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On the Simultaneous Stabilization of Linear Time-Varying Systems

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Abstract In this paper, we consider the simultaneous stabilization of discrete linear time-varying systems in an operator-theoretic framework. We give a criterion of simultaneous stabilization on account of one kind of strong representation and also give a parametrization for all the simultaneous stabilizing controllers.

Keywords strong representation; simultaneous stabilization; time-varying system; nest algebra

MR(2010) Subject Classification 47L35

1. Introduction

Mathematical control theory is a field of combining engineering and mathematics. In control theory, stabilizability and optimalizability of linear systems are hot topics. While stabilizability is the necessity condition for the study. The stability of system means: if the input signals are bounded, then system has bounded output signals. In order to achieve this property of systems, people always study feedback configuration. Feedback stabilization is a basic concept for the analysis of time-varying linear systems. The problem has been studied within various frameworks and many equivalent conditions have been derived for judging whether a system can be stabilized [1–4]. For finite-dimensional systems and infinite-dimensional discrete linear time-varying systems, the stabilizability is equivalent to the existence of both strong left representation and strong right representation (i.e., the doubly coprime factorization) [2–4]. Based on the assumption that a plant admits a doubly coprime factorization, the parametrization theorem gives the forms of all controllers stabilizing the plant in terms of both strong representations [1–4]. As is well known, Youla parametrization is the important basis of robust stabilization, simultaneous stabilization and optimizing problems (take [5–7] for example).

Attracted by the importance of Youla parametrization, several authors present some generalizations of this theorem [8–10]. In [8], the author gives a parametrization method of stabilizing controllers that requires only a singly coprime factorization of the given plant. The new method is basically to extend the original plant by adding zeros to the transfer function to produce an

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extended plant which owns a doubly coprime factorization. Then the classical Youla parametrization can be used to obtain all the stabilizing controllers. An interesting result in [9] shows that for a linear time-invariant single input single output plant defined in the field of fractions of an integral domain with a unity, the existence of a doubly coprime factorization is not necessary for the internal stability any more. And a parametrization for the stabilizing controllers is exhibited in terms of two free parameters by using the fractional ideal approach. In [10], the parametrization is generalized to multi input multi output plants within the lattice approach.

In the context of operator theory, causal linear time-varying systems are represented by the algebra of lower triangular operators. As shown in [3], under this framework, we look at linear systems not pointwise but in an operator space. The approach is purely operator theoretic and does not use any state space realization. There have been numerous papers on time-varying system in the framework of nest algebra. The common basis of these references is the well-known Youla parametrization in terms of doubly coprime factorizations of given plants. An interesting result in [11] shows that a plant is stabilizable if and only if it has one kind of strong representation and gives a parametrization for the stabilizing controllers in terms of the given single strong representation. Motivated by these works, we want to depict the simultaneous stabilization of two plants on account of one kind of strong representation.

This paper is organized as follows. In Section 2, we recall some basic definitions and auxiliary properties. In Section 3, we introduce the stabilization problem for linear systems and the extension of Youla parametrization theorem. In Section 4, we draw our main results about the simultaneous stabilizability of two plants and we give an example to show how our results can be applied.

2. Preliminaries

In this section we introduce some basic concepts and results needed in the sequel. More details can be found in [3,4,12,13].

Let $\mathbb{C}$ denote the set of complex numbers and $\mathbb{Z}_+$ denote the set of nonnegative integers. Let $\mathbb{C}^d$ be the Cartesian product of d copies of C, here d is a fixed positive integer. Let $\mathcal{H}$ be the complex infinite-dimensional sequence space

$$\ell^2(\mathbb{Z}_+ : \mathbb{C}^d) = \Big\{(x_0, x_1, x_2, \ldots) : x_i \in \mathbb{C}^d, \sum_{i=0}^{\infty} \|x_i\|_d^2 < \infty\Big\},$$

where $\|\cdot\|_d$ denotes the standard Euclidean norm on $\mathbb{C}^d$. Obviously, $\mathcal{H}$ is a separable Hilbert space with the standard inner product $\langle \{x_i\}_{i=0}^{\infty}, \{y_i\}_{i=0}^{\infty}\rangle = \sum_{i=1}^{\infty}\langle x_i, y_i\rangle_{c^d}$.

Let $\mathcal{B}(\mathcal{H})$ denote the space of bounded linear operators on $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, $\operatorname{Ran} T$ denotes the range $\{Tx : x \in \mathcal{H}\}$ of T and $KerT$ denotes the kernel $\{x \in \mathcal{H} : Tx = 0\}$ of T.

Let $\mathcal{H}_e$ be the extended space of $\mathcal{H}$

$$\mathcal{H}_e = \{(x_0, x_1, x_2, \ldots) : x_i \in C^d\}.$$

For each $n \geq 0$, we denote by P_n the standard truncation projection on $\mathcal{H}$ and $\mathcal{H}_e$ as

$$P_n(x_0, x_1, x_2, \ldots, x_n, x_{n+1}, \ldots) = (x_0, x_1, x_2, \ldots, x_n, 0, 0, 0, \ldots)$$

with $P_{-1} = 0$, $P_\infty = I$. P_n sets all outputs after time n to zero, so the projection sequence $\{P_n\}_{n=0}^{\infty}$ is crucial to the physical notion of causality for linear systems.

The following definitions show how discrete linear time-varying systems are defined in the operator-theoretic framework.

Definition 2.1 ([3]) *A linear transformation T on $\mathcal{H}_e$ is causal if $P_nT = P_nTP_n$ for each n. A discrete linear time-varying system on $\mathcal{H}_e$ is a causal linear transformation on $\mathcal{H}_e$, which is continuous with respect to the resolution topology. A discrete linear time-varying system T is stable if its restriction on $\mathcal{H}$ is a bounded operator.*

We denote the set of all discrete linear time-varying systems on $\mathcal{H}_e$ by $\mathcal{L}$. It is clear that $\mathcal{L}$ is an algebra with standard addition and multiplication. And any element of $\mathcal{L}$ is an infinite-dimensional lower triangular matrix (with respect to the standard basis of H, see Chapter 5 of [3]). The set of stable discrete linear time-varying systems, denoted by $\mathcal{S}$, is a weakly closed algebra containing the identity. $\mathcal{M}_2(\mathcal{S})$ denote the 2×2 operator matrices with entries in $\mathcal{S}$. Likewise, $\mathcal{C}_2(\mathcal{S})$ denote the 2×1 operator matrices with entries in $\mathcal{S}$, $\mathcal{R}_2(\mathcal{S})$ denote the 1×2 operator matrices with entries in $\mathcal{S}$.

Remark 2.2 Based on the fact that any element of $\mathcal{L}$ is an infinite-dimensional lower triangular matrix, the invertibility in $\mathcal{L}$ is purely algebraic: $T \in \mathcal{L}$ is invertible in $\mathcal{L}$ if and only if it has no singular elements on its diagonal. The following result can be easily obtained:

Given $T \in \mathcal{S}$. For any $\varepsilon > 0$, there exists an operator $\Delta T \in \mathcal{S}$ with $\|\Delta T\| < \varepsilon$ such that $T + \Delta T$ is invertible in $\mathcal{L}$.

For $T \in \mathcal{L}$, the linear manifold $\mathcal{D}(T) = \{x \in \mathcal{H} : Tx \in \mathcal{H}\}$ denotes the domain of operator T. The graph of T is the set

$$G(T) := \left\{ \begin{bmatrix} x \\ Tx \end{bmatrix} : x \in \mathcal{D}(T) \right\} \subset \mathcal{H} \oplus \mathcal{H}.$$

And the inverse graph of T is

$$G^{-1}(T) := \left\{ \begin{bmatrix} Tx \\ x \end{bmatrix} : x \in \mathcal{D}(T) \right\} \subset \mathcal{H} \oplus \mathcal{H}.$$

Here we use the symbol $\mathcal{H} \oplus \mathcal{H}$ to denote the direct sum of spaces $\mathcal{H}$ and $\mathcal{H}$.

3. Stabilization and strong representation

In this section, we introduce the definitions of stabilization, strong representation and the classical Youla parametrization theorem for discrete linear time-varying systems within the nest algebra framework.

Given $L, C \in \mathcal{L}$, the closed-loop system $\{L, C\}$ is defined by the following system of equation:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} I & C \\ L & I \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix},$$

where u_1, u_2 denote the externally applied inputs; e_1, e_2 denote the inputs to the plant and controller, respectively.

And the closed-loop system $\{L, C\}$ is well-posed if the linear transformation

$$\begin{bmatrix} I & C \\ L & I \end{bmatrix} : \mathcal{D}(L) \oplus \mathcal{D}(C) \to \mathcal{H} \oplus \mathcal{H}$$

is invertible. This inverse is given by the transfer matrix

$$H(L, C) = \begin{bmatrix} (I - CL)^{-1} & -C(I - LC)^{-1} \\ -L(I - CL)^{-1} & (I - LC)^{-1} \end{bmatrix}.$$

Definition 3.1 ([3]) *The closed-loop system $\{L, C\}$ is stable if $\begin{bmatrix} I & C \\ L & I \end{bmatrix}$ has a bounded causal inverse defined on $\mathcal{H} \oplus \mathcal{H}$, i.e., $H(L, C) \in \mathcal{M}_2(\mathcal{S})$. A plant L is stabilizable if there exists a controller $C \in \mathcal{L}$ such that $\{L, C\}$ is stable.*

A geometrical interpretation for the stability of $\{L, C\}$ is given as follows.

Proposition 3.2 ([3]) *$\{L, C\}$ is stable if and only if $\mathcal{H} \oplus \mathcal{H}$ can be decomposed as the algebraic direct sum $G(L) + G^{-1}(C)$.*

For the purpose of characterizing the stabilizable linear systems, we need the following notions of representations for a linear system.

Definition 3.3 ([3]) *A plant L has a right representation $\begin{bmatrix} M \\ N \end{bmatrix}$ if $M, N \in \mathcal{S}$ such that $G(L) = \mathrm{Ran} \begin{bmatrix} M \\ N \end{bmatrix}$. Moreover, a right representation $\begin{bmatrix} M \\ N \end{bmatrix}$ is strong if $\begin{bmatrix} M \\ N \end{bmatrix}$ has a causal bounded left inverse, i.e., there exist $X, Y \in \mathcal{S}$ such that*

$$[\, Y \quad X \,] \begin{bmatrix} M \\ N \end{bmatrix} = I.$$

L has a left representation $[\, -\hat{N} \quad \hat{M} \,]$ if $\hat{N}, \hat{M} \in \mathcal{S}$ such that $G(L) = \mathrm{Ker}\, [\, -\hat{N} \quad \hat{M} \,]$. Moreover, a left representation $[\, -\hat{N} \quad \hat{M} \,]$ is strong if it has a causal bounded right inverse, i.e., there exist $\hat{X}, \hat{Y} \in \mathcal{S}$ such that

$$[\, -\hat{N} \quad \hat{M} \,] \begin{bmatrix} -\hat{X} \\ \hat{Y} \end{bmatrix} = I.$$

Theorem 3.4 ([3]) *Suppose $L \in \mathcal{L}$ has a strong right representation $\begin{bmatrix} M \\ N \end{bmatrix}$. Then any strong right representation $\begin{bmatrix} M_1 \\ N_1 \end{bmatrix}$ of L is of the form $\begin{bmatrix} M \\ N \end{bmatrix} S$ with S invertible in $\mathcal{S}$. Similarly, $[\, -\hat{N} \quad \hat{M} \,]$ is a strong left representation of L, then any strong left representation of L is of the form $T\, [\, -\hat{N} \quad \hat{M} \,]$ with T invertible in $\mathcal{S}$.*

We have the following proposition to characterize strong right representations of some $L \in \mathcal{L}$.

Proposition 3.5 ([3]) *Let $M, N \in \mathcal{S}$. Then $\begin{bmatrix} M \\ N \end{bmatrix}$ is a strong right representation of $L \in \mathcal{L}$ if* *and only if the following statements hold:*

(i) There exist $X, Y \in \mathcal{S}$ *such that* $[Y \quad X]\begin{bmatrix} M \\ N \end{bmatrix} = I$.

(ii) M *is invertible in* $\mathcal{L}$.

The stabilizability of linear systems is closely related to the existence of strong right and strong left representations. The most famous theorem characterizing the relationship is the classical Youla parametrization theorem [3, Theorem 6.4.8]. The following theorem extends Youla parametrization theorem by only requiring one kind of strong representation but not both.

Theorem 3.6 ([11]) *A linear system* $L \in \mathcal{L}$ *is stabilizable if and only if* L *has a strong right representation. Moreover, if* $\begin{bmatrix} M \\ N \end{bmatrix} \in \mathcal{C}_2(\mathcal{S})$ *is a strong right representation for* L *with causal bounded left inverse* $[Y \quad X] \in \mathcal{R}_2(\mathcal{S})$*, then a controller* $C \in \mathcal{L}$ *stabilizes* L *if and only if it has a strong right representation*

$$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} W^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} - \begin{bmatrix} N \\ M \end{bmatrix} Q$$

for some $Q \in \mathcal{S}$*, where* W *is an invertible operator matrix in* $\mathcal{M}_2(\mathcal{S})$ *satisfying*

$$W \begin{bmatrix} M \\ N \end{bmatrix} [Y \quad X] W^{-1} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

The advantage of this result is that the stabilizability of a plant and the parametrization form of all stabilizing controllers only depend on the existence of one kind of strong representation. Hence, it is natural to ask whether or not it is possible to depict simultaneous stabilizability of two plants only depending on the existence of one kind of strong representation. The question will be answered in the next section.

4. Main results

A family of plants $L_0, L_1, \ldots, L_n$ are simultaneously stabilizable if there exists a common linear system $C \in \mathcal{L}$ such that $\{L_i, C\}$ is stable for all $i = 0, 1, \ldots, n$. In this section, we will give a necessary and sufficient condition for simultaneous stabilization in terms of only one kind of strong representation, and give a parameter for all stabilizing controllers.

Theorem 4.1 *If* $L_i \in \mathcal{L}$ *has a strong right representation* $\begin{bmatrix} M_i \\ N_i \end{bmatrix}$ *with causal bounded left inverse* $[Y_i \quad X_i]$, $i = 0, 1$*. Then* L_0 *and* L_1 *are simultaneously stabilizable if and only if there exists* $Q \in \mathcal{S}$ *such that*

$$[0 \quad I] W_1 W_0^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} - [0 \quad I] W_1 \begin{bmatrix} M_0 \\ N_0 \end{bmatrix} Q$$

is invertible in $\mathcal{S}$*, where* W_i *is an invertible operator matrix in* $\mathcal{M}_2(\mathcal{S})$ *such that*

$$W_i \begin{bmatrix} M_i \\ N_i \end{bmatrix} [Y_i \quad X_i] W_i^{-1} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}. \tag{1}$$

Proof Necessity. Suppose that L_0 and L_1 are simultaneously stabilized by a controller $C \in \mathcal{L}$, by Theorem 3.6, C has a strong right representation

$$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} W_0^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} - \begin{bmatrix} N_0 \\ M_0 \end{bmatrix} Q$$

for some $Q \in \mathcal{S}$, where W_0 is an invertible operator matrix in $\mathcal{M}_2(\mathcal{S})$ satisfying

$$W_0 \begin{bmatrix} M_0 \\ N_0 \end{bmatrix} [\, Y_0 \quad X_0 \,] W_0^{-1} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

Again applying Theorem 3.6, C has a strong right representation

$$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} W_1^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} - \begin{bmatrix} N_1 \\ M_1 \end{bmatrix} T$$

for some $T \in \mathcal{S}$, where W_1 is an invertible operator matrix in $\mathcal{M}_2(\mathcal{S})$ satisfying

$$W_1 \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} [\, Y_1 \quad X_1 \,] W_1^{-1} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} W_0^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} - \begin{bmatrix} N_0 \\ M_0 \end{bmatrix} Q$ and $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} W_1^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} - \begin{bmatrix} N_1 \\ M_1 \end{bmatrix} T$ are both strong right representations of C by Theorem 3.4, there exists an invertible element $Z \in \mathcal{S}$ such that

$$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} W_0^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} - \begin{bmatrix} N_0 \\ M_0 \end{bmatrix} Q = \left(\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} W_1^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} - \begin{bmatrix} N_1 \\ M_1 \end{bmatrix} T \right) Z.$$

By multiplying $[\, 0 \quad I \,] W_1 \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ to the preceding equation, we obtain that

$$[\, 0 \quad I \,] W_1 \left(W_0^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} - \begin{bmatrix} M_0 \\ N_0 \end{bmatrix} Q \right) = \left(I - [\, 0 \quad I \,] W_1 \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} T \right) Z.$$

By Eq. (1), we have

$$\begin{aligned} \left(I - [\, 0 \quad I \,] W_1 \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} T \right) Z &= \left(I - [\, 0 \quad I \,] W_1 \begin{bmatrix} M_1 \\ N1 \end{bmatrix} [\, Y_1 \quad X_1 \,] \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} T \right) Z \\ &= \left(I - [\, 0 \quad I \,] \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} W_1 \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} T \right) Z \\ &= Z. \end{aligned}$$

Since Z is invertible in $\mathcal{S}$, $[\, 0 \quad I \,] W_1 W_0^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} - [\, 0 \quad I \,] W_1 \begin{bmatrix} M_0 \\ N_0 \end{bmatrix} Q$ is invertible in $\mathcal{S}$.

Sufficiency. Assume that there exists $\hat{Q} \in \mathcal{S}$ such that $[\, 0 \quad I \,] W_1 W_0^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} - [\, 0 \quad I \,] W_1 \begin{bmatrix} M_0 \\ N_0 \end{bmatrix} \hat{Q}$ is invertible in $\mathcal{S}$. Since the set of invertible elements in $\mathcal{S}$ is an open set in $\mathcal{S}$, and by Remark 2.2, there exists an operator $Q \in \mathcal{S}$ satisfying $[\, 0 \quad I \,] W_1 W_0^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} - [\, 0 \quad I \,] W_1 \begin{bmatrix} M_0 \\ N_0 \end{bmatrix} Q$ invertible in $\mathcal{S}$ and $[\, 0 \quad I \,] W_0^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} - \begin{bmatrix} N_0 \\ M_0 \end{bmatrix} Q$ invertible in $\mathcal{L}$. Define the operator matrix

$$\begin{bmatrix} M \\ N \end{bmatrix} := \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} W_0^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} - \begin{bmatrix} N_0 \\ M_0 \end{bmatrix} Q.$$

Note that

$$\begin{aligned}
&[\,0 \quad I\,]\, W_0 \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} \\
&= I - [\,0 \quad I\,]\, W_0 \begin{bmatrix} M_0 \\ N_0 \end{bmatrix} Q \\
&= I - [\,0 \quad I\,]\, W_0 \begin{bmatrix} M_0 \\ N_0 \end{bmatrix} [\,Y_0 \quad X_0\,] \begin{bmatrix} M_0 \\ N_0 \end{bmatrix} Q \\
&= I - [\,0 \quad I\,] \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} W_0 \begin{bmatrix} M_0 \\ N_0 \end{bmatrix} Q \\
&= I.
\end{aligned}$$

Hence $\begin{bmatrix} M \\ N \end{bmatrix} \in \mathcal{C}_2(\mathcal{S})$ has a causal bounded left inverse. According to Proposition 3.4, $\begin{bmatrix} M \\ N \end{bmatrix}$ is a strong right representation of some linear system $C \in \mathcal{L}$. Next, we will show that C is a common stabilizing controller for L_0 and L_1. By Theorem 3.6, it is obvious that C stabilizes L_0. To prove C stabilizes L_1, it suffices to show

$$G^{-1}(C) + G(L_1) = \mathcal{H} \oplus \mathcal{H}.$$

Since

$$G^{-1}(C) = \mathrm{Ran}\left(\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} \right) = \mathrm{Ran}\left(W_0^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} - \begin{bmatrix} M_0 \\ N_0 \end{bmatrix} Q \right),$$

then for any $z \in G^{-1}(C) \bigcap G(L_1)$, there exist $x, y \in \mathcal{H}$ such that

$$z = \left(W_0^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} - \begin{bmatrix} M_0 \\ N_0 \end{bmatrix} Q \right) x = \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} y.$$

Observe that

$$\begin{aligned}
[\,0 \quad I\,]\, W_1 \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} y &= [\,0 \quad I\,]\, W_1 \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} [\,Y_1 \quad X_1\,] \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} y \\
&= [\,0 \quad I\,] \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} W_1 \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} y \\
&= 0.
\end{aligned}$$

Combining the fact that $[\,0 \quad I\,]\, W_1 (W_0^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} - \begin{bmatrix} M_0 \\ N_0 \end{bmatrix} Q)$ is invertible in $\mathcal{S}$, we have $x = 0$, hence $z = 0$. Therefore, $G^{-1}(C) \bigcap G(L_1) = \{0\}$.

Next, we will prove the fact that $G^{-1}(C) + G(L_1) = \mathcal{H} \oplus \mathcal{H}$. Since $[\,0 \quad I\,]\, W_1 (W_0^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} - \begin{bmatrix} M_0 \\ N_0 \end{bmatrix} Q)$ is invertible in $\mathcal{S}$, there exists $P \in \mathcal{S}$ such that

$$[\,0 \quad I\,]\, W_1 \left(W_0^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} - \begin{bmatrix} M_0 \\ N_0 \end{bmatrix} Q \right) P = I.$$

Then the operator $W_1^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix}$ can be rewritten as

$$\begin{aligned}
W_1^{-1}\begin{bmatrix}0\\I\end{bmatrix} &= W_1^{-1}\begin{bmatrix}0\\I\end{bmatrix}[\,0\quad I\,]\,W_1(W_0^{-1}\begin{bmatrix}0\\I\end{bmatrix}-\begin{bmatrix}M_0\\N_0\end{bmatrix}Q)P\\
&= W_1^{-1}\begin{bmatrix}0&0\\0&I\end{bmatrix}W_1\left(W_0^{-1}\begin{bmatrix}0\\I\end{bmatrix}-\begin{bmatrix}M_0\\N_0\end{bmatrix}Q\right)P\\
&= W_1^{-1}\left(\begin{bmatrix}I&0\\0&I\end{bmatrix}-\begin{bmatrix}I&0\\0&0\end{bmatrix}\right)W_1\left(W_0^{-1}\begin{bmatrix}0\\I\end{bmatrix}-\begin{bmatrix}M_0\\N_0\end{bmatrix}Q\right)P\\
&= \left(W_0^{-1}\begin{bmatrix}0\\I\end{bmatrix}-\begin{bmatrix}M_0\\N_0\end{bmatrix}Q\right)P-W_1^{-1}\begin{bmatrix}I&0\\0&0\end{bmatrix}W_1\left(W_0^{-1}\begin{bmatrix}0\\I\end{bmatrix}-\begin{bmatrix}M_0\\N_0\end{bmatrix}Q\right)P\\
&= \left(W_0^{-1}\begin{bmatrix}0\\I\end{bmatrix}-\begin{bmatrix}M_0\\N_0\end{bmatrix}Q\right)P-\begin{bmatrix}M_1\\N_1\end{bmatrix}[\,Y_1\quad X_1\,]\left(W_0^{-1}\begin{bmatrix}0\\I\end{bmatrix}-\begin{bmatrix}M_0\\N_0\end{bmatrix}Q\right)P.
\end{aligned}$$

For any $\begin{bmatrix}x\\y\end{bmatrix}$ in $\mathcal{H}\oplus\mathcal{H}$,

$$\begin{aligned}
\begin{bmatrix}x\\y\end{bmatrix} &= W_1^{-1}\begin{bmatrix}I&0\\0&0\end{bmatrix}W_1\begin{bmatrix}x\\y\end{bmatrix}+W_1^{-1}\begin{bmatrix}0&0\\0&I\end{bmatrix}W_1\begin{bmatrix}x\\y\end{bmatrix}\\
&= \begin{bmatrix}M_1\\N_1\end{bmatrix}[\,Y_1\quad X_1\,]\begin{bmatrix}x\\y\end{bmatrix}+W_1^{-1}\begin{bmatrix}0\\I\end{bmatrix}[\,0\quad I\,]\,W_1\begin{bmatrix}x\\y\end{bmatrix}\\
&= \begin{bmatrix}M_1\\N_1\end{bmatrix}[\,Y_1\quad X_1\,]\begin{bmatrix}x\\y\end{bmatrix}+\left(W_0^{-1}\begin{bmatrix}0\\I\end{bmatrix}-\begin{bmatrix}M_0\\N_0\end{bmatrix}Q\right)P\,[\,0\quad I\,]\,W_1\begin{bmatrix}x\\y\end{bmatrix}\\
&\quad -\begin{bmatrix}M_1\\N_1\end{bmatrix}[\,Y_1\quad X_1\,]\left(W_0^{-1}\begin{bmatrix}0\\I\end{bmatrix}-\begin{bmatrix}M_0\\N_0\end{bmatrix}Q\right)P\,[\,0\quad I\,]\,W_1\begin{bmatrix}x\\y\end{bmatrix},
\end{aligned}$$

thus, $\begin{bmatrix}x\\y\end{bmatrix}\in G(L_1)+G^{-1}(C)$. This implies $\mathcal{H}\oplus\mathcal{H}\subseteq G(L_1)+G^{-1}(C)$. The opposite inclusion is obvious. Hence $\mathcal{H}\oplus\mathcal{H}=G(L_1)+G^{-1}(C)$.

Similarly to Theorem 4.1, we immediately get the following theorem for the simultaneous stabilization problem of $L_0,L_1,\ldots,L_n\in\mathcal{L}$.

Theorem 4.2 *If $L_i\in\mathcal{L}$ has a strong right representation $\begin{bmatrix}M_i\\N_i\end{bmatrix}$ with a causal bounded left inverse $[\,Y_i\quad X_i\,]$, $i=0,1,2,\ldots,n$. Then $L_0,L_1,\ldots,L_n$ are simultaneously stabilizable if and only if there exists $Q\in\mathcal{S}$ such that*

$$[\,0\quad I\,]\,W_iW_0^{-1}\begin{bmatrix}0\\I\end{bmatrix}-[\,0\quad I\,]\,W_i\begin{bmatrix}M_0\\N_0\end{bmatrix}Q \tag{2}$$

is invertible in $\mathcal{S}$, where W_i is an invertible operator matrix in $\mathcal{M}_2(\mathcal{S})$ such that

$$W_i\begin{bmatrix}M_i\\N_i\end{bmatrix}[\,Y_i\quad X_i\,]\,W_i^{-1}=\begin{bmatrix}I&0\\0&0\end{bmatrix}.$$

Moreover, C stabilizes L if and only if C has a strong right representation $\begin{bmatrix}0&I\\I&0\end{bmatrix}W_0^{-1}\begin{bmatrix}0\\I\end{bmatrix}-\begin{bmatrix}N_0\\M_0\end{bmatrix}Q$ for some Q satisfying (2).

At the end of the paper, we give an example to show the effectiveness and applicability of the proposed method in Theorem 4.1.

Example 4.3 Consider the following two linear time-varying systems:

$$L_0 = \begin{bmatrix} 2 & & & & \\ -4 & 2 & & & \\ 8 & -4 & 2 & & \\ -16 & 8 & -4 & 2 & \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

and

$$L_1 = \begin{bmatrix} 3 & & & & \\ -9 & 3 & & & \\ 27 & -9 & 3 & & \\ -81 & 27 & -9 & 3 & \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

It can be easily checked that strong right representation $\begin{bmatrix} M_i \\ N_i \end{bmatrix}$ $i = 0, 1$ as

$$M_0 = \begin{bmatrix} \frac{1}{2} & & & & & & \\ 1 & \frac{1}{2} & & & & & \\ & 1 & \frac{1}{2} & & & & \\ & & 1 & \frac{1}{2} & & & \\ & & & 1 & \frac{1}{2} & & \\ & & & & \ddots & \ddots & \end{bmatrix}, \ M_1 = \begin{bmatrix} \frac{1}{3} & & & & & & \\ 1 & \frac{1}{3} & & & & & \\ & 1 & \frac{1}{3} & & & & \\ & & 1 & \frac{1}{3} & & & \\ & & & 1 & \frac{1}{3} & & \\ & & & & \ddots & \ddots & \end{bmatrix}, \ N_1 = N_0 = I.$$

Set $W_i = \begin{bmatrix} 0 & I \\ -I & M_i \end{bmatrix}$, clearly, $W_i^{-1} = \begin{bmatrix} M_i & -I \\ I & 0 \end{bmatrix}$ and W_i satisfies

$$W_i \begin{bmatrix} M_i \\ N_i \end{bmatrix} [\, Y_i \quad X_i \,] \, W_i^{-1} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

where $X_i = I, Y_i = 0$. Choose $Q \in \mathcal{S}$ with $\| Q \| < 6$, then

$$\begin{aligned}
&[\, 0 \quad I \,] \, W_1 W_0^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} - [\, 0 \quad I \,] \, W_1 \begin{bmatrix} M_0 \\ N_0 \end{bmatrix} Q \\
&= [\, 0 \quad I \,] \begin{bmatrix} 0 & I \\ -I & \hat{M}_1 \end{bmatrix} \begin{bmatrix} M_0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} 0 \\ I \end{bmatrix} - [\, 0 \quad I \,] \begin{bmatrix} 0 & I \\ -I & \hat{M}_1 \end{bmatrix} \begin{bmatrix} M_0 \\ I \end{bmatrix} Q \\
&= [\, -I \quad \hat{M}_1 \,] \begin{bmatrix} -I \\ 0 \end{bmatrix} - [\, -I \quad \hat{M}_1 \,] \begin{bmatrix} M_0 \\ I \end{bmatrix} Q \\
&= I - (\hat{M}_1 - M_0) Q \\
&= I - \begin{bmatrix} -\frac{1}{6} & & & \\ & -\frac{1}{6} & & \\ & & -\frac{1}{6} & \\ & & & \ddots \end{bmatrix} Q \\
&= I + \frac{1}{6} Q,
\end{aligned}$$

hence,

$$[\, 0 \quad I \,] \, W_1 W_0^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} - [\, 0 \quad I \,] \, W_1 \begin{bmatrix} M_0 \\ N_0 \end{bmatrix} Q$$

is invertible. By Theorem 4.1, L_0 and L_1 can be simultaneously stabilized by some linear system $C \in \mathcal{L}$ with a strong right representation:

$$\begin{aligned}\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} W_0^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} - \begin{bmatrix} N_0 \\ M_0 \end{bmatrix} Q &= \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} M_0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} 0 \\ I \end{bmatrix} - \begin{bmatrix} I \\ M_0 \end{bmatrix} Q \\ &= \begin{bmatrix} I & 0 \\ M_0 & I \end{bmatrix} \begin{bmatrix} 0 \\ I \end{bmatrix} - \begin{bmatrix} I \\ M_0 \end{bmatrix} Q \\ &= \begin{bmatrix} 0 \\ I \end{bmatrix} - \begin{bmatrix} I \\ M_0 \end{bmatrix} Q.\end{aligned}$$

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