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Stable Hypersurfaces in a 4-Dimensional Sphere

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Abstract We study complete noncompact 1-minimal stable hypersurfaces in a 4-dimensional sphere $\mathbb{S}^4$. We show that there is no complete noncompact 1-minimal stable hypersurfaces in $\mathbb{S}^4$ with polynomial volume growth and the restriction of the mean curvature and Gauss-Kronecker curvature. These results are partial answers to the conjecture of Alencar, do Carmo and Elbert when the ambient space is a 4-dimensional sphere.

Keywords constant scalar curvature; 1-minimal stable hypersurfaces in space forms

MR(2010) Subject Classification 53C21; 54C42

1. Introduction

Cheng and Yau [1] proved that any complete noncompact hypersurface in the Euclidean space with constant scalar curvature and nonnegative sectional curvature must be a generalized cylinder. It is natural to study the global properties of hypersurfaces in space forms with constant scalar curvature. Alencar, do Carmo and Elbert posed the following question: Is there any complete 1-minimal stable hypersurfaces in $\mathbb{R}^4$ with nonzero Gauss-Kronecker curvature? In [2], it was proved that there is no complete noncompact 1-minimal stable hypersurface M in $\mathbb{R}^4$ with nonzero Gauss-Kronecker curvature and finite total curvature. Silva Neto [3] showed that there is no complete 1-minimal stable hypersurface in $\mathbb{R}^4$ with zero scalar curvature, polynomial volume growth and the restriction of the mean curvature and the Gauss-Kronecker curvature.

Motivatived by our recent work of hypersurfaces in spheres in [4,5], we study the global properties of complete noncompact 1-minimal stable hypersurfaces in a 4-dimensional sphere $\mathbb{S}^4$ in this paper. A Riemannian manifold M^3 has polynomial volume growth, if there exists $\gamma \in (0,3]$ such that $\lim_{r\to\infty} \frac{\mathrm{vol} B_r(p)}{r^\gamma} < +\infty$, for all $p \in M$, where $B_r(p)$ is the geodesic ball of radius r in M. We show two non-existence results as follows:

Theorem 1.1 *There is no stable complete noncompact 1-minimal hypersurface M^3 in $\mathbb{S}^4$ with polynomial volume growth and such that the mean curvature H satisfying*

$$|H| \le \delta_1, \quad \big|\nabla(\frac{1}{H})\big| \le \delta_2,$$

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for any positive constants δ_1 and δ_2.

Theorem 1.2 *There is no stable complete noncompact 1-minimal hypersurface M^3 in $\mathbb{S}^4$ with polynomial volume growth and such that*

$$\frac{-K}{H^3} \geq \delta_1, \left|\nabla(\frac{1}{H})\right| \leq \delta_2,$$

for any positive constants δ_1 and δ_2, where H and K are the mean curvature and the Gauss-Kronecker curvature, respectively.

2. Preliminaries

Let M^3 be a complete Riemannian manifold and let $x: M^3 \to \mathbb{S}^4$ be an isometric immersion into the sphere $\mathbb{S}^4$ with constant scalar curvature. We choose a unit normal field N to M and define the shape operator A associated with the second fundamental form of M, i.e., for any $p \in M$

$$A: \ T_pM \to T_pM$$

satisfies $\langle A(X), Y\rangle = -\langle \bar{\nabla}_X N, Y\rangle$, where $\bar{\nabla}$ is the Riemannian connection in $\mathbb{S}^4$. Let $\lambda_1, \lambda_2, \lambda_3$ denote the eigenvalues of A. The r-th symmetric function of $\lambda_1, \lambda_2, \lambda_3$, denoted by S_r, is defined by

$$\begin{aligned}
&S_1 = \lambda_1 + \lambda_2 + \lambda_3,\\
&S_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3,\\
&S_3 = \lambda_1\lambda_2\lambda_3.
\end{aligned}$$

With the above notations, we call $H_r = \frac{S_r}{C_3^r}$ the r-mean curvature of the immersion. Obviously, $H_1 = H$ is the mean curvature and $K = H_3$ is the Gauss-Kronecker curvature. H_2 is, modulo a constant 1, the scalar curvature of M. The hypersurface M is called r-minimal if $H_{r+1} \equiv 0$.

It is well known that hypersurfaces with constant scalar curvature in space forms are critical point for a geometric variational problem, namely, that associated to the functional

$$\mathcal{A}_1(M) = \int_M S_1$$

under compactly supported variations that preserves the volume. Let

$$P_1 = S_1 Id - A : T_pM \to T_pM.$$

Obviously,

$$\mathrm{trace}(P_1) = 2S_1.$$

We obtain the second variational formula for hypersurfaces in $\mathbb{S}^4$ with constant 2-mean curvature [6]:

$$\frac{\mathrm{d}^2\mathcal{A}_1}{\mathrm{d}t}\Big|_{t=0} = \int_M \langle P_1(\nabla f), \nabla f\rangle - \int_M (S_1S_2 - 3S_3 + 2S_1)f^2,$$

for each $f \in C_c^\infty(M)$. It is known that M^3 is stable if and only if

$$\int_M (S_1S_2 - 3S_3 + 2S_1)f^2 \leq \int_M \langle P_1(\nabla f), \nabla f\rangle, \tag{2.1}$$

for each $f \in C_c^\infty(M)$.

3. Proof of main results

In this section, we will give the proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1 Suppose by contradiction there exists a complete noncompact stable hypersurface satisfying the condition of Theorem 1.1. By assumption, $S_1 = 3H$ is nonzero. We can choose an orientation such that $S_1 = 3H > 0$. There is a fact that $2S_1S_3 \le S_2^2$ which implies that $S_3 \le 0$. The operator P_1 is positive definite since H is positive [7]. Stability and 1-minimality of the hypersurface M imply that there is the following inequality:

$$\int_M (2S_1 - 3S_3) f^2 \le \int_M \langle P_1(\nabla f), \nabla f\rangle, \tag{3.1}$$

for each $f \in C_c^\infty(M)$. Choose $f = {S_1}^q \varphi$ for a positive constant q to be determined and $\varphi \in C_c^\infty(M)$. Since

$$\nabla f = q{S_1}^{q-1}\varphi \nabla S_1 + {S_1}^q \nabla\varphi,$$

we get that

$$\begin{aligned}\langle P_1(\nabla f), \nabla f\rangle =& \langle q{S_1}^{q-1}\varphi P_1(\nabla S_1) + {S_1}^q P_1(\nabla\varphi), (1+q){S_1}^{q-1}\varphi\nabla S_1 + {S_1}^q\nabla\varphi\rangle\\ =& q^2 {S_1}^{2q-2}\varphi^2\langle P_1(\nabla S_1), \nabla S_1\rangle + 2q{S_1}^{2q-1}\varphi\langle P_1(\nabla S_1), \nabla\varphi\rangle +\\ & {S_1}^{2q}\langle P_1(\nabla\varphi), \nabla\varphi\rangle. \end{aligned}\tag{3.2}$$

Since P_1 is positive definite, we obtain that

$$\begin{aligned}& 2q{S_1}^{2q-1}\varphi\langle P_1(\nabla S_1), \nabla\varphi\rangle = {S_1}^{2q-2}\langle P_1(\varphi\nabla S_1), S_1\nabla\varphi\rangle\\ &= 2q{S_1}^{2q-2}\langle \sqrt{P_1}(\varphi\nabla S_1), \sqrt{P_1}(S_1\nabla\varphi)\rangle\\ &\le q{S_1}^{2q-2}(|\sqrt{P_1}(\varphi\nabla S_1)|^2 + |\sqrt{P_1}(S_1\nabla\varphi)|^2)\\ &= q{S_1}^{2q-2}\varphi^2\langle P_1(\nabla S_1, \nabla S_1)\rangle + q{S_1}^{2q}\langle P_1(\nabla\varphi), \nabla\varphi\rangle. \end{aligned}\tag{3.3}$$

By (3.1)–(3.3) and the fact $\langle P_1(X), X\rangle \le 2S_1|X|^2$, we get the following inequality:

$$\begin{aligned}\int_M (2S_1 - 3S_3){S_1}^{2q}\varphi^2 \le& (q^2+q)\int_M {S_1}^{2q-2}\varphi^2\langle P_1(\nabla S_1), \nabla S_1\rangle + \int_M (1+q){S_1}^{2q}\langle P_1(\nabla\varphi), \nabla\varphi\rangle\\ \le& 2(q^2+q)\int_M {S_1}^{2q-1}\varphi^2|\nabla S_1|^2 + 2(1+q)\int_M {S_1}^{2q+1}|\nabla\varphi|^2. \end{aligned}\tag{3.4}$$

We choose $\varphi = \phi^{\frac{3+2q}{2}}$ and get that

$$|\nabla\varphi|^2 = \frac{(3+2q)^2}{4}\phi^{1+2q}|\nabla\phi|^2. \tag{3.5}$$

Combining (3.4) with (3.5), we obtain that

$$\begin{aligned}\int_M (2S_1 - 3S_3){S_1}^{2q}\phi^{3+2q} \le& 2(q^2+q)\int_M {S_1}^{2q-1}\phi^{3+2q}|\nabla S_1|^2 +\\ & \frac{(1+q)(3+2q)^2}{2}\int_M {S_1}^{1+2q}\phi^{1+2q}|\nabla\phi|^2. \end{aligned}\tag{3.6}$$

Using Young's inequality, we have

$$
\begin{aligned}
{S_1}^{1+2q}\phi^{1+2q}|\nabla\phi|^2 &= (b{S_1}^{1+2q}\phi^{1+2q})\cdot(\frac{|\nabla\phi|^2}{b}) \\
&\le \frac{1+2q}{3+2q}b^{\frac{3+2q}{1+2q}}S_1^{3+2q}\phi^{3+2q}+\frac{2}{3+2q}b^{-\frac{3+2q}{2}}|\nabla\phi|^{3+2q},
\end{aligned}
\tag{3.7}
$$

for a positive constant b to be determined. Combining with (3.6), we have

$$
\begin{aligned}
&\int_M (2S_1-3S_3){S_1}^{2q}\phi^{3+2q}-2(q^2+q)\int_M {S_1}^{2q-1}\phi^{3+2q}|\nabla S_1|^2 \\
&\le \frac{(1+q)(3+2q)}{2}\int_M((1+2q)b^{\frac{3+2q}{1+2q}}S_1^{3+2q}\phi^{3+2q}+2b^{-\frac{3+2q}{2}}|\nabla\phi|^{3+2q}).
\end{aligned}
\tag{3.8}
$$

That is,

$$
\int_M \mathcal{A}S_1^{3+2q}\phi^{3+2q}\le\mathcal{B}\int_M|\nabla\phi|^{3+2q},
\tag{3.9}
$$

where

$$
\mathcal{A}=\frac{2}{S_1^2}+\frac{-3S_3}{S_1^3}-2(q^2+q)|\nabla(\frac{1}{S_1})|^2-\frac{(1+q)(3+2q)(1+2q)b^{\frac{3+2q}{1+2q}}}{2}
$$

and

$$
\mathcal{B}=(1+q)(3+2q)b^{-\frac{3+2q}{2}}>0.
$$

Since

$$
|H|\le\delta_1, \big|\nabla(\frac{1}{H})\big|\le\delta_2,
$$

we have

$$
|S_1|\le 3\delta_1, \big|\nabla(\frac{1}{S_1})\big|\le\frac{\delta_2}{3},
$$

which imply that

$$
\mathcal{A}\ge\frac{2}{9\delta_1^2}+\frac{-3S_3}{S_1^3}-\frac{2(q^2+q)\delta_2^2}{9}-\frac{(1+q)(3+2q)(1+2q)b^{\frac{3+2q}{1+2q}}}{2}.
\tag{3.10}
$$

Choosing q and b sufficiently small such that

$$
\frac{2}{9\delta_1^2}-\frac{2(q^2+q)\delta_2^2}{9}-\frac{(1+q)(3+2q)(1+2q)b^{\frac{3+2q}{1+2q}}}{2}>0.
$$

Combining (3.10) with the fact that $\frac{-3S_3}{S_1^3}\ge 0$, we get

$$
\mathcal{A}>0.
$$

Let ϕ be a function depending on the distance r with respect to a fixed point p,

$$
\phi(x)=\begin{cases} 1, & \text{on } B(R), \\ \dfrac{2R-r}{R}, & \text{on } B(2R)\setminus B(R), \\ 0, & \text{on } M\setminus B(2R). \end{cases}
$$

Combining with (3.9), we obtain that

$$
\int_{B(R)}\mathcal{A}{S_1}^{3+2q}\le\mathcal{B}\int_{B(2R)\setminus B(R)}\frac{1}{R^{3+2q}}\le\mathcal{B}\frac{\mathrm{vol}(B(2R))}{R^{3+2q}}.
\tag{3.11}
$$

Noting that M has polynomial volume growth and taking $R \to +\infty$, we obtain that $S_1 = 0$. This contradicts $S_1 \neq 0$. $\Box$

Proof of Theorem 1.2 Suppose by contradiction there exists a complete noncompact stable hypersurface satisfying the condition of Theorem 1.2. Following the same step as proof of Theorem 1.1, we still obtain the inequality (3.9). Since

$$\frac{-K}{H^3} \geq \delta_1, \big|\nabla(\frac{1}{H})\big| \leq \delta_2,$$

we get

$$\frac{-S_3}{S_1^3} \geq \frac{\delta_1}{27}, \big|\nabla(\frac{1}{S_1})\big| \leq \frac{\delta_2}{3}.$$

Thus,

$$\mathcal{A} = \frac{2}{S_1^2} + \frac{\delta_1}{9} - \frac{2(q^2+aq)\delta_2}{3} - \frac{(a+q)(3+2q)(1+2q)b^{\frac{3+2q}{1+2q}}}{2a}.$$

Choosing q and b sufficiently small such that

$$\frac{\delta_1}{9} - \frac{2(q^2+aq)\delta_2}{3} - \frac{(a+q)(3+2q)(1+2q)b^{\frac{3+2q}{1+2q}}}{2a} > 0.$$

Thus $\mathcal{A} > 0$. Let ϕ be a function depending on the distance r with respect to a fixed point p,

$$\phi(x) = \begin{cases} 1, & \text{on } B(R), \\ \dfrac{2R-r}{R}, & \text{on } B(2R)\setminus B(R), \\ 0, & \text{on } M\setminus B(2R). \end{cases}$$

Combining with (3.9), we obtain that

$$\int_{B(R)} \mathcal{A}{S_1}^{3+2q} \leq \mathcal{B}\int_{B(2R)\setminus B(R)} \frac{1}{R^{3+2q}} \leq \mathcal{B}\frac{\mathrm{vol}(B(2R))}{R^{3+2q}}. \tag{3.12}$$

Noting that M has polynomial volume growth and taking $R \to +\infty$, we obtain that $S_1 = 0$. This contradicts $S_1 \neq 0$. $\Box$

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