Journal of Mathematical Research with Applications
Sept., 2016, Vol. 36, No. 5, pp. 568–574
DOI:10.3770/j.issn:2095-2651.2016.05.008
Http://jmre.dlut.edu.cn

On (α,β)-Metrics with Reversible Geodesics

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Abstract In this paper, we get necessary and sufficient conditions for a Finsler space endowed with an (α,β)-metric where its geodesic coefficients $G^i(x,y)$ and the reverse of geodesic coefficients $G^i(x,-y)$ have the same Douglas curvature. They are the conditions such that (α,β)-metrics have reversible geodesics.

Keywords (α,β)-metric; geodesic coefficient; reversible geodesic; Douglas curvature

MR(2010) Subject Classification 53B40; 53C60

1. Introduction

In Finsler geometry, a Finsler metric is said to be reversible if $F(x,y)=F(x,-y)$ for any $y\in T_xM\setminus\{0\}$. In general, the Finsler metrics might not be reversible, such as: when $\phi(s)\neq\phi(-s)$, an (α,β)-metric $F=\alpha\phi(s)$, $s=\beta/\alpha$ is not reversible, where α is a Riemannian metric and β is a 1-form. In special, when $\phi(s)=1+s$, $F=\alpha+\beta$ is a Randers metric. If $\beta\neq 0$, the Randers metric is not reversible. This leads to the irreversibility of geodesics. Let $G^i(x,y)$ be geodesic coefficient of a Finsler metric F. We call F has reversible geodesics, if for an oriented geodesic curve the same path traversed in the opposite sense is also a geodesic. In other words, its geodesic coefficients $G^i(x,y)$ are projectively related to $G^i(x,-y)$, i.e., $G^i(x,y)=G^i(x,-y)+Py^i$, where $P:=P(x,y)$ is a scalar function on $TM\setminus\{0\}$ with $P(x,\lambda y)=\lambda P(x,y)$, $\lambda>0$. If $P=0$, F is said to have strictly reversible geodesics. Bryant [1] showed that a Finsler metric on S^2 of constant flag curvature $K=1$ with reversible geodesics is Riemannian. Crampin [2] showed that a Randers metric $F=\alpha+\beta$ has reversible geodesics if and only if β is parallel to α. Later, Masca-Sabau-Shimada gave the necessary and sufficient conditions for (α,β)-metrics to have reversible geodesics and strictly reversible geodesics, respectively [3].

Let

$$D_j{}^i{}_{kl}:=\frac{\partial^3}{\partial y^j\partial y^k\partial y^l}\big(G^i-\frac{1}{n+1}\frac{\partial G^m}{\partial y^m}y^i\big). \tag{1.1}$$

The tensor $\mathbf{D}:=D_j{}^i{}_{kl}\frac{\partial}{\partial x^i}\otimes \mathrm{d}x^j\otimes\mathrm{d}x^k\otimes\mathrm{d}x^l$ is called the Douglas tensor of F. A Finsler metric is called Douglas metric if the Douglas tensor vanishes. One can check easily that the Douglas tensor is a projectively invariant. Thus if F has reversible geodesics, its geodesic coefficients

Received October 1, 2015; Accepted January 13, 2016
Supported by the National Natural Science Foundation of China (Grant No. 11471246) and the Jiangxi Provincial Science and Technology Project (Grant No. 20161BAB211021).
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$G^i(x, y)$ and the reverse of geodesic coefficient $G^i(x, -y)$ have the same Douglas curvature. In fact, let G^i_1 and G^i_2 be the geodesic coefficients of Finsler metrics F_1 and F_2, respectively. It is well-known that G^i_1 is projectively related to G^i_2 (i.e., $G^i_3 := G^i_1 - G^i_2$ is projectively flat) if and only if G^i_1 and G^i_2 have the same Douglas curvature and the Weyl curvature (definition see section 2) of G^i_3 vanishes [4]. In this paper, we study (α, β)-metrics whose geodesic coefficients $G^i(x, y)$ and the reverse of geodesic coefficients $G^i(x, -y)$ have the same Douglas curvature. The condition under which the geodesic coefficients $G^i(x, y)$ and the reverse of geodesic coefficients $G^i(x, -y)$ have the same Douglas curvature is weaker than the condition under which F has reversible geodesics. The following theorem can be regarded as a generalization of the main theorem in [3].

Theorem 1.1 *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$ be a irreversible (α, β)-metric on a manifold M of dimension $n \geq 3$. Then its geodesic coefficients $G^i(x, y)$ and the reverse of geodesic coefficients $G^i(x, -y)$ have the same Douglas curvature if and only if the following situations hold*

(1) $\phi(s) = k_1\phi(-s) + k_2 s$, where $k_1 (\neq 0)$, k_2 are constants and β is closed, but β is not parallel to α,

(2) β is not parallel with respect to α, in this case, F is a Berwald metric.

It is a surprise that the two situations in Theorem 1.1 are just the necessary and sufficient conditions for an (α, β)-metric to have reversible geodesics [3]. Further, we have

Corollary 1.2 *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$ be a irreversible (α, β)-metric on a manifold M of dimension $n \geq 3$. Then F has reversible geodesics if and only if its geodesic coefficient $G^i(x, y)$ and the reverse of geodesic coefficient $G^i(x, -y)$ have the same Douglas curvature.*

2. Preliminaries

For a given Finsler metric $F = F(x, y)$, the geodesics of F are characterized locally by a system of 2nd ODEs as follows [5]

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + 2G^i(x, \frac{\mathrm{d}x}{\mathrm{d}t}) = 0,$$

where

$$G^i = \frac{1}{4} g^{il}\{[F^2]_{x^m y^l} y^m - [F^2]_{x^l}\},$$

and $(g^{ij}) := (g_{ij})^{-1}$, $g_{ij} := \frac{1}{2}\frac{\partial^2 [F]^2}{\partial y^i \partial y^j}$. G^i are called geodesic coefficients of F.

The Riemann curvature is introduced via geodesics. Let

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^m \partial y^k} y^m + 2G^m \frac{\partial^2 G^i}{\partial y^m \partial y^k} - \frac{\partial G^i}{\partial y^m}\frac{\partial G^m}{\partial y^k}. \tag{2.1}$$

Define Riemann curvature $\mathbf{R}_y = R^i{}_k \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^k$. $\mathbf{R}_y$ is well-defined satisfying $\mathbf{R}_y(y) = 0$ (see [4]). Ricci curvature $\mathrm{Ric} = (n-1)R(y)$ is the trace of the Riemann curvature expressed by

$$R(y) := \frac{1}{n-1} R^m{}_m. \tag{2.2}$$

For a vector $y \in T_xM \setminus \{0\}$, define $\mathbf{W}_y = W^i{}_k \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^k$ by [4]

$$W^i{}_k := A^i{}_k - \frac{1}{n+1}\frac{\partial A^m{}_k}{\partial y^m} y^i, \tag{2.3}$$

where $A^i{}_k := R^i{}_k - R\delta^i{}_k$. We call $\mathbf{W}_y$ Weyl curvature. It is easy to check the Weyl curvature is projectively invariant. Weyl curvature is a Riemannian quantity.

The Douglas metrics can be also characterized by the following equations [6]

$$G^i y^j - G^j y^i = \frac{1}{2}(\Gamma^i_{kl} y^j - \Gamma^j_{kl} y^i) y^k y^l, \tag{2.4}$$

where $\Gamma^i_{kl} := \Gamma^i_{kl}(x)$ are scalar functons on M.

By definition, an (α, β)-metric is a Finsler metric expressed in the following form

$$F = \alpha\phi(s), \quad s = \frac{\beta}{\alpha},$$

where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form with $\|\beta_x\|_\alpha < b_0,\ x \in M$. It was proved [5] that $F = \alpha\phi(\beta/\alpha)$ is a positive definite Finsler metric if and only if the function $\phi = \phi(s)$ is a C^∞ positive function on an open interval $(-b_0, b_0)$ satisfying

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad |s| \le b < b_0.$$

Let G^i and G^i_α denote the geodesic coefficients of F and α, respectively. Denote

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}),$$
$$b^i := a^{il} b_l, \ s^i{}_j := a^{il} s_{lj}, \quad s_i := b^j s_{ji},$$

where $(a^{ij}) := (a_{ij})^{-1}$ and $b_{i|j}$ denote the covariant derivative of β with respect to α. Then we have

Lemma 2.1 ([5]) *Let $b := \|\beta\|_\alpha$ denote the norm of β with respect to α. The geodesic coefficients of G^i are related to G^i_α by*

$$G^i = G^i_\alpha + \alpha Q s^i{}_0 + \{-2Q\alpha s_0 + r_{00}\}\{\Psi b^i + \Theta\alpha^{-1} y^i\}, \tag{2.5}$$

where

$$Q := \frac{\phi'}{\phi - s\phi'}, \ \Theta := \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi\big[(\phi - s\phi') + (b^2 - s^2)\phi''\big]}, \ \Psi := \frac{\phi''}{2\big[(\phi - s\phi') + (b^2 - s^2)\phi''\big]}$$

and $s^i{}_0 := s^i{}_j y^j$, $s_0 := s_i y^i$, $r_{00} := r_{ij} y^i y^j$, etc.

3. β is closed

In this section, we will show that β is closed for irreversible (α, β)-metrics whose geodesic coefficients $G^i(x, y)$ and the reverse of geodesic coefficients $G^i(x, -y)$ have the same Douglas curvature.

Let $F = \alpha\phi(s)$, $s = \beta/\alpha$ be an irreversible (α, β)-metric on a manifold M of dimension $n \ge 3$ whose geodesic coefficients $G^i(x, y)$ and the reverse of geodesic coefficients $G^i(x, -y)$ have the same Douglas curvature. Denote $\bar{G}^i(x, y) := G^i(x, y) - G^i(x, -y)$. Noting that Douglas curvature is linear with respect to $G^i(x,y)$, then $\bar{G}^i(x,y)$ have vanishing Douglas curvature. By (2.4), we have

$$\bar{G}^i y^j - \bar{G}^j y^i = \frac{1}{2}(\Gamma^i_{kl} y^j - \Gamma^j_{kl} y^i) y^k y^l, \tag{3.1}$$

By Lemma 2.1, we obtain

$$\begin{aligned}\bar{G}^i(x,y) =& G^i(x,y) - G^i(x,-y)\\ =&\alpha[Q(s)+Q(-s)]s^i{}_0 + \{[\Psi(s)-\Psi(-s)]r_{00} - 2\alpha[Q(s)\Psi(s)+\\ &Q(-s)\Psi(-s)]s_0\}b^i + \{-2\alpha s_0[\Theta(s)Q(s) - \Theta(-s)Q(-s)]+\\ &r_{00}[\Theta(s)+\Theta(-s)]\}\alpha^{-1}y^i.\end{aligned} \tag{3.2}$$

Plugging it into (3.1) yields

$$\begin{aligned}&\alpha[Q(s)+Q(-s)](s^i{}_0 y^j - s^j{}_0 y^i) + \{[\Psi(s)-\Psi(-s)]r_{00} -\\ &\quad 2\alpha[Q(s)\Psi(s)+Q(-s)\Psi(-s)]s_0\}(b^i y^j - b^j y^i)\\ =&\frac{1}{2}(\Gamma^i_{kl} y^j - \Gamma^j_{kl} y^i) y^k y^l.\end{aligned} \tag{3.3}$$

To simplify the computations, we take an orthonormal basis at x with respect to α such that

$$\alpha = \sqrt{\sum_{i=1}^{n} (y^i)^2}, \quad \beta = by^1,$$

and take the following coordinate transformation [7] in T_xM, $\psi : (s,u^A) \to (y^i) : y^1 = \frac{s}{\sqrt{b^2-s^2}}\bar{\alpha}$, $y^A = u^A$, where $\bar{\alpha} = \sqrt{\sum_{i=2}^{n}(u^A)^2}$. Here, our index conventions are

$$1 \le i,j,k,\cdots \le n, \quad 2 \le A,B,C,\cdots \le n.$$

We have $\alpha = \frac{b}{\sqrt{b^2-s^2}}\bar{\alpha}$, $\beta = \frac{bs}{\sqrt{b^2-s^2}}\bar{\alpha}$. Further

$$s_1 = bs^1{}_1 = 0,\ s_0 = \bar{s}_0,\ s^1{}_0 = \bar{s}^1{}_0,\ s^A{}_0 = \frac{s\bar{\alpha}}{\sqrt{b^2-s^2}} s^A{}_0 + \bar{s}^A{}_0,$$

$$r_{00} = \frac{s^2\bar{\alpha}^2 r_{11}}{b^2-s^2} + \frac{2s\bar{\alpha}\bar{r}_{10}}{\sqrt{b^2-s^2}} + \bar{r}_{00},$$

where

$$\bar{s}^1{}_0 := \sum_{A=2}^{n} s^1{}_A y^A,\ \bar{s}^A{}_0 := \sum_{A=2}^{n} s^a{}_A y^A,\ \bar{r}_{00} := \sum_{A,B=2}^{n} r_{AB} y^A y^B.$$

Let

$$\bar{\Gamma}^1_{10} := \sum_{A=2}^{n} \Gamma^1_{1A} y^A,\ \bar{\Gamma}^1_{01} := \sum_{A=2}^{n} \Gamma^1_{A1} y^A,\ \bar{\Gamma}^1_{00} := \sum_{A,B=2}^{n} \Gamma^1_{AB} y^A y^B,$$

$$\bar{\Gamma}^B_{10} := \sum_{A=2}^{n} \Gamma^B_{1A} y^A,\ \bar{\Gamma}^B_{01} := \sum_{A=2}^{n} \Gamma^B_{A1} y^A,\ \bar{\Gamma}^C_{00} := \sum_{A,B=2}^{n} \Gamma^C_{AB} y^A y^B.$$

For $i=1$, $j=A$, by (3.3), we get

$$[Q(s)+Q(-s)]\big(\bar{s}^1{}_0 by^A - s^A{}_1\frac{bs^2\bar{\alpha}^2}{b^2-s^2}\big) + \{2[\Psi(s)-\Psi(-s)]\bar{r}_{10}s-$$

$$\begin{aligned}&2b[Q(s)\Psi(s)+Q(-s)\Psi(-s)]\bar{s}_0\}by^A\\&=-\frac{1}{2}\Gamma^A_{11}\frac{s^3\bar{\alpha}^2}{b^2-s^2}+\frac{1}{2}[(\bar{\Gamma}^1_{10}y^A+\bar{\Gamma}^1_{01}y^A)-\bar{\Gamma}^A_{00}]s,\end{aligned}\tag{3.4}$$

$$\begin{aligned}&[Q(s)+Q(-s)]\bar{s}^A{}_0\frac{bs\bar{\alpha}^2}{b^2-s^2}-[\Psi(s)-\Psi(-s)]\big(r_{11}\frac{s^2\bar{\alpha}^2}{b^2-s^2}+\bar{r}_{00}\big)by^A\\&=\frac{1}{2}[(\bar{\Gamma}^A_{10}+\bar{\Gamma}^A_{01})-\Gamma^1_{11}y^A]\frac{s^2\bar{\alpha}^2}{b^2-s^2}-\frac{1}{2}\bar{\Gamma}^1_{00}y^A.\end{aligned}\tag{3.5}$$

For $i=A$, $j=B$, by (3.3), we get

$$[Q(s)+Q(-s)](s^A{}_1y^B-s^B{}_1y^A)\frac{bs\bar{\alpha}^2}{b^2-s^2}=\frac{1}{2}[(\Gamma^A_{11}y^B-\Gamma^B_{11}y^A)\frac{s^2\bar{\alpha}^2}{b^2-s^2}+\frac{1}{2}(\bar{\Gamma}^A_{00}y^B-\bar{\Gamma}^B_{00}y^A),\tag{3.6}$$

$$[Q(s)+Q(-s)](\bar{s}^A{}_0y^B-\bar{s}^B{}_0y^A)b=\frac{1}{2}[(\bar{\Gamma}^A_{10}+\bar{\Gamma}^A_{01})y^B-(\bar{\Gamma}^B_{10}+\bar{\Gamma}^B_{01})y^A]s.\tag{3.7}$$

Taking $s=0$ in (3.6), we have $\bar{\Gamma}^A_{00}y^B-\bar{\Gamma}^B_{00}y^A=0$. Then (3.6) can be reduced to

$$[Q(s)+Q(-s)](s^A{}_1y^B-s^B{}_1y^A)b=\frac{1}{2}(\Gamma^A_{11}y^B-\Gamma^B_{11}y^A)s.\tag{3.8}$$

Replacing s in (3.8) by $-s$ yields

$$[Q(s)+Q(-s)](s^A{}_1y^B-s^B{}_1y^A)b=-\frac{1}{2}(\Gamma^A_{11}y^B-\Gamma^B_{11}y^A)s.\tag{3.9}$$

(3.8)+(3.9) yields

$$[Q(s)+Q(-s)](s^A{}_1y^B-s^B{}_1y^A)b=0.\tag{3.10}$$

Replacing s in (3.7) by $-s$ yields

$$[Q(s)+Q(-s)](\bar{s}^A{}_0y^B-\bar{s}^B{}_0y^A)b=-\frac{1}{2}[(\bar{\Gamma}^A_{10}+\bar{\Gamma}^A_{01})y^B-(\bar{\Gamma}^B_{10}+\bar{\Gamma}^B_{01})y^A]s.\tag{3.11}$$

(3.7)+(3.11) yields

$$[Q(s)+Q(-s)](\bar{s}^A{}_0y^B-\bar{s}^B{}_0y^A)b=0.\tag{3.12}$$

To show that β is closed, we firstly need the following

Lemma 3.1 *For an (α,β)-metric $F=\alpha\phi(s)$, $s=\beta/\alpha$ on a manifold M. If $Q(s)+Q(-s)=0$, then F is reversible.*

Proof By the formulation of Q, we have

$$\frac{\phi'(s)}{\phi(s)-s\phi'(s)}=\frac{-\phi'(-s)}{\phi(-s)+s\phi'(-s)},$$

i.e., $\phi'(s)\phi(-s)+\phi(s)\phi'(-s)=0$, where $\phi'(-s):=\frac{\mathrm{d}\phi(t)}{\mathrm{d}t}|_{t=-s}=-\frac{\mathrm{d}\phi(-s)}{\mathrm{d}s}$. Then above equation can be written as $[\frac{\phi(s)}{\phi(-s)}]'=0$. It implies that $\phi(s)=k\phi(-s)$, where $k=$ constant. Noting that $\phi(s)>0$, taking $s=0$, we get $k=1$. Then F is reversible. $\Box$

Putting all these together, we can prove

Proposition 3.2 *Let $F=\alpha\phi(s)$, $s=\beta/\alpha$ be an irreversible (α,β)-metric on a manifold M of dimension $n\geq 3$. If its geodesic coefficients $G^i(x,y)$ and the reverse of geodesic coefficients $G^i(x,-y)$ have the same Douglas curvature, then β is closed.*

Proof By Lemma 3.1, (3.10) and (3.12), we have

$$s^A{}_1y^B - s^B{}_1y^A = 0,\quad \bar{s}^A{}_0y^B - \bar{s}^B{}_0y^A = 0.$$

Contracting the two equations with $y_B := \delta^B_A y^A$ yields

$$s^A{}_1\bar{\alpha}^2 = \bar{s}_{01}y^A,\quad \bar{s}^A{}_0\bar{\alpha}^2 = 0.$$

Because of the manifold M with dimension $n \geq 3$, then $s_{1A} = s_{AB} = 0$. Thus $s_{ij} = 0$, i.e., β is closed. □

4. Determining $\phi(s)$

Under the assumption in Theorem 1.1, by Proposition 3.2, we know that β is closed. Then (3.4) and (3.5) can be reduced to

$$2[\Psi(s) - \Psi(-s)]\bar{r}_{10}by^A = -\frac{1}{2}\Gamma^A_{11}\frac{s^2\bar{\alpha}^2}{b^2-s^2} + \frac{1}{2}[(\bar{\Gamma}^1_{10}y^A + \bar{\Gamma}^1_{01}y^A) - \bar{\Gamma}^A_{00}]. \tag{4.1}$$

$$-[\Psi(s) - \Psi(-s)]\big(r_{11}\frac{s^2\bar{\alpha}^2}{b^2-s^2} + \bar{r}_{00}\big)by^A = \frac{1}{2}[(\bar{\Gamma}^A_{10} + \bar{\Gamma}^A_{01}) - \Gamma^1_{11}y^A]\frac{s^2\bar{\alpha}^2}{b^2-s^2} - \frac{1}{2}\bar{\Gamma}^1_{00}y^A. \tag{4.2}$$

Replacing s in (4.1) by $-s$ yields

$$2[\Psi(s) - \Psi(-s)]\bar{r}_{10}by^A = \frac{1}{2}\Gamma^A_{11}\frac{s^2\bar{\alpha}^2}{b^2-s^2} - \frac{1}{2}[(\bar{\Gamma}^1_{10}y^A + \bar{\Gamma}^1_{01}y^A) - \bar{\Gamma}^A_{00}]. \tag{4.3}$$

(4.1)+(4.3) yields

$$[\Psi(s) - \Psi(-s)]\bar{r}_{10}b = 0. \tag{4.4}$$

Replacing s in (4.2) by $-s$ yields

$$[\Psi(s) - \Psi(-s)]\big(r_{11}\frac{s^2\bar{\alpha}^2}{b^2-s^2} + \bar{r}_{00}\big)by^A = \frac{1}{2}[(\bar{\Gamma}^A_{10} + \bar{\Gamma}^A_{01}) - \Gamma^1_{11}y^A]\frac{s^2\bar{\alpha}^2}{b^2-s^2} - \frac{1}{2}\bar{\Gamma}^1_{00}y^A. \tag{4.5}$$

(4.5)−(4.2) yields

$$[\Psi(s) - \Psi(-s)]\big(r_{11}\frac{s^2\bar{\alpha}^2}{b^2-s^2} + \bar{r}_{00}\big)b = 0. \tag{4.6}$$

Then we have

Proposition 4.1 *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$ be an irreversible (α,β)-metric on a manifold M of dimension $n \geq 3$ satisfying that β is not parallel with respect to α. If its geodesic coefficients $G^i(x,y)$ and the reverse of geodesic coefficients $G^i(x,-y)$ have the same Douglas curvature, then $\phi(s) = k_1\phi(-s) + k_2s$, where $k_1(\neq 0)$, k_2 are constants.*

Proof If $\bar{r}_{10} \neq 0$, by (4.4), we have

$$\Psi(s) - \Psi(-s) = 0. \tag{4.7}$$

If $\bar{r}_{10} = 0$, by assumption, β is not parallel with respect to α, we have $(\bar{r}_{00}, r_{11}) \neq (0,0)$. Then (4.6) implies that (4.7) still holds. Thus whether $\bar{r}_{10} = 0$ or not, we always have $\Psi(s) - \Psi(-s) = 0$, i.e.,

$$\frac{\phi''(s)}{\phi(s) - s\phi'(s) + (b^2-s^2)\phi''(s)} = \frac{\phi''(-s)}{\phi(-s) + s\phi'(-s) + (b^2-s^2)\phi''(-s)},$$

where $\phi''(-s) := \frac{\mathrm{d}^2\phi(t)}{\mathrm{d}t^2}\mid_{t=-s}= \frac{\mathrm{d}^2\phi(-s)}{\mathrm{d}s^2}$. Then we have

$$\frac{\phi''(s)}{\phi(s)-s\phi'(s)} = \frac{\phi''(-s)}{\phi(-s)+s\phi'(-s)}. \tag{4.8}$$

Denote $P(s) := \phi(s) - s\phi'(s)$. (4.8) can be written as $P'(s)P(-s) + P'(-s)P(s) = 0$. It implies $P(s) = k_1P(-s)$, where $k_1 =$ constant. Then we obtain

$$\phi(s) - k_1\phi(-s) = [\phi'(s) + k_1\phi'(-s)]s = [\phi(s) - k_1\phi(-s)]'s.$$

Thus there is a constant k_2 such that $\phi(s) - k_1\phi(-s) = k_2s$.

Acknowledgements We thank the referees for their time and comments.

References

[1] R. BRYANT. *Geodesically reversible Finsler 2-spheres on constant curvature*. in Inspired by S. S. Chern, ed. Phillip A. Griffiths, Nankai Tracts in Math., 2006, **11**: 95–111.
[2] M. CRAMPIN. *Randers spaces with reversible geodesics*. Publ. Math. Debrecen, 2005, **67**(3-4): 401–409.
[3] I. M. MASCA, S. V. SABAU, H. SHIMADA. *Reversible geodesics for (α,β)-metrics*. Internat. J. Math., 2010, **21**(8): 1071–1094.
[4] Zhongmin SHEN. *Differential Geometry of Spray and Finsler Spaces*. Kluwer Academic Publishers, Dordrecht, 2001.
[5] S. S. CHERN, Zhongmin SHEN. *Rieman-Finsler Geometry*. Nankai Tracts in Mathematics, Vol. 6, World Scientific, 2005.
[6] S. BACSO, M. MATSUMOTO. *On Finsler spaces of Douglas type–a generalization of the notion of Berwald space*. Publ. Math. Debrecen, 1997, **51**(3-4): 385–406.
[7] Xinyue CHENG, Zhongmin SHEN. *A class of Finsler metrics with isotropic S-curvature*. Israel J. Math., 2009, **169**(1): 317–340.