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Necessary and Sufficient Conditions for Boundedness of Commutators of Bilinear Fractional Integral Operators on Morrey Spaces

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Abstract In this paper, we obtain that $b \in \mathrm{BMO}(\mathbb{R}^n)$ if and only if the commutator $[b, I_\alpha]$ is bounded from the Morrey spaces $L^{p_1,\lambda_1}(\mathbb{R}^n) \times L^{p_2,\lambda_2}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$, for some appropriate indices p, q, λ, μ. Also we show that $b \in \mathrm{Lip}_\beta(\mathbb{R}^n)$ if and only if the commutator $[b, I_\alpha]$ is bounded from the Morrey spaces $L^{p_1,\lambda_1}(\mathbb{R}^n) \times L^{p_2,\lambda_2}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$, for some appropriate indices p, q, λ, μ.

Keywords fractional integral operator; Morrey spaces; commutators; BMO; Lipschitz

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1. Introduction

Let $I_\alpha,\ 0 < \alpha < n$, be the fractional integral operator of order α, defined by

$$I_\alpha := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \mathrm{d}y.$$

For a locally integrable function b, the commutator $[b, I_\alpha]$ is defined by Chanillo [1] as follows,

$$[b, I_\alpha]f(x) := b(x)I_\alpha - I_\alpha(bf)(x).$$

Meanwhile, Komori [2] obtained $[b, I_\alpha]$ is a bounded map of $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $1/q = 1/p - \alpha/n$ and $0 < \alpha < n$, if and only if $b \in \mathrm{BMO}(\mathbb{R})$ and operator norm $\|[b, I_\alpha]\|_{L^p \to L^q} \approx \|b\|_*$, this is equivalent Characterizations of BMO Spaces. Then, Paluszyński [3] obtained that $b \in \mathrm{Lip}_\beta(\mathbb{R}^n)$ if and only if the commutator $[b, I_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $1 < p < q < \infty, 0 < \beta < 1$ with $1/q = 1/p - (\beta + \alpha)/n$, and $1/p - (\alpha + \beta) > 0$. In addition, Lip_β spaces also could be characterized by the boundedness of the commutators.

Later, Di Fazio and Ragusa [4] certificated that if $b \in \mathrm{BMO}(\mathbb{R}^n)$, then the commutator $[b, I_\alpha]$ is bounded from the Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$. Moreover for some appropriate indices p, q, λ, μ and α, if the commutator $[b, I_\alpha]$ is bounded from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$, then $b \in \mathrm{BMO}(\mathbb{R}^n)$. In addition, Shirai [5] showed that $b \in \mathrm{Lip}_\beta(\mathbb{R}^n)$ if and only if the commutator $[b, I_\alpha]$ is bounded from the Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$, where α and β satisfy some approximate conditions.

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The multilinear fractional integral operators were first considered by Kenig and Stein [6], who obtained the boundedness of it on Lebesgue spaces with suitable indexes. Mo and Zhang [7] showed the boundedness of the commutators generated by it and Lipschitz functions on Lebesgue spaces. Afterwards, Chaffee [8] showed the boundedness of commutators of bilinear fractional integral operators on Lebesgue spaces. Jiang, Pan and Wang [9] proved that necessary and sufficient conditions for boundedness of commutators multilinear fractional integral operators on Lebesgue spaces.

Because Morrey space is a generalization of Lebesgue spaces, a natural problem is whether the BMO spaces, and Lip_β spaces could be characterized by the boundedness of the bilinear fractional integral commutators on Morrey spaces. In this paper, We affirmatively answer it.

The aim of this paper is to prove that $b \in \mathrm{BMO}(\mathbb{R}^n)$ if the commutator $[b, I_\alpha]$ generated by bilinear fractional integral operators I_α and the symbols b is bounded from the classical Morrey spaces $L^{p_1,\lambda_1}(\mathbb{R}^n) \times L^{p_2,\lambda_2}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$, for some appropriate indices p, q, λ, μ. Also we show that $b \in \mathrm{Lip}_\beta(R^n)$ if and only if the commutators $[b, I_\alpha]$ generated by bilinear fractional integral operators I_α and the symbols b is bounded from the classical Morrey spaces $L^{p_1,\lambda_1}(\mathbb{R}^n) \times L^{p_2,\lambda_2}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$, for some appropriate indices p, q, λ, μ.

Throughout this paper, the letter C always denotes a constant which is independent of main variables and may change from one occurrence to another. All cubes are assumed to have their sides parallel to the coordinate axes. We use $Q = Q(x_0, r)$ to denote a cube centered at x_0 with side length r. Given a Lebesgue measurable set E, χ_E will denote the characteristic function of E and $|E|$ denotes the Lebesgue measure of E.

2. Some definitions and lemmas

Let us first recall several definitions and lemmas.

Definition 2.1 (Bilinear Fractional Integral Operator) *For $0 < \alpha < 2n$, the bilinear fractional integral operator I_α is defined by*

$$I_\alpha(f, g)(x) = \int_{R^n} \int_{R^n} \frac{f(y)g(z)}{(|x-y| + |x-z|)^{2n-\alpha}} \mathrm{d}y\mathrm{d}z.$$

Definition 2.2 (Commutator) *Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ or $b \in \mathrm{Lip}_\beta(\mathbb{R}^n)$. The commutators $[b, I_\alpha]_i$ ($i = 1, 2$) generated by the symbol b and the bilinear fractional integral operator I_α are defined by*

$$[b, I_\alpha]_1(f, g)(x) = \int_{R^n} \int_{R^n} \frac{b(y) - b(x)}{(|x-y| + |x-z|)^{2n-\alpha}} f(y)g(z)\mathrm{d}y\mathrm{d}z,$$

and

$$[b, I_\alpha]_2(f, g)(x) = \int_{R^n} \int_{R^n} \frac{b(z) - b(x)}{(|x-y| + |x-z|)^{2n-\alpha}} f(y)g(z)\mathrm{d}y\mathrm{d}z.$$

Now, we recall the definitions of the Morrey spaces, BMO space and Lipschitz spaces.

Definition 2.3 (Morrey space) *Let $1 \le p < \infty, \lambda \ge 0$. We define the classical Morrey space by*

$$L^{p,\lambda}(\mathbb{R}^n) := \{f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{L^{p,\lambda}} < \infty\},$$

where

$$\|f\|_{L^{p,\lambda}} := \sup_{\substack{x_0\in\mathbb{R}^n\\ t>0}} \Big(\frac{1}{t^\lambda}\int_{Q(x_0,t)}|f(x)|^p\mathrm{d}x\Big)^{1/p}.$$

For classical Morrey space $L^{p,\lambda}(\mathbb{R}^n)$, the next results are well-known:

Suppose that $1\le p<\infty$, then we have $L^{p,0}(\mathbb{R}^n)=L^p(\mathbb{R}^n)$ and $L^{p,n}(\mathbb{R}^n)=L^{p,\infty}(\mathbb{R}^n)$ when $\lambda=n$, and if $n<\lambda$, then we get $L^{p,\lambda}(\mathbb{R}^n)=\{0\}$. Hence we consider the case only $0<\lambda<n$.

Definition 2.4 (BMO space) *A locally integrable function f is said to belong to BMO space if there exists a constant $C>0$ such that for any cube $Q\in\mathbb{R}^n$,*

$$\|f\|_* := \sup_Q \frac{1}{|Q|}\int_Q |f(x)-f_Q|\mathrm{d}x,$$

where $f_Q := \frac{1}{|Q|}\int_Q f(y)\mathrm{d}y$ and the supremum is taken over all cubes Q in $(\mathbb{R}^n)$.

Definition 2.5 (Lipschitz space) *We define the (homogeneous) Lipschitz space of order of β, $0<\beta<1$, by*

$$\mathrm{Lip}_\beta(\mathbb{R}^n) := \{f : |f(x)-f(y)|\le C|x-y|^\beta\}$$

and the smallest constant $C>0$ is the Lipschitz norm $\|\cdot\|_{\mathrm{Lip}_\beta}$.

Remark 2.6 This remark can be found in [3]. For $0<\beta<1$ and $1<q\le\infty$, we get

$$\mathrm{Lip}_\beta(\mathbb{R}^n) \approx \sup_Q \frac{1}{|Q|^{1+\frac{\beta}{n}}}\int_Q |f(x)-f_Q|\mathrm{d}x \approx \sup_Q \frac{1}{|Q|^{\frac{\beta}{n}}}\Big(\int_Q |f(x)-f_Q|^q\mathrm{d}x\Big)^{\frac{1}{q}},$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$.

The blocks and the spaces generated by blocks were introduced by Long [10].

Definition 2.7 (blocks) *Suppose that $1\le q<r\le\infty$, a function $g(x)$ on $\mathbb{R}^n$ is called a (q,r)-block, if there exists a cube $Q(x_0,t)$ such that*

(i) $\mathrm{supp}(g)\subset Q(x_0,t)$;

(ii) $\|g\|_{L^r}\le t^{n(\frac{1}{r}-\frac{1}{q})}$.

Definition 2.8 (blocks spaces) *Let $1\le q<r\le\infty$. We define the space generated by blocks by*

$$h_{q,r}(\mathbb{R}^n) := \Big\{f=\sum_{j=1}^{\infty} m_j g_j : g_j \text{ are } (q,r)\text{-blocks},\ \ \|f\|_{h_{q,r}} = \inf\sum\nolimits_{j=1}^{\infty}|m_j|<\infty\Big\},$$

where the infimum extends over all representations $f=\sum_{j=1}^{\infty} m_j g_j$.

We observe that each (q,r)-blocks $g_j\in L^q(\mathbb{R}^n)$ and $\|g_j\|_q\le 1$. So the series of blocks $\sum_j m_j g_j$ converges in $L^q(\mathbb{R}^n)$ and absolutely almost everywhere if $\sum_j |m_j|<\infty$, therefore each space $h_{q,r}(\mathbb{R}^n)$ is a Banach space.

Lemma 2.9 ([11]) *Let $1\le p<\infty, 0<\lambda<n$ and $1\le q<r\le\infty$. Then we obtain*

(i) $\|\chi_{Q(x_0,t)}\|_{L^{p,\lambda}}\le C_n t^{(n-\lambda)/p}$;

(ii) $\|\chi_{Q(x_0,t)}\|_{h_{q,r}}\le C_n t^{n/q}$,

where C_n is a positive constant depending only on n.

Lemma 2.10 ([10,11]) *Let $1 \le q < p' \le \infty$, $q = np/(np-n+\lambda)$ and $1/p+1/p'=1$. Then the Banach space dual of $h_{q,p'}(\mathbb{R}^n)$ is isomorphic to $L^{p,\lambda}(\mathbb{R}^n)$.*

3. Main results

The following statements are our main results.

Theorem 3.1 *For $0<\alpha<2n$, $0<\lambda,\lambda_1,\lambda_2<n$ and $1<p,p_1,p_2<\infty$ with $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$. Let $1<q<\infty$ with $\frac{\lambda}{p}=\frac{\lambda_1}{p_1}+\frac{\lambda_2}{p_2}$ and $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n-\lambda}$. The following statements are equivalent:*

(i) $b\in \mathrm{BMO}(\mathbb{R}^n)$;

(ii) $[b,I_\alpha]_i(f,g)(x)$ *is bounded from* $L^{p_1,\lambda_1}(\mathbb{R}^n)\times L^{p_2,\lambda_2}(\mathbb{R}^n)$ *to* $L^{q,\lambda}(\mathbb{R}^n)$.

Theorem 3.2 *For $0<\alpha<2n$, $0<\beta<1$, $0<\alpha+\beta<2n(\frac{1}{p}-\frac{1}{q})<2n$, and $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$. Let $1<p<q<\infty$ with $\frac{\lambda}{p}=\frac{\lambda_1}{p_1}+\frac{\lambda_2}{p_2}$ and $\frac{1}{q}=\frac{1}{p}-\frac{\alpha+\beta}{n-\lambda}$. The following statements are equivalent:*

(i) $b\in \mathrm{Lip}_\beta(\mathbb{R}^n)$;

(ii) $[b,I_\alpha]_i(f,g)(x)$ *is bounded from* $L^{p_1,\lambda_1}(\mathbb{R}^n)\times L^{p_2,\lambda_2}(\mathbb{R}^n)$ *to* $L^{q,\lambda}(\mathbb{R}^n)$.

Proof of Theorem 3.1 (i)$\Longrightarrow$ (ii). This result was proved by Ding and Mei [12].

(ii)$\Longrightarrow$ (i). We use the same argument as Janson [13]. Choose $0\neq z_0\in(\mathbb{R}^n)$ such that $0\notin Q(z_0,2)$. Then for $x\in Q(z_0,2)$, $|x|^{n-\alpha}\in C^\infty(Q(z_0,2))$. Therefore, considering a cut function on the cube $Q(z_0,2+\delta)$ for sufficiently small $\delta>0$, $|x|^{n-\alpha}$ can be written as the absolutely convergent Fourier series

$$|x|^{n-\alpha}=\sum_{m\in\mathbb{Z}^n} a_m e^{i\langle v_m,x\rangle}$$

with $\sum_m |a_m|<\infty$, where the exact form of the vectors v_m is unrelated. We do not care about the specific vectors $v_m\in\mathbb{R}^{2n}$, but we will at times express them as $v_m=(v_m^1,v_m^2)\in\mathbb{R}^n\times\mathbb{R}^n$.

For any $x_0\in\mathbb{R}^n$ and $r>0$, let $Q=Q(x_0,r)$ and $Q_{z_0}=Q(x_0+z_0r,r)$. Let $\sigma(x)=\mathrm{sgn}(b(x)-b_{Q_{z_0}})$. We then have the following,

$$\begin{aligned}
&\int_Q |b(x)-b_{Q_{z_0}}|\mathrm{d}x\\
&=\int_Q (b(x)-b_{Q_{z_0}})\sigma(x)\mathrm{d}x=\frac{1}{|Q_{z_0}|^2}\int_Q\int_{Q_{z_0}}\int_{Q_{z_0}}(b(x)-b_{Q_{z_0}})\sigma(x)\mathrm{d}z\mathrm{d}y\mathrm{d}x\\
&=\frac{1}{|Q_{z_0}|^2}\int_Q\int_{Q_{z_0}}\int_{Q_{z_0}}\frac{(b(x)-b(y))r^{2n-\alpha}}{(|x-y|+|x-z|)^{2n-\alpha}}\Big(\frac{|x-y|+|x-z|}{r}\Big)^{2n-\alpha}\sigma(x)\mathrm{d}z\mathrm{d}y\mathrm{d}x\\
&=r^{-\alpha}\int_Q\int_{Q_{z_0}}\int_{Q_{z_0}}\frac{(b(x)-b(y))}{(|x-y|+|x-z|)^{2n-\alpha}}\Big(\frac{|x-y|+|x-z|}{r}\Big)^{2n-\alpha}\sigma(x)\mathrm{d}z\mathrm{d}y\mathrm{d}x\\
&=r^{-\alpha}\int_Q\int_{Q_{z_0}}\int_{Q_{z_0}}\frac{(b(x)-b(y))}{(|x-y|+|x-z|)^{2n-\alpha}}\sum_{m\in\mathbb{Z}^n}a_m e^{i\langle v_m,\frac{|x-y|+|x-z|}{r}\rangle}\sigma(x)\mathrm{d}z\mathrm{d}y\mathrm{d}x\\
&=r^{-\alpha}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(b(x)-b(y))}{(|x-y|+|x-z|)^{2n-\alpha}}\times\\
&\quad\sum_{m\in\mathbb{Z}^n}a_m e^{i\langle v_m,\frac{|x-y|+|x-z|}{r}\rangle}\sigma(x)\chi_Q(x)\chi_{Q_{z_0}}(y)\chi_{Q_{z_0}}(z)\mathrm{d}z\mathrm{d}y\mathrm{d}x
\end{aligned}$$

Let

$$f_m(y) = e^{-iv_m^1 \cdot y}\chi_{Q_{z_0}}(y), \quad g_m(z) = e^{-iv_m^2 \cdot z}\chi_{Q_{z_0}}(z)$$

and

$$h_m(x) = e^{iv_m <x,x>}\sigma(x)\chi_Q(x).$$

Note that each of the above function has an L^q norm of $|Q|^{1/q}$ for any $q \geqslant 1$. Because Q, Q_{z_0} all have side length r, we will have that $Q \cap Q_{z_0} = \varnothing$. Continuing with our above calculation, we get

$$\begin{aligned}
&\int_Q |b(x) - b_{Q_{z_0}}|\mathrm{d}x \\
&= r^{-\alpha} \sum_{m\in\mathbb{Z}^n} a_m \int_{\mathbb{R}^n} h_m(x) \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(b(x)-b(y))}{(|x-y|+|x-z|)^{2n-\alpha}} f_m(y)g_m(z)\mathrm{d}z\mathrm{d}y\mathrm{d}x \\
&= r^{-\alpha} \sum_{m\in\mathbb{Z}^n} a_m \int_{\mathbb{R}^n} h_m(x)[b, I_\alpha](f_m, g_m)(x)\mathrm{d}x. \\
&\le r^{-\alpha} \sum_{m\in\mathbb{Z}^n} |a_m| \int_{\mathbb{R}^n} |h_m(x)||[b, I_\alpha](f_m, g_m)(x)|\mathrm{d}x.
\end{aligned}$$

Applying the Hölder inequality, we obtain

$$\begin{aligned}
&\int_Q |b(x) - b_{Q_{z_0}}|\mathrm{d}x \\
&\le r^{-\alpha} \sum_{m\in\mathbb{Z}^n} a_m \Big(\int_{\mathbb{R}^n} |h_m(x)|^{q'}\Big)^{\frac{1}{q'}} \Big(\int_{\mathbb{R}^n} |[b, I_\alpha](f_m, g_m)(x)|^q\mathrm{d}x\Big)^{\frac{1}{q}} \\
&= r^{-\alpha} \sum_{m\in\mathbb{Z}^n} a_m \Big(\int_{\mathbb{R}^n} |h_m(x)|^{q'}\Big)^{\frac{1}{q'}} r^{\frac{\lambda}{q}} \Big(\int_{\mathbb{R}^n} \frac{1}{r^\lambda}|[b, I_\alpha](f_m, g_m)(x)|^q\mathrm{d}x\Big)^{\frac{1}{q}} \\
&= \sum_{m\in\mathbb{Z}^n} a_m |Q|^{\frac{-\alpha}{n}} \|h_m\|_{L^{q'}} |Q|^{\frac{\lambda}{qn}} \|[b, I_\alpha]\|_{L^{p_1,\lambda_1}\times L^{p_2,\lambda_2}\longrightarrow L^{p,\lambda}} \|f_m\|_{L^{p_1,\lambda_1}} \|f_m\|_{L^{p_2,\lambda_2}} \\
&= \|[b, I_\alpha]\|_{L^{p_1,\lambda_1}\times L^{p_2,\lambda_2}\longrightarrow L^{p,\lambda}} \sum_{m\in\mathbb{Z}^n} a_m |Q|^{\frac{-\alpha}{n}} |Q|^{\frac{1}{q'}} |Q|^{\frac{\lambda}{qn}} |Q|^{\frac{1}{p_1}} |Q|^{\frac{\lambda_1}{p_1 n}} |Q|^{\frac{1}{p_2}} |Q|^{\frac{\lambda_2}{p_2 n}} \\
&= \|[b, I_\alpha]\|_{L^{p_1,\lambda_1}\times L^{p_2,\lambda_2}\longrightarrow L^{p,\lambda}} \sum_{m\in\mathbb{Z}^n} a_m |Q|^{\frac{-\alpha}{n}+\frac{1}{q'}+\frac{\lambda}{qn}+\frac{1}{p_1}+\frac{1}{p_2}+\frac{\lambda_1}{p_1 n}+\frac{\lambda_1}{p_2 n}} \\
&\le C\|[b, I_\alpha]\|_{L^{p_1,\lambda_1}\times L^{p_2,\lambda_2}\longrightarrow L^{p,\lambda}}|Q|.
\end{aligned}$$

Therefore we have

$$\frac{1}{|Q|}\int_Q |b(x) - b_Q|\mathrm{d}x \le \frac{2}{|Q|}\int_Q |b(x) - b_{Q_{z_0}}|\mathrm{d}x \le C\|[b, I_\alpha]\|_{L^{p_1,\lambda_1}\times L^{p_2,\lambda_2}\longrightarrow L^{p,\lambda}}.$$

This implies that $b \in \mathrm{BMO}(\mathbb{R}^n)$, thus we complete the proof of Theorem 3.1. □

Proof of Theorem 3.2 (i)$\Longrightarrow$(ii). Let $b \in \mathrm{Lip}_\beta(\mathbb{R}^n)$. Then we have

$$\begin{aligned}
|[b, I_\alpha]_1(f, g)(x)| &= \Big|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{b(y)-b(x)}{(|x-y|+|x-z|)^{2n-\alpha}} f(y)g(z)\mathrm{d}y\mathrm{d}z\Big| \\
&\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|b(y)-b(x)|}{||x-y|+|x-z||^{2n-\alpha}} |f(y)||g(z)|\mathrm{d}y\mathrm{d}z
\end{aligned}$$

$$\begin{aligned}&\le C\|b\|_{\mathrm{Lip}_\beta}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(y)||g(z)|}{||x-y|+|x-z||^{2n-(\alpha+\beta)}}\mathrm{d}y\mathrm{d}z\\&=C\|b\|_{\mathrm{Lip}_\beta}I_{\alpha+\beta}(|f|,|g|)(x).\end{aligned}$$

For any $x\in\mathbb{R}^n$, by [14] we get

$$\|[b,I_\alpha]_1(f,g)\|_{L^{q,\lambda}}\le C'\|b\|_{\mathrm{Lip}_\beta}\|I_{\alpha+\beta}(|f|,|g|)\|_{L^{q,\lambda}}\le C\|b\|_{\mathrm{Lip}_\beta}\|f\|_{L^{p_1,\lambda_1}}\times\|g\|_{L^{p_2,\lambda_2}}.$$

(ii)$\Longrightarrow$(i). We can prove by using an argument similar to the proof of Theorem 3.1. Below we give a completeness proof. Suppose that Q and Q_{z_0} are the same cubes as in the proof (ii)$\Longrightarrow$(i) in Theorem 3.1. Then we have

$$\begin{aligned}&\int_Q|b(x)-b_{Q_{z_0}}|\mathrm{d}x\\&=r^{-\alpha}\sum_{m\in\mathbb{Z}^n}a_m\int_{\mathbb{R}^n}h_m(x)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(b(x)-b(y))}{(|x-y|+|x-z|)^{2n-\alpha}}f_m(y)g_m(z)\mathrm{d}z\mathrm{d}y\mathrm{d}x\\&=r^{-\alpha}\sum_{m\in\mathbb{Z}^n}a_m\int_{\mathbb{R}^n}h_m(x)[b,I_\alpha](f_m,g_m)(x)\mathrm{d}x\\&\le r^{-\alpha}\sum_{m\in\mathbb{Z}^n}|a_m|\Big|\int_{\mathbb{R}^n}|h_m(x)||[b,I_\alpha](f_m,g_m)(x)\Big|\mathrm{d}x.\end{aligned}$$

It follows from Lemmas 2.9 and 2.10 that

$$\begin{aligned}&\int_Q|b(x)-b_{Q_{z_0}}|\mathrm{d}x\\&\le r^{-\alpha}\sum_{m\in\mathbb{Z}^n}|a_m|\|[b,I_\alpha](f_m,g_m)\|_{L^{q,\lambda}}\|h_m\|_{h_{nq/(nq-n+\lambda),q'}}\\&\le r^{-\alpha}\sum_{m\in\mathbb{Z}^n}|a_m|\|[b,I_\alpha]\|_{L^{p_1,\lambda_1}\times L^{p_2,\lambda_2}\longrightarrow L^{q,\lambda}}\|f_m\|_{L^{p_1,\lambda_1}}\|g_m\|_{L^{p_2,\lambda_2}}\|h_m\|_{h_{nq/(nq-n+\lambda),q'}}\\&\le r^{-\alpha}\sum_{m\in\mathbb{Z}^n}|a_m|\|[b,I_\alpha]\|_{L^{p_1,\lambda_1}\times L^{p_2,\lambda_2}\longrightarrow L^{q,\lambda}}C_n'r^{\frac{n-\lambda_1}{p_1}}C_n''r^{\frac{n-\lambda_2}{p_2}}C_nr^{\frac{nq-n+\lambda}{q}}\\&\le C\|[b,I_\alpha]\|_{L^{p_1,\lambda_1}\times L^{p_2,\lambda_2}}|Q|^{1+\frac{\beta}{n}}.\end{aligned}$$

Therefore we obtain

$$\frac{1}{|Q|^{1+\frac{\beta}{n}}}\int_Q|b(x)-b_Q|\mathrm{d}x\le\frac{2}{|Q|}\int_Q|b(x)-b_{Q_{z_0}}|\mathrm{d}x\le C\|[b,I_\alpha]\|_{L^{p_1,\lambda_1}\times L^{p_2,\lambda_2}\longrightarrow L^{p,\lambda}},$$

which implies that $b\in\mathrm{Lip}_\beta(\mathbb{R}^n)$. This completes the proof of Theorem 3.2. □

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References

[1] S. CHANILLO. *A note on commutators*. Indiana Univ. Math. J., 1982, **31**(1): 7–16.

[2] Y. KOMORI. *The factorization of H^P and the commutators*. Tokyo J. Math., 1983, **6**(2): 435–445.

[3] P. PALUSZYŃSKI. *Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss*. Indiana Univ. Math. J., 1995, **44**(1): 1–17.

[4] G. DI FAZIO, M. A. RAGUSA. *Commutators and Morrey spaces*. Boll. Un. Mat. Ital. A (7), 1991, **5**(3): 323–332.

[5] S. SHIRAI. *Necessary and sufficient conditions for boundedness of commutators of fractional integral operators on classical Morrey spaces*. Hokkaido Math. J., 2006, **35**(3): 683–696.

[6] C. KENIG, E. M. STEIN. *Multilinear estimates and fractional integration*. Math. Res. Lett., 1999, **6**(1): 1–15.

[7] Huixia MO, Zhiying ZHANG. *Boundedness of commutators generated by multilinear fractional integrals and Lipschitz functions* Acta Math. Sci. Ser. A Chin. Ed., 2011, **31**(5): 1447–1458.

[8] L. CHAFFEE. *Characterizations of BMO through commutators of bilinear singular integral operators*. arXiv:1410.4587v3 [math.CA] 9 Dec 2014.

[9] Songbai WANG, Jibin PAN, Yinsheng JIANG. *Necessary and sufficient conditions for boundedness of commutators of multilinear fractional integral operators*. Acta Math. Sci. Ser. A Chin. Ed., 2015, **35**(6): 1106–1114. (in Chinese)

[10] Ruilin LONG. *The spaces generated by blocks*. Sci. Sinica Ser. A, 1984, **27**(1): 16–26.

[11] Y. KOMORI, T. MIZUHARA. *Notes on commutators and Morrey spaces*. Hokkaido Math. J., 2003, **32**(2): 345–353.

[12] Yong DING, Ting MEI. *Boundedness and compactness for the commutators of bilinear operators on Morrey spaces*. Potential Anal., 2015, **42**(3): 717–748.

[13] S. JANSON. *Mean oscillation and commutators of singular integral operators*. Ark. Mat., 1978, **16**(2): 263–270.

[14] Ling TANG. *Endpoint estimates for multilinear fractional integrals*. J. Aust. Math. Soc., 2008, **84**(3): 419–429.