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Quasicontinuity on Weighted Sobolev Spaces with Variable Exponent

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Abstract In the paper we give some precise characterizations for the quasicontinuity on weighted Sobolev spaces with variable exponent. Moreover, under the quasicontinuous meanings we obtain the uniqueness result in the weighted Sobolev spaces with variable exponent.

Keywords variable exponent; Muckenhoupt weight; Sobolev space; quasicontinuity

MR(2010) Subject Classification 31B05; 42B25; 46E30

1. Introduction

In the paper we are mainly concerned with some properties of weighted function space with variable exponent. Specifically speaking, our aim is to give some precise characterizations for the quasicontinuity of functions in weighted Sobolev spaces with variable exponent. Since Kováčik and Rákosník [1] introduced the variable exponent Lebesgue space and Sobolev space in higher dimensional Euclidean spaces, many mathematicians were deeply involved into this field and obtained a very great deal of results. There are some books and surveys to deal with systemly them in the nonweighted and weighted variable exponent cases, we may refer to [2–5]. Diening [6] proved the boundedness of maximal operator in variable exponent Lebesgue space, and Hästö [7] introduced a simple and convenient method to pass from local to global results in variable exponent function spaces, which greatly promoted the theoretical development of variable exponent function spaces. From these interesting theoretical considerations, Aydin [8] introduced the weighted Sobolev capacity with variable exponent, and generalized some results from Harjulehto et al. [9–11], Kilpeläinen [12] and Samko [5] to the weighted case with variable exponent. In addition, the strong $p(\cdot)$-Laplacian operator related to Sobolev spaces with variable exponent has also been investigated by Hästö et al. [13,14]. Heinonen [15] and Turesson [16] studied quasicontinuity on Sobolev space, which was partly generalized by Aydin (see Lemmas 3.1 and 3.8 below). Inspired by the statements above, we continue to develop Turesson's results in weighted Sobolev spaces with variable exponent. For the better statements about our results, in the next section we will provide some notations and background materials.

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2. Capacity, Sobolev spaces with variable exponents

For a measure function $p(\cdot):\mathbb{R}^N\to[1,\infty)$, write

$$p^-:=\operatorname{essinf}_{x\in\mathbb{R}^N}p(x),\quad p^+:=\operatorname{esssup}_{x\in\mathbb{R}^N}p(x).$$

Let the weight function $\omega:\mathbb{R}^N\to(0,\infty)$ be a positive, measurable and locally integrable function. We define the weight modular with respect to function f by

$$\varrho_{p(\cdot),\omega}(f):=\int_{\mathbb{R}^N}|f(x)|^{p(x)}\omega(x)\mathrm{d}x.$$

Denote by $L^{p(\cdot)}(\mathbb{R}^N,\omega)$ the weighted Lebesgue space with variable exponent endowed with the norm

$$\|f\|_{p(\cdot),\omega}=\|f\omega^{\frac{1}{p(\cdot)}}\|_{p(\cdot)}$$

on $\mathbb{R}^N$, where the Luxemburg norm $\|\cdot\|_{p(\cdot)}$ satisfies

$$\|f\|_{p(\cdot)}:=\inf\big\{\lambda>0:\varrho_{p(\cdot)}(\frac{f}{\lambda})\le 1\big\}.$$

Obviously, when $\omega(x)$ is a constant function, i.e., $\omega(x)\equiv 1$, we have $\|f\|_{p(\cdot),\omega}=\|f\|_{p(\cdot)}$, $\varrho_{p(\cdot),\omega}(f)=\varrho_{p(\cdot)}(f)$ and $L^{p(\cdot)}(\mathbb{R}^N,\omega)=L^{p(\cdot)}(\mathbb{R}^N)$. Similarly, if $p(\cdot)$ is a constant function p, then $L^{p(\cdot)}(\mathbb{R}^N)$ is clearly the usual Lebesgue space $L^p(\mathbb{R}^N)$.

The class $A_{p(\cdot)}$ consists of all weights ω with

$$\|\omega\|_{A_{p(\cdot)}}=\sup_{B\in\mathfrak{B}}|B|^{-p_B}\|\omega\|_{L^1(B)}\|\frac{1}{\omega}\|_{\mathrm{L}^{p'(\cdot)/p(\cdot)}(B)}<\infty,$$

where B denotes the set of all open balls in $\mathbb{R}^n$, and

$$p_{_B}=\Big[\frac{1}{|B|}\int_B\frac{1}{p(x)}\mathrm{d}x\Big]^{-1},\quad \frac{1}{p(\cdot)}+\frac{1}{p'(\cdot)}=1.$$

Note that, when $p(\cdot)\equiv p$, $A_{p(\cdot)}$ is the classical Muckenhoupt class A_p.

For $1<p^-\le p(x)\le p^+<\infty$, $\omega^{-1/(p(\cdot)-1)}\in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and $k\in\mathbb{N}$, we define the weighted Sobolev spaces with variable exponents $W^{k,p(\cdot)}(\mathbb{R}^N,\omega)$ by

$$W^{k,p(\cdot)}(\mathbb{R}^N,\omega):=\big\{f\in L^{p(\cdot)}(\mathbb{R}^N,\omega):D^\alpha f\in L^{p(\cdot)}(\mathbb{R}^N,\omega),0\le|\alpha|\le k\big\}$$

with the norm

$$\|f\|_{k,p(\cdot),\omega}=\sum_{0\le|\alpha|\le k}\|D^\alpha f\|_{p(\cdot),\omega},$$

where $\alpha\in\mathbb{N}_0^N=\mathbb{N}^N\cup\{0\}$ is a multi-index with $|\alpha|=\alpha_1+\alpha_2+\cdots\alpha_N$ and $D^\alpha=\partial^{|\alpha|}/(\partial_{x_1}^{\alpha_1}\cdots\partial_{x_N}^{\alpha_N})$.

For $\alpha>0$, let g_α be the Bessel kernel. Next we introduce the weighted Bessel potential spaces with variable exponent $\mathcal{L}^{\alpha,p(\cdot)}(\mathbb{R}^N,\omega)$ by

$$\mathcal{L}^{\alpha,p(\cdot)}(\mathbb{R}^N,\omega)=\{h=g_\alpha*f:f\in L^{p(\cdot)}(\mathbb{R}^N,\omega)\}$$

equipped with the norm

$$\|h\|_{\alpha,p(\cdot),\omega}=\|f\|_{p(\cdot),\omega}.$$

Obviously, we have $g_0*f=f$ and $\mathcal{L}^{0,p(\cdot)}(\mathbb{R}^N,\omega)=L^{p(\cdot)}(\mathbb{R}^N,\omega)$ when $\alpha=0$.

For $E \subset \mathbb{R}^N$ and $\alpha > 0$, the $B_{\alpha,p(\cdot),\omega}$-capacity in $\mathcal{L}^{\alpha,p(\cdot)}(\mathbb{R}^N, \omega)$ is defined by

$$B_{\alpha,p(\cdot),\omega}(E) := \inf \varrho_{p(\cdot),\omega}(f),$$

where the infimum is taken over all $f \in L^{p(\cdot)}(\mathbb{R}^N, \omega)$ such that the convolution $g_\alpha * f \geq 1$ on E. It is well known that $B_{\alpha,p(\cdot),\omega)}$-capacity satisfies the following properties [8,9]:

- $B_{\alpha,p(\cdot),\omega}(\emptyset) = 0$;
- For $E_1 \subset E_2 \subset \mathbb{R}^N$, $B_{\alpha,p(\cdot),\omega}(E_1) \leq B_{\alpha,p(\cdot),\omega}(E_2)$;
- For any set $E \subset \mathbb{R}^N$, $B_{\alpha,p(\cdot),\omega}(E) = \inf_{E \subset G, G \text{ open}} B_{\alpha,p(\cdot),\omega}(G)$;
- For $E_i \subset \mathbb{R}^N$, $i = 1, 2, \dots$, we have

$$B_{\alpha,p(\cdot),\omega}\Big(\bigcup_{i=1}^{\infty} E_i\Big) \leq \sum_{i=1}^{\infty} B_{\alpha,p(\cdot),\omega}(E_i).$$

A property holds $B_{\alpha,p(\cdot),\omega}$-quasieverywhere (shorten by $B_{\alpha,p(\cdot),\omega}$-q.e.), if it is true except in a set of capacity zero. We say a function f is $B_{\alpha,p(\cdot),\omega}$-quasicontinuous in $\mathbb{R}^N$, provided that, for given $\epsilon > 0$, there exists an open set G such that $B_{\alpha,p(\cdot),\omega}(E) < \epsilon$ and $f|_{G^c}$ is continuous.

3. Statements of main results

For $x \in \mathbb{R}^N$ and $r > 0$, we denote by $B(x, r)$ an open ball with x and radius r. For $f \in L^1_{\text{loc}}(\mathbb{R}^N)$, the centered Hardy-Littlewood maximal operator $\mathcal{M}f$ of the function f is defined by

$$\mathcal{M}f := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| \mathrm{d}y,$$

where the supremum is taken over all the balls $B(x, r)$ in $\mathbb{R}^N$. Let C be a constant. If

$$|p(x) - p(y)| \leq \frac{C}{\log(e + 1/|x - y|)}$$

for $x, y \in \mathbb{R}^N$, then we call that $p(\cdot)$ satisfies the local log-Hölder continuity condition. If

$$|p(x) - p_\infty| \leq \frac{C}{\log(e + |x|)}$$

for some $p_\infty > 1$ and $x \in \mathbb{R}^N$, then we say that $p(\cdot)$ satisfies the log-Hölder decay condition. If $p(\cdot)$ satisfies both the local log-Hölder continuity condition and the log-Hölder decay condition, then we say it is log-Hölder continuous, and denote by $P^{\log}(\mathbb{R}^N)$ the class of variable exponents. Next we start to sate our main theorems and the related lemmas.

Lemma 3.1 ([8, Proposition 3.19]) *Let $1 < p^- < p^+ < \infty$. If $f \in L^{p(\cdot)}(\mathbb{R}^N, \omega)$ and*

$$E = \{x \in \mathbb{R}^N : (g_\alpha * f)(x) = \infty\},$$

then $B_{\alpha,p(\cdot),\omega}(E) = 0$.

Theorem 3.2 *Suppose that the sequence $\{f_n\}_{n=1}^\infty$ converges to f in $L^{p(\cdot)}(\mathbb{R}^N, \omega)$. Then there is a subsequence $\{f_{n_j}\}_{j=1}^\infty$ such that*

$$g_\alpha * f_{n_j} \to g_\alpha * f \quad \text{as} \quad j \to \infty$$

for $B_{\alpha,p(\cdot),\omega}$-q.e. $x \in \mathbb{R}^N$, uniformly outside an open set of arbitrarily small $B_{\alpha,p(\cdot),\omega}$-capacity.

Proof Among the procedure of proof we mainly follow the method of Meyer [17]. Obviously, $g_\alpha * f$, $g_\alpha * f_n (n \in \mathbb{N}) \in \mathcal{L}^{\alpha,p(\cdot)}(\mathbb{R}^N, \omega)$. Since $\{f_n\}_{n=1}^\infty$ converges to f in $L^{p(\cdot)}(\mathbb{R}^N, \omega)$, we may choose a subsequence $\{f_{n_j}\}_{j=1}^\infty$ such that

$$\|g_\alpha * f_{n_j} - g_\alpha * f\|_{\alpha,p(\cdot),\omega} = \|f_{n_j} - f\|_{p(\cdot),\omega} \to 0 \quad \text{as} \quad j \to \infty.$$

From Lemma 3.1 we know that there exist set E_{n_j} for each $j \in \mathbb{N}$ and F such that $B_{\alpha,p(\cdot),\omega}(E_{n_j}) = 0$ and $B_{\alpha,p(\cdot),\omega}(F) = 0$, so we obtain that

$$B_{\alpha,p(\cdot),\omega}\Big(\bigcup_{j=1}^\infty E_{n_j} \cup F\Big) \le \sum_{j=1}^\infty B_{\alpha,p(\cdot),\omega}(E_{n_j}) + B_{\alpha,p(\cdot),\omega}(F) = 0.$$

Set $E = \bigcup_{j=1}^\infty E_{n_j} \cup F$. Therefore, for $x \in E^c$,

$$g_\alpha * f_{n_j}(x) \to g_\alpha * f(x) \quad \text{as} \quad j \to \infty.$$

Then there exists a subsequence $\{f_{n_j}\}_{j=1}^\infty$ such that the previous limit holds uniformly outside an open set of arbitrarily small $B_{\alpha,p(\cdot),\omega}$-capacity so that Theorem 3.2 follows.□

Set

$$\mathcal{D}(\mathbb{R}^N) := \{p(\cdot) : 1 < p^- \le p(x) \le p^+ < \infty, \|\mathcal{M}f\|_{p(\cdot),\omega} \le C\|f\|_{p(\cdot),\omega}\}.$$

For $p(\cdot) \in P^{\log}(\mathbb{R}^N)$ and $1 < p^- \le p^+ < \infty$, $\mathcal{M} : L^{p(\cdot)}(\mathbb{R}^N, \omega) \hookrightarrow L^{p(\cdot)}(\mathbb{R}^N, \omega)$ if and only if $\omega \in A_{p(\cdot)}$ (see [18]). Based on Theorem 3.2 above, we have the following two theorems.

Theorem 3.3 *Let $0 < \alpha < N$, $p(\cdot) \in P^{\log}(\mathbb{R}^N)$ and $\omega \in A_{p(\cdot)}$. If $f \in L^{p(\cdot)}(\mathbb{R}^N, \omega)$, then $g_\alpha * f$ is $B_{\alpha,p(\cdot),\omega}$-quasicontinuous.*

To be convenient to the readers, we state some preliminary lemmas needed in the proof of Theorem 3.3.

Lemma 3.4 ([8, Lemma 3.2]) *Let $p(\cdot) \in P^{\log}(\mathbb{R}^N)$ and $1 < p^- \le p^+ < \infty$. If $\omega \in A_{p(\cdot)}$, then*

(i) $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{k,p(\cdot)}(\mathbb{R}^N, \omega)$ for $k \in \mathbb{N}$;

(ii) the Schwartz class $\mathcal{S}$ is dense in $\mathcal{L}^{\alpha,p(\cdot)}(\mathbb{R}^N, \omega)$ with $\alpha \ge 0$.

Lemma 3.5 ([8, Theorem 3.2]) *Let $p(\cdot) \in \mathcal{D}(\mathbb{R}^N)$ and $k \in \mathbb{N}$. Then*

$$\mathcal{L}^{k,p(\cdot)}(\mathbb{R}^N, \omega) = W^{k,p(\cdot)}(\mathbb{R}^N, \omega)$$

and the corresponding norms are equivalent.

Lemma 3.6 ([8, Proposition 2.5]) *Let $p(\cdot) \in \mathcal{D}(\mathbb{R}^N)$. The class $C_0^\infty(\mathbb{R}^N)$ is dense in $L^{p(\cdot)}(\mathbb{R}^N, \omega)$.*

Proof of Theorem 3.3 From Lemma 3.1 we know that $g_\alpha * f$ is defined and finite $B_{\alpha,p(\cdot),\omega}$-q.e. By (i) in Lemma 3.4, and Lemmas 3.5 and 3.6, we may choose a sequence of functions $\{f_n\}_{n=1}^\infty$ in $C_0^\infty(\mathbb{R}^N)$ such that f_n converges to f in $L^{p(\cdot)}(\mathbb{R}^N, \omega)$. Therefore, from Theorem 3.2, we clearly see that there exists a subsequence of functions $\{f_{n_j}\}_{j=1}^\infty$ such that

$$g_\alpha * f_{n_j} \to g_\alpha * f \quad \text{as} \quad j \to \infty$$

for $B_{\alpha,p(\cdot),\omega}$-q.e. $x \in \mathbb{R}^N$ and uniformly outside an open set of arbitrarily small $B_{\alpha,p(\cdot),\omega}$-capacity. So Theorem 3.3 follows from Lemmas 3.4 and 3.5. □

Theorem 3.7 *Let $0 < \alpha < N$, $1 < p^- < p^+ < \infty$ and $\omega \in A_{p(\cdot)}$. Suppose that O is an open subset of $\mathbb{R}^N$ such that*

$$\limsup_{r\to 0} \frac{|O \cap B_r(x)|}{|B_r(x)|} > 0 \tag{1}$$

for $B_{\alpha,p(\cdot),\omega}$-q.e. $x \in \partial O$. If f_1 and f_2 are two $B_{\alpha,p(\cdot),\omega}$-quasicontinuous functions on $\mathbb{R}^N$, then $f_1 = f_2$ $B_{\alpha,p(\cdot),\omega}$-q.e. on $\overline{O}$.

Lemma 3.8 *Let $1 < p^- < p^+ < \infty$. If $f \in L^{P(\cdot)}(\mathbb{R}^N, \omega)$, then*

$$\lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} (g_\alpha * f)(y)\mathrm{d}y = (g_\alpha * f)(x)$$

for $B_{\alpha,p(\cdot),\omega}$-q.e. $x \in \mathbb{R}^N$.

Here Lemma 3.8 plays a vital role to prove Theorem 3.7, and it is also due to Aydin [8, Proposition 3.20]. In fact, when $p(\cdot) \equiv p$, the uniqueness theorem above or Theorem 3.7 results from Turesson [16]. As for the nonweighted case, we refer to Carlsson [19], which improves the work by Maz'ya and Havin [20]. Carlsson did not consider the positive superior limit (i.e., inequality (1)) but the inferior limit being equal or greater than some positive constant. In addition, Aikawa [21] generalized the results from Maz'ya and Havin into weighted version.

Proof of Theorem 3.7 Since the function $f = f_1 - f_2$ is $B_{\alpha,p(\cdot),\omega}$-quasicontinuous, there exist open sets $G_n, n = 1, 2, \ldots$, such that $B_{\alpha,p(\cdot),\omega}(G_n) \to 0$ as $n \to \infty$, and $f|_{G_n^c}$ is continuous. Let ϕ_n be nonnegative functions in $L^{p(\cdot)}(\mathbb{R}^N, \omega)$ satisfying $\|\phi_n\|_{L^{p(\cdot)}(\mathbb{R}^N,\omega)} \to 0$ and $\psi_n = g_\alpha * \phi_n \geq 1$ on G_n. By Theorem 3.2 we may assume that $\psi_n \to 0 B_{\alpha,p(\cdot),\omega}$-q.e. by passing to a subsequence.

Let $x \in \overline{O}$ be a point such that $\psi_n \to 0$ as $n \to \infty$ the inequality holds. Then we see from Lemma 3.8 that

$$\begin{aligned}\limsup_{r\to 0} \frac{|G_n \cap B_r(x)|}{|B_r(x)|} &\leq \limsup_{r\to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} \chi_{G_n}(y)\psi_n(y)\mathrm{d}y \\ &\leq \lim_{r\to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} \psi_n(y)\mathrm{d}y = \psi_n(x).\end{aligned}$$

Since $\psi_n \to 0$ as $n \to \infty$, we may find some n such that

$$\limsup_{r\to 0} \frac{|G_n \cap B_r(x)|}{|B_r(x)|} < \limsup_{r\to 0} \frac{|O \cap B_r(x)|}{|B_r(x)|}.$$

Hence, for arbitrarily small r it follows that $|O \cap B_r(x) \cap G_n^c| > 0$. Then we may choose a sequence $\{x_i\}_{n=1}^\infty$ of points in $O \cap G_n^c$ such that $x_i \to x$ as $n \to \infty$ and $f(x_i) = 0$ for every i. Due to $x \in G_n^c$, it implies that $f(x) = 0$, which is exactly the desired result. □

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References

[1] O. KOVÁČIK, J. RÁKOSNÍK. *On spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$*. Czechoslovak Mathematical Journal, 1991, **41**(116): 592–618.

[2] L. DIENING, P. HARJULEHTO, P. HÄSTÖ, et al. *Lebesgue and Sobolev Spaces with Variable Exponents*. Lecture Notes in Mathematics, 2017. Springer, Heidelberg, 2011.

[3] L. DIENING, P. HÄSTÖ, A. NEKVINDA. *Open Problems in Variable Exponent Lebesgue and Sobolev Spaces, In: Drabek, P., Rakosnik, J. (eds) FSDONA 04 Proceedings, Milovy, Czech Republic, pp. 38-58*. Academy of Sciences of the Czech Republic, Prague, 2004.

[4] Xianling FAN, Dun ZHAO. *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*. J. Math. Anal. Appl., 2001, **263**(2): 424–446.

[5] S. SAMKO. *On a progress in the theory of Lebesgue spaces with variable exponent: maximal and singular operators*. Integral Transforms Spec. Funct., 2005, **16**(5-6): 461–482.

[6] L. DIENING. *Maximal function on Musielak-Orlicz spaces and generalized Lebesgue spaces*. Bull. Sci. Math., 2005, **129**(8): 657–700.

[7] P. HÄSTÖ. *Local-to-global results in variable exponent spaces*. Math. Res. Lett., 2009, **16**(2): 263–278.

[8] I. AYDIN. *Weighted variable Sobolev spaces and capacity*. J. Funct. Spaces Appl., 2012, Art. ID 132690, 17 pp.

[9] P. GURKA, P. HARJULEHTO, A. NEKVINDA. *Bessel potential spaces with variable exponent*. Math. Inequal. Appl., 2007, **10**(3): 661–676.

[10] P. HARJULEHTO, P. HÄSTÖ. *Lebesgue points in variable exponent spaces*. Ann. Acad. Sci. Fenn. Math., 2004, **29**(2): 295–306.

[11] P. HARJULEHTO, P. HÄSTÖ, M. KOSKENOJA, et al. *Sobolev capacity on the space $W^{1,p(\cdot)}(\mathbb{R}^N)$*. J. Funct. Spaces Appl., 2003, **1**(1): 17–33.

[12] T. KILPELÄINEN. *Weighted Sobolev spaces and capacity*. Ann. Acad. Sci. Fenn. Ser. A I Math., 1994, **19**(1): 95–113.

[13] T. ADAMOWICZ, H. HÄSTÖ. *Harnack's inequality and the strong $p(\cdot)$-Laplacian*. J. Differential Equations, 2011, **250**(3): 1631–1649.

[14] P. HARJULEHTO, P. HÄSTÖ, V. LATVALA. *Harnack's inequality for $p(\cdot)$-harmonic functions with unbounded exponent p*. J. Math. Anal. Appl., 2009, **352**(1): 345–359.

[15] J. HEINONEN, T. KILPELÄINEN, O. MARTIO. *Nonlinear Potential Theory of Degenerate Elliptic Equations*. The Clarendon Press, Oxford University Press, New York, 1993.

[16] B. O. TURESSON. *Nonlinear Potential Theory and Weighted Sobolev Spaces*. Springer-Verlag, Berlin, 2000.

[17] N. G. MEYERS. *A theory of capacities for potentials of functions in Lebesgue classes*. Math. Scand., 1970, **26**: 255–292.

[18] P. HÄSTÖ, L. DIENING. *Muckenhoupt weights in variable exponent spaces*, Preprint. http://www.helsinki.fi/~pharjule/varsob/publications.shtml

[19] A. CARLSSON. *Inequalities of Poincaré-Wirtinger Type*. Department of Mathematics, Linköping University, Linköping, 1990.

[20] V. G. MAZ'JA, V. P. HAVIN. *A nonlinear potential theory*. Uspehi Mat. Nauk, 1972, **27**(6): 67–138.

[21] H. AIKAWA. *On weighted Beppo Levi functions-integral representations and behavior at infinity*. Analysis, 1989, **9**(4): 323–346.