, [f_1, f_3] = -2f_3, [f_2, f_3] = f_1, [f_1, f_4] = 0, [f_1, f_5] = 2f_5, [f_1, f_6] = -2f_6,$$
$$[f_2, f_4] = -2f_5, [f_2, f_5] = 0, [f_2, f_6] = f_4, [f_3, f_4] = 2f_6, [f_3, f_5] = -f_4, [f_3, f_6] = 0,$$
$$[f_4, f_5] = 2f_5, [f_4, f_6] = -2f_6, [f_5, f_6] = f_4.$$

Denote $G_1 = \mathrm{span}\{f_1, f_2, f_3\}$, $G_2 = \mathrm{span}\{f_4, f_5, f_6\}$, then we have

$$G = G_1 \oplus G_2, G_1 \cong A_1, [G_1, G_2] \subset G_2, \tag{43}$$

and G_1, G_2 are all single Lie subalgebras. Based on the Lie algebra G and Eq. (43), we introduce a linear space T:

$$T = \mathrm{span}\{t_1, t_2, \ldots, t_6\}, \tag{44}$$

whose commutative relations are defined as

$$[t_1, t_2] = 2t_3, [t_1, t_3] = -2t_1 - 4t_2, [t_2, t_3] = 2t_2, [t_1, t_4] = 0, [t_1, t_5] = 2t_6,$$
$$[t_1, t_6] = [t_4, t_6] = -2t_4 - 4t_5, [t_2, t_4] = -2t_6, [t_2, t_5] = 0, [t_2, t_6] = 2t_5, [t_3, t_4] = -[t_1, t_6],$$
$$[t_3, t_5] = -2t_5, [t_3, t_6] = 0, [t_4, t_5] = 2t_6, [t_5, t_6] = 2t_5.$$

It can be verified that the above relations satisfy

$$[a, b] = -[b, a], [\alpha a + \beta b, c] = \alpha[a, c] + \beta[b, c], [[a, b], c] + [[b, c], a] + [[c, a], b] = 0,$$
$$\forall a, b, c \in T, \alpha, \beta \in C.$$

Therefore, T becomes a Lie algebra. If we denote $T_1 = \mathrm{span}\{t_1, t_2, t_3\}, T_2 = \mathrm{span}\{t_4, t_5, t_6\}$, then T, T_1 and T_2 satisfy the relation (43). Furthermore, we have $T_1 \cong H$. Hence, the Lie algebra T is an enlarged one of the Lie algebra H. We now define a loop algebra of the Lie algebra T:

$$\tilde{T} = \mathrm{span}\{t_1(n), t_2(n), \ldots, t_6(n)\}, \tag{45}$$

where

$$t_i(n) = t_i\lambda^{2n+1}, i = 1, 2, 4, 5; t_j(n) = t_j\lambda^{2n}, \quad j = 3, 6. \tag{46}$$

According to (44) and (46), we get the commutative relations

$$[t_1(m), t_2(n)] = 2t_3(m + n + 1), [t_1(m), t_3(n)] = -2t_1(m + n) - 4t_2(m + n),$$
$$[t_2(m), t_3(n)] = 2t_2(m + n), [t_1(m), t_4(n)] = 0, [t_1(m), t_5(n)] = 2t_6(m + n + 1),$$
$$[t_1(m), t_6(n)] = [t_4(m), t_6(n)] = -2t_4(m + n) - 4t_5(m + n),$$
$$[t_2(m), t_4(n)] = -2t_6(m + n + 1), [t_2(m), t_5(n)] = 0, [t_2(m), t_6(n)] = 2t_5(m + n),$$
$$[t_3(m), t_4(n)] = 2t_4(m + n) + 4t_5(m + n), [t_3(m), t_5(n)] = -2t_5(m + n),$$
$$[t_3(m), t_6(n)] = 0, [t_4(m), t_5(n)] = 2t_6(m + n + 1), [t_5(m), t_6(n)] = 2t_5(m + n), \; m, n \in \mathbf{Z}.$$

Set

$$\tilde{T}_1 = \mathrm{span}\{t_1(n), t_2(n), t_3(n)\}, \; \tilde{T}_2 = \mathrm{span}\{t_4(n), t_5(n), t_6(n)\},$$

then we have $\tilde{T} = \tilde{T}_1 \oplus \tilde{T}_2, [\tilde{T}_1, \tilde{T}_2] \subset \tilde{T}_2$. For illustration, we first present an application of the loop algebra $\tilde{T}_1$ in the following example. Consider

$$\begin{cases} \psi_x = U_1\psi, U_1 = t_3(1) + qt_1(0) + rt_2(0), \\ \psi_t = V_1\psi, V_1 = \sum\limits_{m\geq 0}(\sum\limits_{j=1}^{3} V_{j,m}t_j(-m)). \end{cases} \tag{47}$$

According to the Tu scheme, we solve the following stationary zero curvature equation for $V_{j,m}$

$$(V_1)_x = [U_1, V_1] \tag{48}$$

to yield

$$\begin{cases} (V_{1,m})_x = 2V_{1,m+1} - 2qV_{3,m}, \\ (V_{2,m})_x = 4V_{1,m+1} - 2V_{2,m+1} - 4qV_{3,m} + 2rV_{3,m}, \\ (V_{3,m})_x = 2qV_{2,m+1} - 2rV_{1,m+1}, \end{cases}$$

which is equivalent to

$$\begin{cases} V_{1,m+1} = \frac{1}{2}(V_{1,m})_x + qV_{3,m}, \\ V_{2,m+1} = -\frac{1}{2}(V_{2,m})_x + (V_{1,m})_x + rV_{3,m}, \\ (V_{3,m})_x = -q(V_{2,m})_x + (2q-r)(V_{1,m})_x. \end{cases} \tag{49}$$

Given the initial values $V_{3,0} = \alpha, V_{1,0} = V_{2,0} = 0$, Eq. (49) leads to

$$V_{1,1} = \alpha q, V_{2,1} = \alpha r, V_{3,1} = \alpha(q^2 - qr), V_{1,2} = \alpha(\frac{1}{2}q_x + q^3 - q^2r),$$
$$V_{2,2} = \alpha(-\frac{1}{2}r_x + q_x + rq^2 - qr^2), V_{3,2} = \alpha(\frac{1}{2}qr_x - rq_x - 3q^3r + \frac{3}{2}q^4 + \frac{3}{2}q^2r^2), \ldots.$$

We decompose Eq. (48) into the following:

$$-(V_{1,+}^{(n)})_x + [U_1, V_{1,+}^{(n)}] = (V_{1,-}^{(n)})_x - [U_1, V_{1,-}^{(n)}], \tag{50}$$

where

$$V_{1,+}^{(n)} = \sum_{m=0}^{n}\sum_{j=1}^{3} V_{j,m}t_j(-m)\lambda^{2n},\ V_{1,-}^{(n)} = \lambda^{2n}V - V_{1,+}^{(n)}.$$

The degree of the left-hand side in (50) is greater than zero, whilst the right-hand side is smaller than 1. Thus, we obtain that

$$-(V_{1,+}^{(n)})_x + [U_1, V_{1,+}^{(n)}] = -2V_{1,n+1}t_1(0) + (2V_{2,n+1} - 4V_{1,n+1})t_2(0) + (2rV_{1,m+1} - 2qV_{2,n+1})t_3(0).$$

Denote $V_1^{(n)} = V_{1,+}^{(n)} - V_{3,n}t_3(0)$, we then have

$$-(V_1^{(n)})_x + [U_1, V_1^{(n)}] = -(V_{1,n})_x t_1(0) - (V_{2,n})_x t_2(0).$$

Therefore, we obtain the zero curvature equation $(U_1)_{t_n} - (V_1^{(n)})_x + [U_1, V_1^{(n)}] = 0$, which is a compatibility condition of the following Lax pair

$$\varphi_x = U_1\varphi, \varphi_{t_n} = V_1^{(n)}\varphi, \tag{51}$$

which admits that

$$\begin{pmatrix} q \\ r \end{pmatrix}_{t_n} = \begin{pmatrix} (V_{1,n})_x \\ (V_{2,n})_x \end{pmatrix} = J\begin{pmatrix} -4V_{1,n} + 2V_{2,n} \\ 2V_{1,n} \end{pmatrix}, \tag{52}$$

where $J = \begin{pmatrix} 0 & \frac{\partial}{2} \\ \frac{\partial}{2} & \partial \end{pmatrix}$ is an obvious Hamiltonian operator. We consider the Hamiltonian structure of Eq. (52). Since

$$U_1 = \begin{pmatrix} \lambda^2 & \lambda q \\ -2\lambda q + 2\lambda r & -\lambda^2 \end{pmatrix},\ V_1 = \begin{pmatrix} \bar{V}_3 & \lambda \bar{V}_1 \\ -2\lambda \bar{V}_1 + 2\lambda \bar{V}_2 & -\bar{V}_3 \end{pmatrix},$$

where $\bar{V}_i = \sum_{m\geq 0} V_{i,m}\lambda^{-2m}$, $i = 1, 2, 3$, it is easy to derive that

$$\langle V_1, \frac{\partial U_1}{\partial q}\rangle = -4\lambda^2\bar{V}_1 + 2\lambda^2\bar{V}_2,\ \langle V_1, \frac{\partial U_1}{\partial r}\rangle = 2\lambda^2\bar{V}_1,$$
$$\langle V_1, \frac{\partial U_1}{\partial \lambda}\rangle = -4\lambda\bar{V}_3 + \lambda(2r\bar{V}_1 - 4q\bar{V}_1) - 2\lambda q\bar{V}_2.$$

Substituting the above equations into the trace identity yields

$$\frac{\delta}{\delta u}[-4\bar{V}_3 + 2r\bar{V}_1 - 4q\bar{V}_1 - 2q\bar{V}_2]\lambda = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\begin{pmatrix} (-4\bar{V}_1 + 2\bar{V}_2)\lambda^2 \\ 2\bar{V}_1\lambda^2 \end{pmatrix}, \tag{53}$$

where $\frac{\delta}{\delta u} = (\frac{\delta}{\delta q}, \frac{\delta}{\delta r})^T$. Comparing the coefficients of λ^{-2n+1} in Eq. (53), we have

$$\frac{\delta}{\delta u}(-4V_{3,n} + 2rV_{1,n} - 4qV_{1,n} - 2qV_{2,n}) = (2 - 2n + \gamma)\begin{pmatrix} -4V_{1,n} + 2V_{2,n} \\ 2V_{1,n} \end{pmatrix}.$$

Using the initial values in Eq. (49), we have $\gamma = 2$. Thus, we obtain

$$\begin{pmatrix} -4V_{1,n} + 2V_{2,n} \\ 2V_{1,n} \end{pmatrix} = \frac{\delta}{\delta u}(\frac{-2V_{3,n} + (r - 2q)V_{1,n} - qV_{2,n}}{2 - n}) \equiv \frac{\delta H_n}{\delta u}.$$

Therefore, the Hamiltonian structure of the integrable hierarchy (52) is given as:

$$u_{t_n} = \begin{pmatrix} q \\ r \end{pmatrix}_{t_n} = J\frac{\delta H_n}{\delta u}, \tag{54}$$

where $H_n = \frac{1}{n-2}[2V_{3,n} + (2q - r)V_{1,n} + qV_{2,n}]$ are the Hamiltonian densities of Eq. (44). When $n = 2$, Eq. (52) can be reduced to

$$\begin{cases} q_{t_2} = \frac{\alpha}{2}q_{xx} + 3\alpha q^2 q_x - \alpha(q^2 r)_x, \\ r_{t_2} = -\frac{\alpha}{2}r_{xx} + \alpha q_{xx} + \alpha(q^2 r)_x - \alpha(qr^2)_x, \end{cases} \tag{55}$$

which is a kind of generalized nonlinear Schrödinger equation. Obviously, Eq. (52) is not the classical KN hierarchy but a new integarbel system.

To tackle the more complicated case of variable-coefficient nonlinear integrable equations, we will devise in the following an integrable coupling of Eq. (52) and deduce its Hamiltonian structure by using the variational identity [19]. Let

$$\begin{cases} \varphi_x = U\varphi, U = U_1 + U_2, \\ \varphi_t = V\varphi, V = V_1 + V_2, \end{cases} \tag{56}$$

where

$$U_1 = t_3(1) + qt_1(0) + rt_2(0),\ U_2 = t_6(1) + u_1 t_4(0) + u_2 t_5(0),$$

$$V_1=\sum_{m\geq 0}\Big(\sum_{j=1}^{3}V_{j,m}t_j(-m)\Big),\ V_2=\sum_{m\geq 0}\Big(\sum_{j=4}^{6}V_{j,m}t_j(-m)\Big).$$

The compatibility condition of Eq. (56) gives

$$U_{1,t}+U_{2,t}-V_{1,x}-V_{2,x}+[U_1,V_1]+[U_1,V_2]+[U_2,V_1]+[U_2,V_2]=0. \tag{57}$$

The corresponding stationary zero curvature equation of Eq. (57) obtains

$$-V_{1,x}-V_{2,x}+[U_1,V_1]+[U_1,V_2]+[U_2,V_1]+[U_2,V_2]=0, \tag{58}$$

whose sufficient condition deduces that

$$-V_{1,x}+[U_1,V_1]=0, \tag{59}$$

$$-V_{2,x}+[U_1,V_2]+[u_2,V_1+V_2]=0. \tag{60}$$

It can be seen that Eq. (59) is the same as Eq. (48). Again, we find that (60) is equivalent to

$$\begin{aligned}
(V_{4,m})_x=&4V_{4,m+1}-(2q+2u_1)V_{6,m}-(2u_1+2q)V_{3,m}+(V_{1,m})_x,\\
(V_{5,m})_x=&8V_{4,m+1}-4V_{5,m+1}+(-4q+2r-4u_1+2u_2)V_{6,m}+\\
&(-4u_1+2u_2+4q-2r)V_{3,m}+(V_{2,m})_x,\\
(V_{6,m})_x=&(2q+2u_1)V_{5,m+1}-(2r+2u_2)V_{4,m+1}+2u_1V_{2,m+1}-2u_2V_{1,m+1},
\end{aligned}$$

which can be written as

$$\begin{cases}
V_{4,m+1}=\frac{1}{4}(V_{4,m})_x+\frac{1}{2}(q+u_1)V_{6,m}+\frac{1}{2}(u_1-q)V_{3,m}-\frac{1}{4}(V_{1,m})_x,\\
V_{5,m+1}=-\frac{1}{4}(V_{5,m})_x+\frac{1}{4}(V_{2,m})_x+\frac{1}{2}(V_{4,m})_x-\frac{1}{2}(V_{1,m})_x+\frac{1}{2}(r+u_2)V_{6,m}+\frac{1}{2}(u_2-r)V_{3,m},\\
(V_{6,m})_x=-\frac{1}{2}(q+u_1)(V_{5,m})_x+(q+u_1-\frac{1}{2}r-\frac{1}{2}u_2)(V_{4,m})_x+\frac{1}{2}(q-u_1)(V_{2,m})_x+\\
\qquad (u_1-q+\frac{1}{2}r-\frac{1}{2}u_2)(V_{1,m})_x.
\end{cases} \tag{61}$$

Note that

$$V_+^{(n)}=V_{1,+}^{(n)}+\sum_{m=0}^{n}(\sum_{j=4}^{6}V_{j,m}t_j(-m))\lambda^{2n}=\lambda^{2n}V-V_-^{(n)}.$$

A direct computation leads to

$$\begin{aligned}
&-(V_+^{(n)})_x+[U,V_+^{(n)}]\\
&=-2V_{1,n+1}t_1(0)+(2V_{2,n+1}-4V_{1,n+1})t_2(0)+(2rV_{1,n+1}-2qV_{2,n+1})t_3(0)-\\
&\quad(4V_{4,n+1}+2V_{1,n+1})t_4(0)+(-8V_{4,n+1}+4V_{5,n+1}-4V_{1,n+1}+2V_{2,n+1})t_5(0)+\\
&\quad(-2qV_{5,n+1}-2u_1V_{2,n+1}+2u_2V_{1,n+1}-2u_1V_{5,n+1}+2u_2V_{4,n+1}+2rV_{4,n+1})t_6(0).
\end{aligned}$$

Denote

$$V^{(n)}=V_+^{(n)}-V_{3,n}t_3(0)-V_{6,n}t_6(0).$$

From Eqs. (49) and (61), we have:

$$\begin{aligned}
&-V_x^{(n)}+[U,V^{(n)}]\\
&=-(V_{1,n})_xt_1(0)-(V_{2,n})_xt_2(0)+[(2q+2u_1)V_{6,n}+2u_1V_{3,n}-4V_{4,n+1}-2V_{1,n+1}]t_4(0)+
\end{aligned}$$

$$[(4q-2r+4u_1-2u_2)V_{6,n}+(4u_1-2u_2)V_{3,n}-8V_{4,n+1}+4V_{5,n+1}-4V_{1,n+1}+2V_{2,n+1}]t_5(0)$$
$$=-(V_{1,n})_x t_1(0)-(V_{2,n})_x t_2(0)-(V_{4,n})_x t_4(0)-(V_{5,n})_x t_5(0).$$

Thus, the zero curvature equation $U_{t_n}-V_x^{(n)}+[U,V^{(n)}]=0$ admits that

$$Q_{t_n}\equiv\begin{pmatrix} q\\ r\\ u_1\\ u_2\end{pmatrix}_{t_n}=\begin{pmatrix}(V_{1,n})_x\\ (V_{2,n})_x\\ (V_{4,n})_x\\ (V_{5,n})_x\end{pmatrix}. \tag{62}$$

When $u_1=u_2=0$, Eq. (62) reduces to Eq. (52). Hence, according to the theory on integrable couplings [19–23], Eq. (62) is an integrable coupling of Eq. (52). Using the following initial values in Eqs. (49) and (61):

$$V_{3,0}=\alpha,\ V_{1,0}=V_{2,0}=V_{4,0}=V_{5,0}=V_{6,0}=0, \tag{63}$$

we obtain

$$\begin{aligned}
V_{1,1}=&\alpha q, V_{2,1}=\alpha r, V_{3,1}=\alpha(q^2-qr), V_{4,1}=\frac{\alpha}{2}(u_1-q), V_{5,1}=\frac{\alpha}{2}(u_2-r),\\
V_{6,1}=&\frac{\alpha}{4}(-qu_2+qu_1-u_1r-u_1u_2+qr+u_1^2-q^2), V_{1,2}=\frac{\alpha}{2}q_x+\alpha q^3-\alpha q^2r,\\
V_{2,2}=&-\frac{\alpha}{2}qr_x+\alpha q_x+\alpha rq^2-\alpha qr^2,\\
V_{3,2}=&\frac{\alpha}{2}qr_x-\frac{\alpha}{2}rq_x-3\alpha q^3r+\frac{3\alpha}{2}q^4+\frac{3\alpha}{2}q^2r^2,\\
V_{4,2}=&\frac{\alpha}{8}u_{1,x}-\frac{3\alpha}{8}q_x-\frac{\alpha}{8}q^2u_2-\frac{\alpha}{4}qu_1u_2-\frac{\alpha}{2}qru_1-\frac{5\alpha}{8}q^3+\frac{5\alpha}{8}q^2r+\frac{\alpha}{4}qu_1^2-\\
&\frac{\alpha}{8}u_1^2r+\frac{\alpha}{2}u_1q^2+\frac{\alpha}{8}u_1^3-\frac{\alpha}{8}u_1^2u_2,\\
V_{5,2}=&-\frac{\alpha}{8}u_{2,x}+\frac{3\alpha}{8}r_x-\frac{3\alpha}{4}q_x+\frac{\alpha}{4}u_{1,x}+\frac{\alpha}{8}qru_1-\frac{\alpha}{8}u_1r^2-\frac{\alpha}{4}ru_1u_2+\frac{5\alpha}{8}qr^2+\\
&\frac{\alpha}{8}u_1^2r-\frac{5\alpha}{8}rq^2-\frac{\alpha}{8}qu_2^2+\frac{\alpha}{8}qu_1u_2-\frac{\alpha}{8}u_1u_2^2+\frac{\alpha}{8}u_1^2u_2+\frac{3\alpha}{8}q^2u_2-\frac{\alpha}{2}qru_2,\ldots.
\end{aligned}$$

When $n=2$, Eq. (62) reduces to

$$\begin{cases} q_{t_2}=\frac{\alpha}{2}q_{xx}+3\alpha q^2q_x-\alpha(q^2r)_x,\\ r_{t_2}=-\frac{\alpha}{2}r_{xx}+\alpha q_{xx}+\alpha(rq^2)_x-\alpha(qr^2)_x.\end{cases} \tag{64}$$

$$\begin{cases}
(u_1)_{t_2}=\frac{\alpha}{8}u_{1,xx}-\frac{3\alpha}{8}q_{xx}-\frac{\alpha}{8}(q^2u_2)_x-\frac{\alpha}{4}(qu_1u_2)_x-\frac{\alpha}{2}(qru_1)_x-\frac{15\alpha}{8}q^2q_x+\frac{5\alpha}{8}(q^2r)_x+\\
\qquad\frac{\alpha}{4}(qu_1^2)_x-\frac{\alpha}{8}(u_1^2r)_x+\frac{\alpha}{2}(q^2u_1)_x+\frac{3\alpha}{8}u_1^2u_{1,x}-\frac{\alpha}{8}(u_1^2u_2)_x,\\
(u_2)_{t_2}=-\frac{\alpha}{8}u_{2,xx}+\frac{3\alpha}{8}r_{xx}-\frac{3\alpha}{4}q_{xx}+\frac{\alpha}{4}u_{1,xx}+\frac{\alpha}{8}(qru_1)_x-\frac{\alpha}{8}(u_1r^2)_x-\frac{\alpha}{4}(ru_1u_2)_x+\\
\qquad\frac{5\alpha}{8}(qr^2)_x+\frac{\alpha}{8}(u_1^2r)_x-\frac{5\alpha}{8}(rq^2)_x-\frac{\alpha}{8}(qu_2^2)_x+\frac{\alpha}{8}(qu_1u_2)_x-\frac{\alpha}{8}(u_1u_2^2)_x+\\
\qquad\frac{\alpha}{8}(u_1^2u_2)_x+\frac{3\alpha}{8}(q^2u_2)_x-\frac{\alpha}{2}(qru_2)_x.
\end{cases} \tag{65}$$

Obviously, Eq. (65) is a variable-coefficient nonlinear integrable coupled equation under the constrained equation (64). When $q = r = 0$, Eq. (65) reduces to

$$\begin{cases} (u_1)_{t_2} = \frac{\alpha}{8}u_{1,xx} + \frac{3\alpha}{8}u_1^2 u_{1,x}, \\ (u_2)_{t_2} = -\frac{\alpha}{8}u_{2,xx} + \frac{\alpha}{4}u_{1,xx} - \frac{\alpha}{8}(u_1 u_2^2)_x + \frac{\alpha}{8}(u_1^2 u_2)_x. \end{cases} \tag{66}$$

Remark 4.1 The first equation in Eq. (66) is similar to both the mKdV equation and the Burgers equation, which can be called a deformed Burgers equation. In addition, we see that Eq. (66) is a new integrable coupling of this deformed equation. The second equation in Eq. (66) can be regarded as a variable-coefficient equation, whose variable coefficients are controlled by the deformed Burgers equation. When $u_1 = 0$, it is trivial that Eq. (66) is the well-known heat equation:

$$(u_2)_{t_2} = -\frac{\alpha}{8}u_{2,xx}. \tag{67}$$

If $u_1 = 1$, Eq. (66) casts into a generalized Burgers equation:

$$(u_2)_{t_2} = -\frac{\alpha}{8}u_{2,xx} - \frac{\alpha}{4}u_2 u_{2,x} + \frac{\alpha}{8}u_{2,x}. \tag{68}$$

Obviously, if we take $q = r = 1$, then Eq. (64) holds and Eq. (65) reduces to the following coupled nonlinear equation with constant coefficients:

$$\begin{cases} (u_1)_{t_2} = \frac{\alpha}{8}u_{1,xx} - \frac{\alpha}{8}u_{2,x} - \frac{\alpha}{4}(u_1u_2)_x + \frac{\alpha}{4}u_1u_{1,x} + \frac{3\alpha}{8}u_1^2u_{1,x} - \frac{\alpha}{8}(u_1^2u_2)_x, \\ (u_2)_{t_2} = -\frac{\alpha}{8}u_{2,xx} + \frac{\alpha}{4}u_{1,xx} + \frac{\alpha}{4}u_1u_{1,x} - \frac{\alpha}{4}u_2u_{2,x} - \frac{\alpha}{8}(u_1u_2)_x - \frac{\alpha}{8}(u_1u_2^2)_x + \\ \qquad \frac{\alpha}{8}(u_1^2u_2)_x - \frac{\alpha}{8}u_{2,x}. \end{cases} \tag{69}$$

We note here that Eq. (69) is not an integrable coupling system, which is an essential difference from Eq. (66).

We now recall the steps for generating Hamiltonian structures of integrable couplings by the variational identity:

(i) A column-vector Lie algebra V, which has the same dimension with the known Lie algebra T, is constructed. V and T are isomorphic to each other.

(ii) In the Lie algebra V, the commutator is exhibited by the following form:

$$[a, b] = a^T R(b), \tag{70}$$

where $a, b \in V$, $R(b)$ is a square matrix with entries $b_i, i = 1, 2, \ldots, p$. Here, p represents the dimension of the Lie algebra V.

(iii) Solve the matrix equation for F:

$$R(b)F = -F(R(b))^T, F = F^T, \tag{71}$$

where F is a matrix with constant entries.

(iv) Introduce a linear functional

$$\{a, b\} = a^T F b. \tag{72}$$

(v) Deduce from the variational identity

$$\frac{\delta}{\delta u}\int^x \{a, \frac{\partial U}{\partial \lambda}\}\mathrm{d}x = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\{a, \frac{\partial Y}{\partial u_i}\} \tag{73}$$

the Hamiltonian structures of the obtained integrable hierarchy, where U is a vector in the loop algebra $\tilde{V}$ of the Lie algebra V. It is remarked here that the loop algebra $\tilde{V}$ is not unique. Based on the above steps, we consider a linear map:

$$\delta : T \to V,\ a = \sum_{i=1}^{6} a_i t_i \in T \to \delta(a) = (a_1, a_2, \ldots, a_6)^T \in V, \tag{74}$$

from which we can prove that δ is an isomorphism. Define

$$\begin{aligned}[a,b] = (&2a_3b_1 - 2a_1b_3, 2a_2b_3 - 2a_3b_2 + 4a_3b_1 - 4a_1b_3, 2a_1b_2 - 2a_2b_1, 2a_6b_1 - 2a_1b_6 + \\ &2a_3b_4 - 2a_4b_3, 4a_6b_1 - 4a_1b_6 + 2a_2b_6 - 2a_6b_2 + 2a_5b_3 - 2a_3b_5 + 4a_3b_4 - \\ &4a_4b_3, 2a_1b_5 - 2a_5b_1 + 2a_4b_2 - 2a_2b_4)^T,\end{aligned} \tag{75}$$

where $a = (a_1, \ldots, a_6)^T, b = (b_1, \ldots, b_6)^T$. It can be verified that V becomes a Lie algebra by combining with Eq. (75) which is rewritten as

$$[a,b] = a^T R(b), \tag{76}$$

where

$$a^T = (a_1, \ldots, a_6),\ R(b) = \begin{pmatrix} R_1 & R_2 \\ 0 & R_3 \end{pmatrix},\ R_1 = \begin{pmatrix} -2b_3 & -4b_3 & 2b_2 \\ 0 & 2b_3 & -2b_1 \\ 2b_1 & 4b_1 - 2b_2 & 0 \end{pmatrix},$$

$$R_2 = \begin{pmatrix} -2b_6 & -4b_6 & 2b_5 \\ 0 & 2b_6 & -2b_4 \\ 2b_4 & 4b_4 - 2b_5 & 0 \end{pmatrix},\ R_3 = \begin{pmatrix} -2b_3 - 2b_6 & -4b_6 - 4b_3 & 2b_2 + 2b_5 \\ 0 & 2b_3 + 2b_6 & -2b_1 - 2b_4 \\ 2b_1 + 2b_4 & -2b_5 + 4b_1 + 4b_4 - 2b_2 & 0 \end{pmatrix}.$$

Using (76), we obtain from Eq. (71):

$$F = \begin{pmatrix} F_1 & F_2 \\ F_2 & F_3 \end{pmatrix}, \tag{77}$$

where

$$F_1 = \begin{pmatrix} -2\eta_1 & \eta_1 & 0 \\ \eta_1 & 0 & 0 \\ 0 & 0 & \eta_1 \end{pmatrix},\ F_2 = \begin{pmatrix} -2\eta_2 & \eta_2 & 0 \\ \eta_2 & 0 & 0 \\ 0 & 0 & \eta_2 \end{pmatrix},\ F_3 = \begin{pmatrix} -2\eta_2 & \eta_2 & 0 \\ \eta_2 & 0 & 0 \\ 0 & 0 & \eta_2 \end{pmatrix}.$$

In terms of Eq. (77), we introduce a linear functional:

$$\begin{aligned}\{a,b\} = \eta_1(&a_2b_1 - 2a_1b_1 + a_1b_2 + a_3b_3 + a_6b_3) + \eta_2(a_5b_1 - 2a_4b_1 + a_4b_2 + \\ &a_6b_3 - 2a_1b_4 + a_2b_4 - 2a_4b_4 + a_5b_4 + a_1b_5 + a_4b_6 + a_3b_6 + a_6b_6).\end{aligned} \tag{78}$$

By making use of the Lie algebra V, we introduce a Lax pair

$$\begin{cases} \psi_x = U\psi, U = (q\lambda, r\lambda, \lambda^2, u_1\lambda, u_2\lambda, \lambda^2)^T, \\ \psi_t = V\psi, V = (V_1\lambda, V_2\lambda, V_3, V_4\lambda, V_5\lambda, V_6)^T. \end{cases} \tag{79}$$

Due to the isomorphic property of δ , Eq. (79) has the same compatibility condition with Eq. (56). Hence, the stationary equation of the compatibility condition Eq. (79) leads to Eqs. (49) and (61).

Denote

$$
\begin{aligned}
V_{+}^{(n)} &= \sum_{m\geq 0}(V_{1,m}\lambda, V_{2,m}\lambda, V_{3,m}, V_{4,m}\lambda, V_{5,m}\lambda, V_{6,m})^T\lambda^{2n-2m} = \lambda^{2n}V - V^{(n)},\\
V^{(n)} &= V^{(n)} + (0,0,-V_{3,n},0,0,-V_{6,n})^T.
\end{aligned}
$$

We can easily obtain

$$
-V_x^{(n)} + [U, V^{(n)}] = (-(V_{1,n})_x,\ -(V_{2,n})_x, 0, -(V_{4,n})_x, -(V_{5,n})_x, 0)^T.
$$

Thus, the zero curvature equation $U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0$ admits the integrable-coupling hierarchy (62). From (79), we have

$$
\begin{aligned}
&\frac{\partial U}{\partial q} = (\lambda,0,0,0,0,0)^T,\ \frac{\partial U}{\partial r} = (0,\lambda,0,0,0,0)^T,\\
&\frac{\partial U}{\partial u_1} = (0,0,0,\lambda,0,0)^T,\ \frac{\partial U}{\partial u_2} = (0,0,0,0,\lambda,0)^T,\\
&\frac{\partial U}{\partial \lambda} = (q, r, 2\lambda, u_1, u_2, 2\lambda)^T.
\end{aligned}
$$

Substituting the above results into Eq. (73) yields that

$$
\begin{aligned}
&\{V, \frac{\partial U}{\partial q}\} = (-2\eta_1V_1 + \eta_1V_2 - 2\eta_2V_4 + \eta_2V_5)\lambda^2,\ \{V, \frac{\partial U}{\partial r}\} = (\eta_1V_1 + \eta_2V_4)\lambda^2,\\
&\{V, \frac{\partial U}{\partial u_1}\} = (-2\eta_2V_1 + \eta_2V_2 - 2\eta_2V_4 + \eta_2V_5)\lambda^2,\ \{V, \frac{\partial U}{\partial u_2}\} = \eta_2(V_1 + V_4)\lambda^2,\\
&\{V, \frac{\partial U}{\partial \lambda}\} = \eta_1[qV_2 + 2V_3 + (r-2q)V_1]\lambda + \eta_2[(u_2 - 2u_1)V_1 + u_1V_2 + 2V_3 +\\
&\qquad (r - 2q - 2u_1 + u_2)V_4 + (q + u_1)V_5]\lambda.
\end{aligned}
$$

Putting the above results into the variational identity gives

$$
\begin{aligned}
&\frac{\delta}{\delta Q}\int^x (\{\eta_1[qV_2 + 2V_3 + (r-2q)V_1]\lambda + \eta_2[(u_2 - 2u_1)V_1 + u_1V_2 + 2V_3 +\\
&\quad (r - 2q + u_2 - 2u_1)V_4 + (q+u_1)V_5]\lambda\})\mathrm{d}x\\
&= \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\begin{pmatrix} (-2\eta_1V_1 + \eta_1V_2 - 2\eta_2V_4 + \eta_2V_5)\lambda^2\\ (\eta_1V_1 + \eta_2V_4)\lambda^2\\ (-2\eta_2V_1 + \eta_2V_2 - 2\eta_2V_4 + \eta_2V_5)\lambda^2\\ \eta_2(V_1+V_4)\lambda^2\end{pmatrix}.
\end{aligned}
\tag{80}
$$

Comparing the coefficients of λ^{-2n+1} in (80), we obtain

$$
\begin{aligned}
&\frac{\delta}{\delta Q}\int^x (\{\eta_1[qV_{2,n} + 2V_{3,n} + (r-2q)V_{1,n}] + \eta_2[(u_2 - 2u_1)V_{1,n} + u_1V_{2,n} + 2V_{3,n} +\\
&(r - 2q + u_2 - 2u_1)V_{4,n} + (q+u_1)V_{5,n}]\})\mathrm{d}x\\
&= (2 - n + \gamma)\begin{pmatrix} -2\eta_1V_{1,n} + \eta_1V_{2,n} - 2\eta_2V_{4,n} + \eta_2V_{5,n}\\ \eta_1V_{2,n} + \eta_2V_{4,n}\\ -2\eta_2V_{1,n} + \eta_2V_{2,n} - 2\eta_2V_{4,n} + \eta_2V_{5,n}\\ \eta_2V_{1,n} + \eta_2V_{4,n}\end{pmatrix}.
\end{aligned}
\tag{81}
$$

It can be seen that $\gamma = 2$. Thus, we get

$$\begin{pmatrix} -2\eta_1 V_{1,n} + \eta_1 V_{2,n} - 2\eta_2 V_{4,n} + \eta_2 V_{5,n} \\ \eta_1 V_{2,n} + \eta_2 V_{4,n} \\ -2\eta_2 V_{1,n} + \eta_2 V_{2,n} - 2\eta_2 V_{4,n} + \eta_2 V_{5,n} \\ \eta_2 V_{1,n} + \eta_2 V_{4,n} \end{pmatrix} \equiv \frac{\delta H_n}{\delta Q}, \tag{82}$$

where

$$H_n = \int^x \Big(\frac{1}{4-n}\{\eta_1[qV_{2,n} + 2V_{3,n} + (r-2q)V_{1,n}] + \eta_2[(u_2 - 2u_1)V_{1,n} + u_1 V_{2,n} + 2V_{3,n} + (r - 2q + u_2 - 2u_1)V_{4,n} + (q + u_1)V_{5,n}]\}\Big)\mathrm{d}x.$$

Hence, the integrable coupling (62) can be written as the Hamiltonian structure form:

$$Q_{t_n} = J \begin{pmatrix} -2\eta_1 V_{1,n} + \eta_1 V_{2,n} - 2\eta_2 V_{4,n} + \eta_2 V_{5,n} \\ \eta_1 V_{2,n} + \eta_2 V_{4,n} \\ -2\eta_2 V_{1,n} + \eta_2 V_{2,n} - 2\eta_2 V_{4,n} + \eta_2 V_{5,n} \\ \eta_2 V_{1,n} + \eta_2 V_{4,n} \end{pmatrix} = J\frac{\delta H_n}{\delta Q}, \tag{83}$$

where

$$J = \begin{pmatrix} 0 & \frac{\partial}{\eta_1 - \eta_2} & 0 & -\frac{\partial}{\eta_1 - \eta_2} \\ \frac{\partial}{\eta_1 - \eta_2} & 0 & -\frac{\partial}{\eta_1 - \eta_2} & 0 \\ 0 & -\frac{\partial}{\eta_1 - \eta_2} & 0 & \frac{\eta_1 \partial}{\eta_2(\eta - \eta_2)} \\ -\frac{\partial}{\eta_1 - \eta_2} & 0 & \frac{\eta_1 \partial}{\eta_2(\eta_1 - \eta_2)} & 0 \end{pmatrix},$$

is a Hamiltonian operator, $\partial \equiv \frac{\partial}{\partial x}$. Obviously, we cannot allow η_2 to vanish, otherwise J has no meaning. If $\eta_1 = 0$, we can get a simpler Hamiltonian structure of Eq. (62):

$$Q_{t_n} = \bar{J}\frac{\delta \bar{H}_n}{\delta Q}, \tag{84}$$

where

$$\bar{J} = \begin{pmatrix} 0 & -\frac{\partial}{\eta_2} & 0 & \frac{\partial}{\eta_2} \\ -\frac{\partial}{\eta_2} & 0 & \frac{\partial}{\eta_2} & 0 \\ 0 & \frac{\partial}{\eta_2} & 0 & 0 \\ \frac{\partial}{\eta_2} & 0 & 0 & 0 \end{pmatrix}$$

is an obvious Hamiltonian operator, and

$$\bar{H}_n \equiv \frac{\eta_2}{4-n}[(u_2 - 2u_1)V_{1,n} + u_1 V_{2,n} + 2V_{3,n} + (r - 2q + u_2 - 2u_1)V_{4,n} + (q + u_1)V_{5,n}].$$

Therefore, we obtained two different Hamiltonian structure of Eq. (62) by choosing various parameters η_1 and η_2.

5. A new loop algebra $\bar{H}$ of the Lie algebra H and some applications

In the section, we introduce a higher-degree loop algebra $\bar{H}$ to generate multi-Hamiltonian structures as an application. This result can be further extended to obtain an mKdV equation with complex coefficients. Set

$$\bar{H} = \mathrm{span}\{t_1(i,m), t_2(i,m), t_3(i,m)\}, \tag{85}$$

where $t_j(i,m) = t_j\lambda^{3m+i}, j = 1,2,3; i = 0,1,2; m = 0, \pm 1, \pm 2, \ldots$. In terms of the commutative relations of the Lie algebra H, we get the corresponding operative relations of $\bar{H}$:

$$[t_1(i,m), t_2(j,n)] = \begin{cases} 2t_3(i+j, m+n), i+j < 3, \\ 2t_3(i+j-3, m+n+1), i+j \geq 3, \end{cases}$$

$$[t_1(i,m), t_3(j,n)] = \begin{cases} -2t_1(i+j, m+n) - 4t_2(i+j, m+n), i+j < 3, \\ -2t_1(i+j-3, m+n+1) - 4t_2(i+j-3, m+n+1), i+j \geq 3, \end{cases}$$

$$[t_2(i,m), t_3(j,n)] = \begin{cases} 2t_2(i+j, m+n), i+j < 3, \\ 2t_3(i+j-3, m+n+1), i+j \geq 3, \end{cases}$$

where $i,j = 0,1,2; m,n \in \mathbf{Z}$. Define the degree of each $t_j(i,m)$ to be $\deg(t_j(i,m)) = 3m + i, i = 0,1,2$. We find that the linear space $\bar{H}$ becomes a loop algebra, which can be used to derive a 3-Hamiltonian integrable hierarchy. Set

$$\begin{cases} \varphi_x = U\varphi, U = t_3(1,0) + u_1 t_1(0,0) + u_2 t_2(0,0) + u_3 t_1(2,-1) + u_4 t_2(2,-1) + u_5 t_3(2,-1), \\ \varphi_t = V\varphi, V = \sum\limits_{m\geq 0}\sum\limits_{i=0}^{2} \left[a(i,m)\, t_1(i,-m) + b(i,m)\, t_2(i,-m) + c(i,m)\, t_3(i,-m)\right]. \end{cases} \tag{86}$$

According to the Tu scheme, the stationary zero curvature equation

$$V_x = [U, V] \tag{87}$$

is equivalent to the following equations

$$\begin{cases} a(0,m) = \frac{1}{2}a_x(1,n) + u_1 c(1,m) + u_3 c(2,m) - u_5 a(2,m), \\ b(0,m) = -\frac{1}{2}b_x(1,m) + 2a(0,m) + (u_2 - 2u_1)c(1,m) + (u_4 - 2u_3)c(2,m) + 2u_5 a(2,m) - \\ \qquad u_5 b(2,m), \\ a(2,m+1) = \frac{1}{2}a_x(0,m) + u_1 c(0,m) + u_3 c(1,m) - u_5 a(1,m), \\ b(2,m+1) = -\frac{1}{2}b_x(0,m) + 2a(2,m+1) + (u_2 - 2u_1)c(0,m) + (u_4 - 2u_3)c(1,m) + \\ \qquad 2u_5 a(1,m) - u_5 b(1,m), \\ a(1,m+1) = \frac{1}{2}a_x(2,m+1) + u_3 c(0,m) - u_5 a(0,m) + u_1 c(2,m+1), \\ b(1,m+1) = -\frac{1}{2}b_x(2,m+1) + 2a(1,m+1) + u_2 c(2,m+1) + (u_4 - 2u_3)c(0,m) + \\ \qquad 2u_5 a(0,m) - u_5 b(0,m) - 2u_1 c(2,m+1), \\ c_x(2,m+1) = -u_1 b_x(0,m) + (4u_1 - 2u_2)a(2,m+1) - 4u_1^2 c(0,m) + \\ \qquad (2u_1 u_4 - 4u_1 u_3)c(1,m) + 4u_1 u_5 a(1,m) - 2u_1 u_5 b(1,m) + 2u_1 u_2 c(0,m) + \\ \qquad 2u_3 b(0,m) - 2u_4 a(0,m). \end{cases} \tag{88}$$

Assume the initial values of Eq. (88) as follows:

$$a(0,0) = b(0,0) = c(1,0) = a(1,0) = b(1,0) = a(2,0) = b(2,0) = c(2,0) = 0, c(0,0) = \beta.$$

From Eq. (88) we can get a series of explicit solutions:

$$\begin{aligned}
&a(2,1)=\beta u_1, b(2,1)=\beta u_2, c(2,1)=0, a(1,1)=\frac{\beta}{2}u_{1,x}+\beta u_3,\\
&b(1,1)=\beta u_{1,x}-\frac{\beta}{2}u_{2,x}+\beta u_4, c(1,1)=\beta u_1^2-\beta u_1 u_2,\\
&a(0,1)=\frac{\beta}{4}u_{1,xx}+\frac{\beta}{2}u_{3,x}+\beta u_1^3-\beta u_1^2 u_2-\beta u_1 u_5,\\
&b(0,1)=\frac{\beta}{4}u_{2,xx}-\frac{\beta}{2}u_{4,x}+\beta u_{3,x}+\beta u_1^2 u_2-\beta u_1 u_2^2-\beta u_2 u_5,\\
&c(0,1)=\frac{\beta}{2}u_1 u_{2,x}-\frac{\beta}{2}u_2 u_{1,x}-\beta u_1 u_4+2\beta u_1 u_3-\beta u_2 u_3,\ldots.
\end{aligned}$$

Denote

$$V_+^{(n)}=\sum_{m=0}^{n}\sum_{i=0}^{2}[a(i,m)t_1(i,n-m)+b(i,m)t_2(i,n-m)+c(i,m)t_3(i,n-m)]=\lambda^{3n}V-V_-^{(n)}.$$

Eq. (87) can be decomposed into: $-(V_+^{(n)})_x+[U,V_+^{(n)}]=(V_-^{(n)})_x-[U,V_-^{(n)}]$ whose left-hand side gives

$$\begin{aligned}
&-(V_+^{(n)})_x+[U,V_+^{(n)}]\\
&\quad=-2a(2,n+1)t_1(0,0)+[2b(2,n+1)-4a(2,n+1)]t_2(0,0)+\\
&\quad[a_x(2,n+1)-2a(1,n+1)+2u_1c(2,n+1)]t_1(2,-1)+\\
&\quad[b_x(2,n+1)+4u_1c(2,n+1)-4a(1,n+1)+2b(1,n+1)-2u_2c(2,n+1)]t_2(2,-1)+\\
&\quad[c_x(2,n+1)-2u_1b(2,n+1)+2u_2a(2,n+1)]t_3(2,-1).
\end{aligned}$$

Hence, the compatibility condition of the Lax pair

$$\psi_x=U\psi, \pi_{t_n}=V_+^{(n)}\psi, \tag{89}$$

where U is the same as the one given in Eq. (86), gives directly the following integrable hierarchy with 5-potential functions:

$$\begin{aligned}
u_{t_n}=\begin{pmatrix}u_1\\u_2\\u_3\\u_4\\u_5\end{pmatrix}_{t_n}=&\begin{pmatrix}2a(2,n+1)\\4a(2,n+1)-2b(2,n+1)\\2a(1,n+1)-a_x(2,n+1)-2u_1c(2,n+1)\\-4u_1c(2,n+1)-b_x(2,n+1)+4a(1,n+1)-2b(1,n+1)+2u_2c(2,n+1)\\-c_x(2,n+1)+2u_1b(2,n+1)-2u_2a(2,n+1)\end{pmatrix}\\
\equiv J_1P_1,&
\end{aligned}\tag{90}$$

where

$$J_1=\begin{pmatrix}0&0&0&1&0\\0&0&-1&0&0\\0&1&0&-\frac{\partial}{2}&-u_1\\-1&0&-\frac{\partial}{2}&-\partial&u_2-2u_1\\0&0&u_1&2u_1-u_2&-\frac{\partial}{2}\end{pmatrix}$$

is a Hamiltonian operator,

$P_1 = (-4a(1,n+1)+2b(1,n+1), 2a(1,n+1), -4a(2,n+1)+2b(2,n+1), 2a(2,n+1), 2c(2,n+1))^T.$

In terms of Eq. (88), the integrable hierarchy (90) can be written as

$$u_{t_n} = \begin{pmatrix} 2a(2,n+1) \\ 4a(2,n+1) - 2b(2,n+1) \\ 2u_3c(0,n) - 2u_5a(0,n) \\ 4u_3c(0,n) - 2u_4c(0,n) - 4u_5a(0,n) + 2u_5b(0,n) \\ 2u_4a(0,n) - 2u_3b(0,n) \end{pmatrix} \equiv J_2P_2, \tag{91}$$

where

$$J_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -u_5 & u_3 \\ 0 & 0 & u_5 & 0 & 2u_3 - u_4 \\ 0 & 0 & -u_3 & u_4 - 2u_3 & 0 \end{pmatrix}$$

is a Hamiltonian operator:

$$P_2 = (2b(2,n+1) - 4a(2,n+1), 2a(2,n+1), 2b(0,n) - 4a(0,n), 2a(0,n), 2c(0,n))^T.$$

Similary, using Eq.(88), we can rewrite Eq.(90) as

$$u_{t_n} = \begin{pmatrix} a_x(0,n) + 2u_1c(0,n) + 2u_3c(1,n) - 2u_5a(1,n) \\ M \\ 2u_3c(0,n) - 2u_5a(0,n) \\ 4u_3c(0,n) - 2u_4c(0,n) - 4u_5a(0,n) + 2u_5b(0,n) \\ 2u_4a(0,n) - 2u_3b(0,n) \end{pmatrix}$$
$$\equiv (A, B, C, D, 2u_4a(0,n) - 2u_3b(0,n))^T \equiv J_3P_3, \tag{92}$$

where

$$M = b_x(0,n) + (4u_1 - 2u_2)c(0,n) + (4u_3 - 2u_4)c(1,n) - 4u_5a(1,n) + 2u_5b(1,n),$$

$$J_3 = (J_{31}, J_{32}, J_{33}, J_{34}, J_{35}),$$

$$J_{31} = \begin{pmatrix} 2u_1\partial^{-1}u_1 \\ \frac{\partial}{2} + 4u_1\partial^{-1}u_1 - 2u_2\partial^{-1}u_1 \\ 2u_3\partial^{-1}u_1 \\ u_5 + (4u_3 - 2u_4)\partial^{-1}u_1 \\ -u_3 \end{pmatrix},$$

$$J_{32} = \begin{pmatrix} \frac{\partial}{2} - 2u_1\partial^{-1}u_2 + 4u_1\partial^{-1}u_1 \\ \partial - 4u_1\partial^{-1}u_2 + 2u_2\partial^{-1}u_2 + 8u_1\partial^{-1}u_1 - 4u_2\partial^{-1}u_1 \\ -u_5 - 2u_3\partial^{-1}u_2 + 4u_3\partial^{-1}u_1 \\ (-4u_3 + 2u_4)\partial^{-1}u_2 + (8u_3 - 4u_4)\partial^{-1}u_1 \\ u_4 - 2u_3 \end{pmatrix},$$

$$J_{33}=\begin{pmatrix} 2u_1\partial^{-1}u_3 \\ 4u_1\partial^{-1}u_3-2u_2\partial^{-1}u_3+u_5 \\ 2u_3\partial^{-1}u_3 \\ (4u_3-2u_4)\partial^{-1}u_3 \\ 0 \end{pmatrix},$$

$$J_{34}=\begin{pmatrix} 4u_1\partial^{-1}u_3-2u_1\partial^{-1}u_4-u_5 \\ -4u_1\partial^{-1}u_4+2u_2\partial^{-1}u_4+8u_1\partial^{-1}u_3-4u_2\partial^{-1}u_3 \\ -2u_3\partial^{-1}u_4+4u_3\partial^{-1}u_3 \\ (-4u_3+2u_4)\partial^{-1}u_4+(8u_3-4u_4)\partial^{-1}u_3 \\ 0 \end{pmatrix},$$

$$J_{35}=\begin{pmatrix} u_3 \\ 2u_3-u_4 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{aligned} P_3 =&(2b(0,n)-4a(0,n),2a(0,n),2b(1,n)-4a(1,n),2a(1,n),2c(1,n))^T,\\ A =&(\partial-4u_1\partial^{-1}u_2)a(0,n)+4u_1\partial^{-1}u_1b(0,n)+4u_1\partial^{-1}u_3b(1,n)-\\ &(4u_1\partial^{-1}u_4+2u_5)a(1,n)+2u_3c(1,n),\\ B =&(\partial+8u_1\partial^{-1}u_1-4u_2\partial^{-1}u_1)b(0,n)-(8u_1\partial^{-1}u_2-4u_2\partial^{-1}u_2)a(0,n)+\\ &(8u_1\partial^{-1}u_3-4u_2\partial^{-1}u_3+2u_5)b(1,n)-(8u_1\partial^{-1}u_4-4u_2\partial^{-1}u_4+4u_5)a(1,n)+\\ &(4u_3-2u_4)c(1,n),\\ C =&4u_3\partial^{-1}u_1b(0,n)-(4u_3\partial^{-1}u_2+2u_5)a(0,n)+4u_3\partial^{-1}u_3b(1,n)-4u_3\partial^{-1}u_4a(1,n),\\ D =&[(8u_3-4u_4)\partial^{-1}u_1+2u_5]b(0,n)-[(8u_3-4u_4)\partial^{-1}u_2+4u_5]a(0,n)+\\ &(8u_3-4u_4)\partial^{-1}u_3b(1,n)-(8u_3-4u_4)\partial^{-1}u_4a(1,n). \end{aligned}$$

Due to the fact that J_3 contains the inverse operator ∂^{-1}, it is very tedious for the verification of J_3 to be a Hamiltonian operator. Instead we will discuss the Hamiltonian structures of Eqs. (90)–(92). It is easy to find that the U and V in Eq.(56) can be written as

$$U=\begin{pmatrix} \lambda+\frac{u_5}{\lambda} & u_1+\frac{u_3}{\lambda} \\ 2u_2-2u_1+\frac{2u_4-2u_3}{\lambda} & -\lambda-\frac{u_5}{\lambda} \end{pmatrix},$$

$$V=\begin{pmatrix} c(0)+C(1)\lambda+c(2)\lambda^2 & a(0)+a(1)\lambda+a(2)\lambda^2 \\ 2b(0)-2a(0)+(2b(1)-2a(1))\lambda+(2b(2)-2a(2))\lambda^2 & -c(0)-c(1)\lambda-c(2)\lambda^2 \end{pmatrix},$$

where $a(0)=\sum_{m\geq 0}a(0,m)\lambda^{-3m}$, $a(1)=\sum_{m\geq 0}a(1,m)\lambda^{-3m},\ldots$. Since

$$\frac{\partial U}{\partial u_1}=\begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix},\frac{\partial U}{\partial u_2}=\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix},\frac{\partial U}{\partial u_3}=\begin{pmatrix} 0 & \frac{1}{\lambda} \\ -\frac{2}{\lambda} & 0 \end{pmatrix},\frac{\partial U}{\partial u_4}=\begin{pmatrix} 0 & 0 \\ \frac{2}{\lambda} & 0 \end{pmatrix},$$

$$\frac{\partial U}{\partial u_5}=\begin{pmatrix}\frac{1}{\lambda} & 0\\ 0 & -\frac{1}{\lambda}\end{pmatrix},\frac{\partial U}{\partial \lambda}=\begin{pmatrix}1-\frac{u_5}{\lambda^2} & -\frac{u_3}{\lambda^2}\\ \frac{2u_3-2u_4}{\lambda^2} & -1+\frac{u_5}{\lambda^2}\end{pmatrix},$$

we have

$$\begin{aligned}
\langle V,\frac{\partial U}{\partial u_1}\rangle =& -4a(0)+2b(0)+(2b(1)-4a(1))\lambda+(2b(2)-4a(2))\lambda^2,\\
\langle V,\frac{\partial U}{\partial u_2}\rangle =&2a(0)+2a(1)\lambda+2a(2)\lambda^2,\\
\langle V,\frac{\partial U}{\partial u_3}\rangle =& -4a(1)+2b(1)+(2b(0)-4a(0))\frac{1}{\lambda}+(2b(2)-4a(0))\lambda,\\
\langle V,\frac{\partial U}{\partial u_4}\rangle =&2a(1)+\frac{2}{\lambda}a(0)+2a(2)\lambda,\\
\langle V,\frac{\partial U}{\partial u_5}\rangle =&\frac{2}{\lambda}c(0)+2c(1)+2c(2)\lambda,\\
\langle V,\frac{\partial U}{\partial \lambda}\rangle =&2c(0)-2u_5c(1)+(4u_3-2u_4)a(2)-2u_3b(2)+\\
&\frac{1}{\lambda}[-u_5c(0)+(4u_3-2u_4)a(1)-2u_3b(1)-u_5c(1)]+\\
&[(2c(1)-u_5c(2)]\lambda+2c(2)\lambda^2+\frac{1}{\lambda^2}[(4u_3-2u_4)a(0)-2u_3b(0)-u_5c(0)].
\end{aligned}$$

Substituting the above results into the trace identity yields that

$$\frac{\delta}{\delta u}\langle V,\frac{\partial U}{\partial \lambda}\rangle=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\begin{pmatrix}\langle V,\frac{\partial U}{\partial u_1}\rangle\\ \langle V,\frac{\partial U}{\partial u_2}\rangle\\ \langle V,\frac{\partial U}{\partial u_3}\rangle\\ \langle V,\frac{\partial U}{\partial u_4}\rangle\\ \langle V,\frac{\partial U}{\partial u_5}\rangle\end{pmatrix}. \tag{93}$$

Comparing the coefficients of λ^{-3n-3} in Eq. (93) gives

$$\begin{aligned}
&\frac{\delta}{\delta u}[2c(0,n+1)-2u_5c(1,n+1)+(4u_3-2u_4)a(2,n+1)-2u_3b(2,n+1)]\\
&=(-3n-2+\gamma)\begin{pmatrix}2b(1,n+1)-4a(1,n+1)\\ 2a(1,n+1)\\ 2b(2,n+1)-4a(0,n+1)\\ 2a(2,n+1)\\ 2c(2,n+1)\end{pmatrix}.
\end{aligned} \tag{94}$$

Comparing the coefficients of λ^{-3n-2} in Eq. (93) yields

$$\begin{aligned}
&\frac{\delta}{\delta u}[2c(1,n+1)-u_5c(2,n+1)+(4u_3-2u_4)a(0,n)-2u_3b(0,n)-u_5c(0,n)]\\
&=(-3n-1+\gamma)\begin{pmatrix}2b(2,n+1)-4a(2,n+1)\\ 2a(2,n+1)\\ 2b(0,n)-4a(0,n)\\ 2a(0,n)\\ 2c(0,n)\end{pmatrix}.
\end{aligned} \tag{95}$$

Comparing the coefficients of λ^{-3n-1} leads to

$$\frac{\delta}{\delta u}[2c(2,n+1)+(4u_3-2u_4)a(0,n)-2u_3b(0,n)-u_5c(0,n)]$$
$$=(-3n+\gamma)\begin{pmatrix}2b(0,n)-4a(0,n)\\2a(0,n)\\2b(1,n)-4a(1,n)\\2a(1,n)\\2c(1,n)\end{pmatrix}. \tag{96}$$

In terms of the initial values of Eq. (88), we have $\gamma=\frac{1}{2}$. Thus, Eqs. (94)–(96) can be written as

$$\begin{cases}P_1=\frac{\delta H(1,3n+3)}{\delta u},\\ H(1,3n+3)=-\frac{1}{3n+\frac{3}{2}}[2c(0,n+1)-2u_5c(1,n+1)+\\ \qquad (4u_3-2u_4)a(2,n+1)-2u_3b(2,n+1)],\end{cases} \tag{97}$$

$$\begin{cases}P_2=\frac{\delta H(2,3n+2)}{\delta u},\\ H(2,3n+2)=-\frac{1}{3n+\frac{1}{2}}[2c(1,n+1)-u_5c(2,n+1)+\\ \qquad (4u_3-2u_4)a(0,n)-2u_3b(0,n)-u_5c(0,n)],\end{cases} \tag{98}$$

$$\begin{cases}P_3=\frac{\delta H(3,3n+1)}{\delta u},\\ H(3,3n+1)=-\frac{1}{3n+\frac{3}{2}}[2c(2,n+1)+(4u_3-2u_4)a(0,n)-2u_3b(0,n)-u_5c(0,n)].\end{cases} \tag{99}$$

Therefore, the integrable hierarchy (90) can be written as

$$u_{t_n}=J_1\frac{\delta H(1,3n+3)}{\delta u}=J_2\frac{\delta H(2,3n+2)}{\delta u}=J_3\frac{\delta H(3,3n+1)}{\delta u},\quad n\geq 1. \tag{100}$$

From Eq. (88), we can obtain a recurrence operator $L=(L_1,L_2,\ldots,L_5)$, where

$$L_1=\begin{pmatrix}-\frac{\partial}{2}+(2u_2-4u_1)\partial^{-1}u_1\\2u_1\partial^{-1}u_1\\1\\0\\2\partial^{-1}u_1\end{pmatrix},\ L_2=\begin{pmatrix}-\partial+4u_1-2u_2+(4u_2-8u_1)\partial^{-1}u_1\\\frac{\partial}{2}+4u_1\partial^{-1}u_1-2u_1\partial^{-1}u_2\\0\\1\\4\partial^{-1}u_1-2u_2\end{pmatrix},$$

$$L_3=\begin{pmatrix}(2u_2-4u_1)\partial^{-1}u_3-u_5\\2u_1\partial^{-1}u_3\\0\\0\\2\partial^{-1}u_3\end{pmatrix},\ L_4=\begin{pmatrix}(4u_2-8u_1)\partial^{-1}u_3+(4u_1-2u_2)\partial^{-1}u_4\\4u_1\partial^{-1}u_3+u_3\\0\\0\\4\partial^{-1}u_3-2\partial^{-1}u_4\end{pmatrix},$$

$$L_5=\begin{pmatrix}-u_4+2u_3\\u_5-2u_1\partial^{-1}u_4\\0\\0\\0\end{pmatrix},$$

satisfies the following relations by using the symobolic software Maple:

$$\begin{cases} J_1L = L^*J_1 = J_2, J_2L = L^*J_2 = J_3, \\ \frac{\delta H(1,3n+3)}{\delta u} = L\frac{\delta H(2,3n+2)}{\delta u}, \\ \frac{\delta H(2,3n+2)}{\delta u} = L\frac{\delta H(3,3n+1)}{\delta u}, \\ \frac{\delta H(3,3n+1)}{\delta u} = L\frac{\delta H(1,3n)}{\delta u}. \end{cases} \tag{101}$$

This implies that L is a cyclic operator of Eq. (90). Since $J_1L = L^*J_1 = J_2$, Eq. (90) is Liouville integrable. By the use of Maple, we can verify that the linear combination of J_1, J_2 and J_3 is an identical Hamiltonian operator. Hence, the integrable hierarchy (90) possesses a 3-Hamilotnian styructure. We consider some reductions of Eq. (90) in the following:

Case 1 If $u_3 = u_4 = u_5 = 0$, Eq. (90) reduces to

$$\begin{aligned} v_{t_n} &= \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_{t_n} = \begin{pmatrix} 2a(2,n+1) \\ 4a(2,n+1) - 2b(2,n+1) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 2b(2,n+1) - 4a(2,n+1) \\ 2a(2,n+1) \end{pmatrix} \\ &\equiv \tilde{J} \begin{pmatrix} 2b(2,n+1) - 4a(2,n+1) \\ 2a(2,n+1) \end{pmatrix}. \end{aligned} \tag{102}$$

From the reduced recurrence relations of Eq. (8):

$$\begin{cases} a_x(0,n) = 2a(2,n+1) - 2u_1c(0,n), \\ b_x(0,n) = 4a(2,n+1) - 2b(2,n+1) - 4u_1c(0,n) + 2u_2c(0,n), \\ c_x(0,n) = 2u_1b(0,n) - 2u_2a(0,n), \end{cases} \tag{103}$$

we get

$$\begin{pmatrix} 2b(2,n+1) - 4a(2,n+1) \\ 2a(2,n+1) \end{pmatrix} = \tilde{L} \begin{pmatrix} 2b(2,n) - 4a(2,n) \\ 2a(2,n) \end{pmatrix},$$

where

$$\tilde{L} = \begin{pmatrix} -\frac{\partial}{2} + (2u_2 - 4u_1)\partial^{-1}u_1 & (4u_1 - 2u_2)\partial^{-1}u_2 - \partial + (4u_2 - 8u_1)\partial^{-1}u_1 \\ 2u_1\partial^{-1}u_1 & \frac{\partial}{2} - 2u_1\partial^{-1}u_2 + 2u_1\partial^{-1}u_1 \end{pmatrix}.$$

Thus, Eq. (102) can be written as

$$v_{t_n} = \tilde{J}\tilde{L} \begin{pmatrix} 2b(2,n) - 4a(2,n) \\ 2a(2,n) \end{pmatrix}. \tag{104}$$

It is obvious that Eq. (104) is not an AKNS hierarchy. Set $a(0,0) = b(0,0) = 0, c(0,0) = \beta$, we can obtain from Eq. (103) that

$$a(2,1)=\beta u_1, b(2,1)=\beta u_2, a(0,1)=\frac{\beta}{4}u_{1,xx}+\beta u_1^3-\beta u_1^2u_2,$$
$$b(0,1)=\frac{\beta}{4}u_{2,xx}+\beta u_1^2u_2-\beta u_1u_2^2, c(0,1)=\frac{\beta}{2}u_1u_{2,x}-\frac{\beta}{2}u_2u_{1,x},$$
$$a(2,2)=\frac{\beta}{8}u_{1,xxx}+\frac{3\beta}{2}u_1^2u_{1,x}-\frac{3\beta}{2}u_1u_2u_{1,x},$$
$$b(2,2)=-\frac{\beta}{8}u_{2,xxx}-3\beta u_1u_2u_{1,x}-\frac{3\beta}{2}u_1^2u_{2,x}+\frac{\beta}{4}u_{1,xxx}+3\beta u_1^2u_{1,x}+\frac{3\beta}{2}u_1u_2u_{2,x},\ldots.$$

If $n=1$, Eq. (104) can be reduced to a coupled equation

$$\begin{cases} u_{1,t}=-\frac{\beta}{4}u_{2,xxx}-3\beta u_1^2u_{2,x}+3\beta u_1u_2u_{2,x}, \\ u_{2,t}=\frac{\beta}{4}u_{1,xxx}+3\beta u_1^2u_{1,x}-3\beta u_1u_2u_{1,x}. \end{cases} \tag{105}$$

If $u_1=iu_2, i^2=-1, u_2=q$, Eq. (105) becomes

$$q_t=\frac{i\beta}{4}q_{xxx}+(3\beta-3i\beta)q^2q_x, \tag{106}$$

which is a modified KdV equation with complex coefficients. When $n=0$, Eq.(104) reduces to $u_{1,t}=2\beta u_2-4\beta u_1, u_{2,t}=2\beta u_1$, and thus $u_{1,tt}=4\beta^2u_1-4\beta u_{1,t}$, which is a 2-order linear equation with respect to u_1.

Case 2 If $n=1, t_1=t$, Eq.(90) reduces to the following equations:

$$u_{1,t}=2\beta u_1, \tag{107}$$

$$u_{2,t}=2\beta(2u_1-u_2), \tag{108}$$

$$\begin{aligned} u_{3,t}=&\beta(u_1u_3u_{2,x}-u_2u_3u_{1,x}-2u_1u_3u_4+4u_1u_3^2-2u_2u_3^2-\frac{1}{2}u_5u_{1,xx}-u_5u_{3,x}- \\ &2u_1^3u_5+2u_2u_5u_1^2+2u_1u_5^2), \end{aligned} \tag{109}$$

$$\begin{aligned} u_{4,t}=&\beta(2u_3-4u_4)u_1u_{2,x}-\beta(2u_3-u_4)u_2u_{1,x}-\beta u_1u_4(4u_3-2u_4)+2\beta u_1u_3(4u_3-2u_4)- \\ &\beta u_2u_3(4u_3-2u_4)-\beta u_5u_{1,xx}-2\beta u_5u_{3,x}-4\beta u_5u_1^3+4\beta u_2u_5u_1^2+4\beta u_1u_5^2, \end{aligned} \tag{110}$$

$$\begin{aligned} u_{5,t}=&\frac{\beta}{2}u_4u_{1,xx}+\beta u_4u_{3,x}+2\beta u_4u_1^3-2\beta u_1^2u_2u_4-2\beta u_1u_4u_5-\frac{\beta}{2}u_3u_{2,xx}+ \\ &\beta u_3u_{4,x}-2\beta u_3u_{3,x}-2\beta u_1^2u_2u_3+2\beta u_1u_3u_2^2+2\beta u_2u_3u_5. \end{aligned} \tag{111}$$

If $u_1=u_2=0$, Eqs. (107)–(111) reduce to

$$\begin{cases} u_{3,t}=-\beta u_{3,x}u_5, u_{4,t}=-2\beta u_{3,x}u_5, \\ u_{5,t}=\beta u_{3,x}u_4+\beta u_3u_{4,x}-2\beta u_3u_{3,x}. \end{cases} \tag{112}$$

If $u_4=2u_3=w$, Eq. (112) leads to the following nonlinear evolution equation

$$w_tw_{xt}-w_xw_{tt}=\frac{\beta}{2}ww_x^3. \tag{113}$$

Remark 5.1 By using the loop algebra $\tilde{H}$, we can obtain a non-isospectral integrable hierarchy.

For instance, set

$$\begin{cases} \varphi_x = U\varphi, U = t_1(1) - vt_2(0) + ut_3(0), \\ \varphi_t = V\varphi, V = At_1(0) + Bt_2(0) + Ct_3(0), \end{cases} \tag{114}$$

where

$$A = \sum_{j=0}^{n} a_j \lambda^{n-j} + \sum_{j=0}^{m} \bar{a}_j \lambda^{m-j}, B = \sum_{j=1}^{n} b_j \lambda^{n-j} + \sum_{j=1}^{m} \bar{b}_j \lambda^{m-j}, C = \sum_{j=1}^{n} c_j \lambda^{n-j} + \sum_{j=1}^{m} \bar{c}_j \lambda^{m-j}.$$

The compatibility condition of Eq. (114) gives rise to

$$\begin{cases} U_t - C_x + 2\lambda B + 2vA = 0, \\ \lambda_t - A_x - 2\lambda C - 2uA = 0, \\ \lambda_t + v_t - A_x + B_x + 2\lambda C + 2vC + 2u(B - A) = 0, \end{cases} \tag{115}$$

which postulates that

$$v_t + 4\lambda C + 2vC - 4uA + 2uB + B_x = 0,$$

where we assume $\lambda_t = \sum_{j=0}^{m} k_j(t)\lambda^{m-j}$. It is easy to see that Eq. (115) is equivalent to the following:

$$\begin{cases} va_0 + b_1 = 0, -c_{j,x} + 2b_{j+1} + 2va_j = 0, j = 1, 2, \ldots, n-1, \\ v\bar{a}_0 + \bar{b}_1 = 0, -\bar{c}_{j,x} + 2\bar{b}_{j+1} + 2v\bar{a}_j = 0, j = 1, 2, \ldots, m-1, \end{cases} \tag{116}$$

$$\begin{cases} a_{0,x} + 2c_1 - 2ua_0 = 0, a_{j,x} + 2c_{j+1} - 2ua_j = 0, j = 1, 2, \ldots, n-1, \\ k_0(t) - \bar{a}_{0,x} - 2\bar{c}_1 + 2u\bar{a}_0 = 0, \bar{a}_{j,x} + 2\bar{c}_{j+1} - 2u\bar{a}_j - k_j(t) = 0, j = 1, 2, \ldots, m-1, \end{cases} \tag{117}$$

$$\begin{cases} -ua_0 + c_1 = 0, b_{j,x} + 2vc_j + 2ub_j - 4ua_j + 4c_{j+1} = 0, j = 1, 2, \ldots, n-1, \\ -u_0\bar{a}_0 + \bar{c}_1 = 0, \bar{b}_{j,x} + 2v\bar{c}_j + 2u\bar{b}_j - 4\bar{a}_j + 4\bar{c}_{j+1} = 0, j = 1, \ldots, m-1, \end{cases} \tag{118}$$

from which we get

$$\begin{cases} a_{j,x} = \frac{1}{2}b_{j,x} + ub_j + vc_j, \\ \bar{a}_{j,x} = \frac{1}{2}\bar{b}_{j,x} + u\bar{b}_j + v\bar{c}_j + k_j(t). \end{cases} \tag{119}$$

Based on the above results, we obtain a Lax integrable hierarchy

$$\begin{cases} v_{t_{n,m}} = -b_{n,x} - \bar{b}_{m,x} - 2v(c_n + \bar{c}_m) - 2u(b_n + \bar{b}_m) + 4u(a_n + \bar{a}_m), \\ u_{t_{n,m}} = c_{n,x} + \bar{c}_{m,x} - 2v(a_n + \bar{a}_m). \end{cases} \tag{120}$$

Set $a_0 = \alpha(t), \bar{a}_0 = k_0(t)x, b_0 = c_0 = \bar{b}_0 = \bar{c}_0 = 0$, one infers from Eqs. (116)-(119) that

$$\begin{aligned}
&b_1 = -\alpha(t)v, c_1 = \alpha(t)u, a_1 = -\frac{1}{2}\alpha(t)v, \bar{b}_1 = -k_0(t)xv, \bar{c}_1 = k_0(t)xu, \\
&\bar{a}_1 = -\frac{1}{2}k_0(t)xv + k_1(t)x, b_2 = \frac{1}{2}\alpha(t)u_x + \frac{1}{2}\alpha(t)v^2, c_2 = \frac{1}{4}\alpha(t)v_x - \frac{1}{2}\alpha(t)uv, \\
&a_2 = \frac{1}{4}\alpha(t)u_x + \frac{3}{8}\alpha(t)v^2 + \frac{1}{4}\alpha(t)u^2, \bar{b}_2 = \frac{1}{2}\alpha(t)u_x + \frac{1}{2}k_0(t)xv^2 - k_1(t)xv, \\
&\bar{c}_2 = \frac{1}{4}k_0(t)v + \frac{1}{4}k_0(t)xv_x - \frac{1}{2}k_0(t)xuv + k_1(t)xu, \\
&\bar{a}_2 = \frac{1}{4}\alpha(t)u_x - \frac{1}{2}k_1(t)xv + \frac{1}{4}\alpha(t)u^2 + \frac{1}{2}k_0(t)xv^2 + k_2(t)x, \ldots.
\end{aligned}$$

If $n = m = 1$, Eq. (120) reduces to

$$\begin{cases} v_{t_{1,1}} = (\alpha(t) + k_0(t)x)v_x - 2(\alpha(t) + k_0(t)x)uv + k_0(t)v + 4k_1(t)xu, \\ u_{t_{1,1}} = (\alpha(t) + k_0(t)x)(u_x + v^2) + k_0(t)u - 2k_1(t)xv, \end{cases} \tag{121}$$

which is a new variable-coefficient nonlinear integrable system. If $n = 1, m = 2$, Eq.(120) becomes

$$\begin{cases} v_{t_{1,2}} = (\alpha(t) + k_1(t)x)(v_x - 2uv) - \frac{1}{2}\alpha(t)u_{xx} - k_0(t)v^2 - \frac{3}{2}k_0(t)xvv_x + 4k_2(t)xu + k_1(t)v + \\ \qquad \alpha(t)u^3 + 2k_0(t)xuv^2, \\ u_{t_{1,2}} = (\alpha(t) + k_2(t)x)u_x + \frac{1}{2}k_0(t)v_x + \frac{1}{4}k_0(t)xv_{xx} - \frac{1}{2}k_0(t)uv - \frac{1}{2}(k_0(t)x + \alpha(t))u_x vn - \\ \qquad \frac{1}{2}k_0(t)xuv_x + k_1(t)u + \alpha(t)v^2 + k_1(t)xv^2 - \frac{1}{2}\alpha(t)u^2v - k_0(t)xv^3 - 2k_2(t)xv, \end{cases} \tag{122}$$

which is a more complicated variable-coefficient integrable system. Similarly, if $n = 2, m = 1; n = m = 2$, we obtain respectively the following variable-coefficient integrable systems:

$$\begin{cases} v_{t_{2,1}} = -\frac{1}{2}\alpha(t)u_{xx} + k_0(t)v + \frac{1}{2}\alpha(t)vv_x - 2k_0(t)xuv + \frac{3}{2}\alpha(t)uv^2 + \alpha(t)u^2 + 4k_1(t)xu, \\ u_{t_{2,1}} = \frac{1}{4}\alpha(t)v_{xx} - \alpha(t)u_xv - \frac{1}{2}\alpha(t)uv_x + k_0(t)u - \frac{3}{4}\alpha(t)v^3 - \frac{1}{2}\alpha(t)vu^2 + \\ \qquad k_0(t)xv^2 - 2k_1(t)xv, \end{cases}$$

$$\begin{aligned} v_{t_{2,2}} =& -\alpha(t)u_{xx} - \frac{3}{2}\alpha(t)vv_x - k_0(t)v^2 - (\frac{3}{2}k_0(t) + 2k_1(t))xuv + k_0(t)xuv^2 + \frac{3}{2}\alpha(t)uv^2 + \\ & k_1(t)v + k_1(t)xv_x + 2\alpha(t)u^3 + 4k_2(t)xu, \\ u_{t_{2,2}} =& \frac{1}{4}\alpha(t)v_{xx} - \frac{3}{2}\alpha(t)u_xv - \frac{1}{2}\alpha(t)uv_x + \frac{1}{2}k_0(t)v_x + \frac{1}{4}k_0(t)xv_{xx} - \frac{1}{2}k_0(t)uv - \\ & \frac{1}{2}k_0(t)xu_xv - \frac{1}{2}k_0(t)xuv_x + k_1(t)u + k_1(t)xu_x - \frac{3}{4}\alpha(t)v^4 - \frac{1}{2}\alpha(t)u^2v + k_1(t)xv^2 - \\ & \frac{1}{2}\alpha(t)vu^2 - k_0(t)xv^3 - 2k_0(t)xv. \end{aligned}$$

Remark 5.2 We obtain in this paper some integrable hierarchies of evolution type, which can be reduced to some explicit equations. Related works can be found from the papers [24–27] in which the algebro-geometric solutions of some well-known nonlinear evolution equations were obtained by using some kinds of nonlinearity methods.

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