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Precise Large Deviation for the Difference of Non-Random Sums of NA Random Variables

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Abstract In this paper, we study precise large deviation for the non-random difference $\sum_{j=1}^{n_1(t)} X_{1j} - \sum_{j=1}^{n_2(t)} X_{2j}$, where $\sum_{j=1}^{n_1(t)} X_{1j}$ is the non-random sum of $\{X_{1j}, j \ge 1\}$ which is a sequence of negatively associated random variables with common distribution $F_1(x)$, and $\sum_{j=1}^{n_2(t)} X_{2j}$ is the non-random sum of $\{X_{2j}, j \ge 1\}$ which is a sequence of independent and identically distributed random variables, $n_1(t)$ and $n_2(t)$ are two positive integer functions. Under some other mild conditions, we establish the following uniformly asymptotic relation

$$\lim_{t\to\infty} \sup_{x\ge\gamma(n_1(t))^{p+1}} \Big|\frac{P(\sum_{j=1}^{n_1(t)} X_{1j} - \sum_{j=1}^{n_2(t)} X_{2j} - (\mu_1 n_1(t) - \mu_2 n_2(t)) > x)}{n_1(t)\bar{F}_1(x)} - 1\Big| = 0.$$

Keywords precise large deviation; negative association; consistently varying tail; difference

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1. Introduction

The study of precise large deviations with heavy tails is an important topic in insurance and finance, many researchers have achieved the asymptotic relation of precise large deviations

$$P(S_n - ES_n > x) \sim n\bar{F}(x),$$

which hold uniformly for some x-region T_n, where $\{X_n, n \ge 1\}$ is a sequence of random variables with a common distribution function F and a finite mean μ, and S_n is its n-th non-random sum, $n = 1, 2, \ldots$. The uniformity is understood in the following sense:

$$\lim_{n\to\infty} \sup_{x\in T_n} \Big|\frac{P(S_n - ES_n > x)}{n\bar{F}(x)} - 1\Big| = 0. \tag{1}$$

For recent works of precise large deviations with heavy tails, we refer the reader to [1–4].

We say X (or its distribution F) is heavy-tailed if it has no exponential moments. In risk theory, heavy-tailed distributions are often used to model large claims. Now, we recall some

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important subclasses of heavy-tailed distributions. A distribution F with support on $[0,\infty)$ belongs to the class $\mathcal{L}$, if

$$\lim_{x\to\infty}\frac{\bar{F}(x-y)}{\bar{F}(x)}=1$$

holds for all $y\in(-\infty,\infty)$. We call that F is a long-tailed distribution. A distribution F is said to belong to class $\mathcal{D}$, which consists of all distributions with dominated variation in the sense that the relation

$$\limsup_{x\to\infty}\frac{\bar{F}(xy)}{\bar{F}(x)}<\infty$$

holds for any $y\in(0,1)$ (or equivalently, for $y=1/2$). Another slightly smaller class is $\mathcal{C}$, which consists of all distributions with consistent variation in the sense that there holds the relation

$$\lim_{y\downarrow 1}\liminf_{x\to\infty}\frac{\bar{F}(xy)}{\bar{F}(x)}=1 \quad \text{or, equivalently,} \quad \lim_{y\uparrow 1}\limsup_{x\to\infty}\frac{\bar{F}(xy)}{\bar{F}(x)}=1.$$

It is easy to achieve the following inclusion relationship $\mathcal{C}\subset\mathcal{D}\cap\mathcal{L}$. For a distribution F, we define

$$\gamma(y):=\liminf_{x\to\infty}\frac{\bar{F}(xy)}{\bar{F}(x)} \quad \text{and} \quad \gamma_F:=\inf\Big\{-\frac{\log\gamma(y)}{\log y}:y>1\Big\}. \tag{2}$$

In [5], γ_F is called the upper Matuszewska index of the nonnegative and nondecreasing function $f(x)=(\bar{F}(x))^{-1}$, $x>0$. Without any danger of confusion, we simply call γ_F the upper Matuszewska index of the distribution function F. See Chapter 2.1 of [5] for more details of the Matuszewska index.

Here, we introduce some kinds of dependent structures. A sequence of random variables $\{X_n,n\geq 1\}$ is said to be extended lower negatively dependent ($ELND$) if there is some $M>0$ such that, for each $n=1,2,\ldots$ and all real numbers $x_1,x_2,\ldots,x_n$,

$$P(X_1\leq x_1,\ldots,X_n\leq x_n)\leq M\prod_{i=1}^{n}P(X_i\leq x_i);$$

it is said to be extended upper negatively dependent ($EUND$) if there is some $M>0$ such that, for each $n=1,2,\ldots$ and all real numbers $x_1,x_2,\ldots,x_n$,

$$P(X_1>x_1,\ldots,X_n>x_n)\leq M\prod_{i=1}^{n}P(X_i>x_i);$$

it is said to be extended negatively dependent (END) if they are both ($ELND$) and ($EUND$). When $M=1$, we call extended lower negatively dependent random variables $\{X_n,n\geq 1\}$ lower negatively dependent (LND) , and call extended upper negatively dependent random variables $\{X_n,n\geq 1\}$ upper negatively dependent (UND); if it is both (LND) and (UND), it is said to be negatively dependent (ND). We say random variables $\{X_i,i=1,\ldots,n\}$ Negatively Associated (NA) if for every pair of disjoint subsets A_1,A_2 of $\{1,2,\ldots,n\}$,

$$\mathrm{Cov}\{f_1(X_i,i\in A_1),f_2(X_i,i\in A_2)\}\leq 0,$$

where f_1 and f_2 are increasing functions. See [3] and [6] for more details for these kinds of dependent random variable sequences.

In [7], the precise large deviation (1) called one-risk model large deviations is extended to the multi-risk model, that is

$$\lim_{\substack{n_i\to\infty\\ i=1,2,\ldots,k}} \sup_{\substack{x\geq\max\{\gamma n_i,\\ i=1,2,\ldots,k\}}} \Big|\frac{P(\sum_{i=1}^k\sum_{j=1}^{n_i}X_{ij}-\sum_{i=1}^k n_i\mu_i>x)}{\sum_{i=1}^k n_i\bar{F}_i(x)}-1\Big|=0.$$

We refer the reader to [7] for more details. [8] established asymptotic formula for the multi-risk model in the case where the distribution of X belongs to subexponential distribution, [9] derived lower bounds of large deviation for sums of long-tailed claims in a multi-risk model, and [10] studied local precise large deviations for independent sums in a multi-risk model. In addition, X and Y are independent random variables where X is distributed by F, and Y is non-negative and non-degenerate at 0. From [11], if $F\in\mathcal{L}$, there holds that

$$P(X-Y>x)\sim\bar{F}(x)\ \ \text{as}\ \ x\to\infty. \tag{3}$$

Assume that there are two types of insurance contracts in an insurance company, the i-th related claims are denoted by $\{X_{ij}, j\geq 1\}$ which are independent and identically distributed random variables, $i=1,2$. The relation $|\sum_{j=1}^{n_1}X_{1j}-\sum_{j=1}^{n_2}X_{2j}|>x$ as $x\to\infty$ shows that one type of insurance claim is much larger than another and it is easier to drive the insurance company to ruin. Thus, the insurance company should pay more attention to this contract. Unfortunately, uptill now, there is little research about the asymptotic behaviors of $\sum_{j=1}^{n_1}X_{1j}-\sum_{j=1}^{n_2}X_{2j}$.

Motivated by the above mentioned papers and these facts, it is obvious to consider the difference of the sums of random variables. In the present work, we aim to deal with the asymptotic behaviors of $\sum_{j=1}^{n_1}X_{1j}-\sum_{j=1}^{n_2}X_{2j}$. The rest of the paper is organized as follows. In Section 2, we give some available propositions . The main results are presented in Section 3.

2. Preliminaries

This section presents several useful propositions, some of them will be used in Section 3.

Proposition 2.1 ([6]) *Let $A_1,\ldots,A_m$ be disjoint subsets of $\{1,2,\ldots,k\}$ and $f_1,f_2,\ldots,f_m$ be increasing positive functions. Then $X_1,X_2,\ldots,X_n$ NA implies*

$$E\prod_{i=1}^m f_i(X_j, j\in A_i)\leq\prod_{i=1}^m Ef_i(X_j, j\in A_i).$$

Proposition 2.2 ([6]) *An immediate consequence of Proposition 2.1 is that for A_1,A_2 disjoint subsets of $\{1,2,\ldots,k\}$, and $x_1,\ldots,x_k$ real,*

$$P(X_i\leq x_i, i=1,2,\ldots,k)\leq P(X_i\leq x_i, i\in A_1)P(X_j\leq x_j, j\in A_2),$$

and

$$P(X_i> x_i, i=1,2,\ldots,k)\leq P(X_i> x_i, i\in A_1)P(X_j> x_j, j\in A_2).$$

Remark 2.3 By Proposition 2.2 we know that $X_1,X_2,\ldots,X_n$ NA implies ND.

Proposition 2.4 *Let $\{X_n, n\geq 1\}$ be NA with common distribution function $F\in\mathcal{C}$ and mean*

0 *satisfying the condition*

$$F(-x) = o(\bar{F}(x)), \text{ as } x \to \infty. \tag{4}$$

And there exists some $r > 1$ *such that* $E(X_1^-)^r < \infty$. *Then*

$$P(S_n > x) \sim n\bar{F}(x), \text{ as } n \to \infty$$

holds uniformly for $x \geq \gamma n$, *for any fixed* $\gamma > 0$.

Remark 2.5 This proposition is the case of NA and identically distributed random variables in the theorem 3.1 of [3]. Let $\{Y_k, k = 1, 2, \ldots\}$ be a sequence of nonnegative and NA random variables with common distribution $F \in \mathcal{C}$ and finite mean $\mu > 0$. If $X_k = Y_k - \mu$, then the relation (4) holds automatically. By (5), for any fixed $\gamma > 0$, the relation

$$P\Big(\sum_{k=1}^{n} Y_k - n\mu > x\Big) = P\Big(\sum_{k=1}^{n} X_k > x\Big) \sim n\bar{F}(x)$$

holds uniformly for all $x \geq \gamma n$ as $n \to \infty$.

Proposition 2.6 ([2]) *For* $F \in \mathcal{D}$ *and every* $p > \gamma_F$, *there exist positive* x_0 *and* B *such that, for all* $\theta \in (0, 1]$ *and all* $x \geq \theta^{-1}x_0$, $\frac{\bar{F}(\theta x)}{\bar{F}(x)} \leq B\theta^{-p}$.

Proposition 2.7 ([2]) *For a distribution function* $F \in \mathcal{D}$ *with a finite expectation,* $1 \leq \gamma_F < \infty$ *and for any* $\gamma > \gamma_F$, $x^{-\gamma} = o(\bar{F}(x))$, *as* $x \to \infty$.

3. Main results

In this section, we are ready to state the main results of this paper and their proofs.

Lemma 3.1 *Let* $\{X_i, i = 1, 2, \ldots, n\}$ *be NA random variables, and* $\{Y_i, i = 1, 2, \ldots, k\}$ *be independent and identically distributed random variables,* $1 \leq k \leq n$. *If* $\{X_i, i = 1, 2, \ldots, n\}$ *and* $\{Y_i, i = 1, 2, \ldots, k\}$ *are mutually independent, then* $\{X_1 - Y_1, \ldots, X_k - Y_k, X_{k+1}, \ldots, X_n\}$ *are* NA, *and they are also* ND.

Proof Let A_1, A_2 be disjoint subsets of $\{1, 2, \ldots, n\}$. We have

$$\begin{aligned}
&\operatorname{cov}\{f_1(X_i - Y_i, i \in A_1), f_2(X_j - Y_j, j \in A_2)\} \\
&\quad = E\left[f_1(X_i - Y_i, i \in A_1) f_2(X_j - Y_j, j \in A_2)\right] - E\left[f_1(X_i - Y_i, i \in A_1)\right] E\left[f_2(X_j - Y_j, j \in A_2)\right] \\
&\quad = E\left\{E\left[f_1(X_i - Y_i, i \in A_1) f_2(X_j - Y_j, j \in A_2) | Y_1, Y_2, \ldots, Y_k\right]\right\} - \\
&\qquad E\left[f_1(X_i - Y_i, i \in A_1)\right] E\left[f_2(X_j - Y_j, j \in A_2)\right] \\
&\quad \leq E\left\{E\left[f_1(X_i - Y_i, i \in A_1) | Y_1, Y_2, \ldots, Y_k\right]\right\} E\left\{E\left[f_2(X_j - Y_j, j \in A_2) | Y_1, Y_2, \ldots, Y_k\right]\right\} - \\
&\qquad E\left[f_1(X_i - Y_i, i \in A_1)\right] E\left[f_2(X_j - Y_j, j \in A_2)\right] \\
&\quad = E\left\{E\left[f_1(X_i - Y_i, i \in A_1) | Y_i, i \in A_1\right]\right\} E\left\{E\left[f_2(X_j - Y_j, j \in A_2) | Y_j, j \in A_2\right]\right\} - \\
&\qquad E\left[f_1(X_i - Y_i, i \in A_1)\right] E\left[f_2(X_j - Y_j, j \in A_2)\right] \\
&\quad = E\left[f_1(X_i - Y_i, i \in A_1)\right] E\left[f_2(X_j - Y_j, j \in A_2)\right] -
\end{aligned}$$

$$
\begin{aligned}
&E\left[f_1(X_i-Y_i,i\in A_1)\right]E\left[f_2(X_j-Y_j,j\in A_2)\right]\\
=&0.
\end{aligned}
$$

By the definition of NA and Remark 2.3, we know that $\{X_1-Y_1,\dots,X_k-Y_k,X_{k+1},\dots,X_n\}$ are NA, and then $\{X_1-Y_1,\dots,X_k-Y_k,X_{k+1},\dots,X_n\}$ are ND.

Lemma 3.2 *Let X be a nonnegative random variable with a distribution $F_1\in\mathcal{C}$ and finite mean $\mu_1>0$, and Let Y be a nonnegative random variable with a distribution F_2. For some $\rho>\gamma_{F_1}\geq 1$, $EY^\rho<\infty$, where γ_{F_1} is defined as (2). If X and Y is independent, X satisfies the condition (4), then the relation*

$$xP\left(X-Y-(\mu_1-\mu_2)<-x\right)=o(P\left(X-Y-(\mu_1-\mu_2)>x\right))$$

holds as $x\to\infty$.

Proof For any fixed v $(0<v<1)$, we have

$$
\begin{aligned}
\frac{xP(X-Y-(\mu_1-\mu_2)<-x)}{P(X-Y-(\mu_1-\mu_2)>x)}=&\frac{x\int_0^{vx}P(X\leq -x+(\mu_1-\mu_2)+u)\mathrm{d}F_2(u)}{P(X-Y-(\mu_1-\mu_2)>x)}+\\
&\frac{x\int_{vx}^{+\infty}P(X\leq -x+(\mu_1-\mu_2)+u)\mathrm{d}F_2(u)}{P(X-Y-(\mu_1-\mu_2)>x)}\\
=&K_1+K_2.
\end{aligned}
$$

To estimate K_1, it follows from the condition (4) and $F_1\in\mathcal{C}\subset\mathcal{D}$ that

$$
\begin{aligned}
K_1\leq&\frac{xP(X\leq -x+(\mu_1-\mu_2)+vx)F_2(vx)}{P(X-Y-(\mu_1-\mu_2)>x,Y\leq vx)}\\
\leq&\frac{xF_1((v-1)x+(\mu_1-\mu_2))}{\bar F_1((1-v)x+(\mu_1-\mu_2))}\cdot\frac{\bar F_1((1-v)x+(\mu_1-\mu_2))}{\bar F_1((v+1)x+(\mu_1-\mu_2))}\to 0,\quad\text{as } x\to\infty.
\end{aligned}
$$

Using $EY^{\rho+1}<\infty$ and Proposition 2.7 gives

$$
\begin{aligned}
K_2\leq&\frac{x\bar F_2(vx)}{P(X-Y-(\mu_1-\mu_2)>x,X\geq x+\mu_1)}\\
\leq&\frac{x\bar F_2(vx)}{F_2(\mu_2)\bar F_1(x+\mu_1)}\\
\leq&\frac{x^{\rho+1}\bar F_2(vx)}{F_2(\mu_2)x^\rho\bar F_1(x+\mu_1)}\to 0,\quad\text{as } x\to\infty.
\end{aligned}
$$

So, the proof of the lemma is completed. □

Remark 3.3 Lemma 3.2 means that the left tail of $X-Y$ is lighter than its right tail.

Theorem 3.4 *Let $\{X_{1j},j\geq 1\}$ be a sequence of NA and nonnegative and identically distributed random variables with common distribution function $F_1\in\mathcal{C}$, finite mean $0<\mu_1<\infty$, and $\gamma_{F_1}\geq 1$. Also let $\{X_{2j},j\geq 1\}$ be a sequence of nonnegative and independent and identically distributed random variables with common distribution function F_2. For some $p>\gamma_{F_1}$, $EX_{2j}^{p+1}<\infty$. For $i=1,2$, $n_i(t)$ is a positive integer function, $n_i(t)\to\infty$ as $t\to\infty$. We assume that $\{X_{ij},j\geq 1\}_{i=1}^2$ are mutually independent, $S^i_{1,n_i(t)}=\sum_{j=1}^{n_i(t)}X_{ij}$ is partial sum of $\{X_{ij},j\geq 1\}$, $i=1,2$. If $n_1(t)$*

and $n_2(t)$ *satisfy*

$$\lim_{t\to\infty}\frac{n_2(t)}{n_1(t)}=a,\quad 0\le a<\infty, \tag{5}$$

then for any fixed $\gamma>0$ *and some* $p>\gamma_{F_1}$, *we have*

$$P(S^1_{1,n_1(t)}-S^2_{1,n_2(t)}-(\mu_1 n_1(t)-\mu_2 n_2(t))>x)\sim n_1(t)\bar F_1(x) \tag{6}$$

holds uniformly for all $x\ge\gamma(n_1(t))^{p+1}$ *as* $t\to\infty$. *That is,*

$$\lim_{t\to\infty}\sup_{x\ge\gamma(n_1(t))^{p+1}}\Big|\frac{P(S^1_{1,n_1(t)}-S^2_{1,n_2(t)}-(\mu_1 n_1(t)-\mu_2 n_2(t))>x)}{n_1(t)\bar F_1(x)}-1\Big|=0.$$

Proof Throughout this section, all limit relationships, unless otherwise stated, are as $t\to\infty$. Here we divide our proof into two parts. Firstly, we give the proof under the condition $0\le a<1$. Secondly, we give the proof under the condition $1\le a<\infty$.

Firstly, we deal with the case of $0\le a<1$. From (5), there exists a large enough $t_0>0$, when $t>t_0>0$, we have

$$n_1(t)\ge n_2(t). \tag{7}$$

Under this condition, the asymptotic relation (6) holds if and only if for any fixed $\gamma>0$,

$$\liminf_{t\to\infty}\inf_{x\ge\gamma(n_1(t))^{p+1}}\frac{P(S^1_{1,n_1(t)}-S^2_{1,n_2(t)}-(\mu_1 n_1(t)-\mu_2 n_2(t))>x)}{n_1(t)\bar F_1(x)}\ge 1 \tag{8}$$

and

$$\limsup_{t\to\infty}\sup_{x\ge\gamma(n_1(t))^{p+1}}\frac{P(S^1_{1,n_1(t)}-S^2_{1,n_2(t)}-(\mu_1 n_1(t)-\mu_2 n_2(t))>x)}{n_1(t)\bar F_1(x)}\le 1. \tag{9}$$

Now, we prove (8). Let

$$S^{[1-2]}_{1,n_2(t)}=\sum_{j=1}^{n_2(t)}(X_{1j}-X_{2j}),$$

$$S^1_{n_2(t)+1,n_1(t)}=\sum_{j=n_2(t)+1}^{n_1(t)}X_{1j},\, n^{[1-2]}(t)=n_1(t)-n_2(t),\ \mu^{[1-2]}=\mu_1-\mu_2.$$

Using (7), for any $0<\varepsilon<1$, we get

$$\begin{aligned}
&P(S^1_{1,n_1(t)}-S^2_{1,n_2(t)}-(\mu_1 n_1(t)-\mu_2 n_2(t))>x)\\
&=P\Big(S^{[1-2]}_{1,n_2(t)}+S^1_{n_2(t)+1,n_1(t)}>x+\mu^{[1-2]}n_2(t)+\mu_1\cdot n^{[1-2]}(t)\Big)\\
&\ge P\Big(\{S^{[1-2]}_{1,n_2(t)}>(1+\varepsilon)x+\mu^{[1-2]}n_2(t),S^1_{n_2(t)+1,n_1(t)}>-\varepsilon x+\mu_1\cdot n^{[1-2]}(t)\}\bigcup\\
&\quad\{S^1_{n_2(t)+1,n_1(t)}>(1+\varepsilon)x+\mu_1\cdot n^{[1-2]}(t),S^{[1-2]}_{1,n_2(t)}>-\varepsilon x+\mu^{[1-2]}n_2(t)\}\Big)\\
&\ge P\Big(S^{[1-2]}_{1,n_2(t)}>(1+\varepsilon)x+\mu^{[1-2]}n_2(t),S^1_{n_2(t)+1,n_1(t)}>-\varepsilon x+\mu_1\cdot n^{[1-2]}(t)\Big)+\\
&\quad P\Big(S^1_{n_2(t)+1,n_1(t)}>(1+\varepsilon)x+\mu_1\cdot n^{[1-2]}(t),S^{[1-2]}_{1,n_2(t)}>-\varepsilon x+\mu^{[1-2]}n_2(t)\Big)-\\
&\quad P\Big(\{S^{[1-2]}_{1,n_2(t)}>(1+\varepsilon)x+\mu^{[1-2]}n_2(t)x\}\bigcap\{S^1_{n_2(t)+1,n_1(t)}>(1+\varepsilon)x+\mu_1\cdot n^{[1-2]}(t)\}\Big)
\end{aligned}$$

$$\begin{aligned}&\geq P\Big(S_{1,n_2(t)}^{[1-2]}>(1+\varepsilon)x+\mu^{[1-2]}n_2(t)\Big)+P\Big(S_{n_2(t)+1,n_1(t)}^{1}>(1+\varepsilon)x+\mu_1\cdot n^{[1-2]}(t)\Big)-\\&\quad P\Big(S_{1,n_2(t)}^{[1-2]}>(1+\varepsilon)x+\mu^{[1-2]}n_2(t),S_{n_2(t)+1,n_1(t)}^{1}\leq-\varepsilon x+\mu_1\cdot n^{[1-2]}(t)\Big)-\\&\quad P\Big(S_{n_2(t)+1,n_1(t)}^{1}>(1+\varepsilon)x+\mu_1\cdot n^{[1-2]}(t),S_{1,n_2(t)}^{[1-2]}\leq-\varepsilon x+\mu^{[1-2]}n_2(t)\Big)-\\&\quad P\Big(S_{1,n_2(t)}^{[1-2]}>(1+\varepsilon)x+\mu^{[1-2]}n_2(t)\Big)P\Big(S_{n_2(t)+1,n_1(t)}^{1}>(1+\varepsilon)x+\mu_1\cdot n^{[1-2]}(t)\Big)\\&:=I_1+I_2-I_3-I_4-I_1\cdot I_2,\end{aligned}$$

where we use Lemma 3.1 and Proposition 2.1 in the third inequality. Since $F_1\in C$, we have

$$\lim_{\varepsilon\downarrow0}\lim_{t\to\infty}\limsup_{x\geq\gamma n_1(t)}\Big|\frac{\bar F_1(1+\varepsilon)x}{\bar F_1(x)}-1\Big|=0.$$

Then for any $0<\delta<1$, all sufficiently small ε and (7), we have

$$\bar F_1((1+\varepsilon)x)>(1-\delta)\bar F_1(x)\tag{10}$$

holds uniformly for $x\geq\gamma n_1(t)$. By Lemmas 3.1 and 3.2, Proposition 2.4, (3), (10) and (7), for any $0<\delta<1$ and any fixed $\gamma>0$, we have

$$(1+\delta)n_2(t)\bar F_1((1+\varepsilon)x)>I_1>(1-\delta)n_2(t)\bar F_1((1+\varepsilon)x)>(1-\delta)^2n_2(t)\bar F_1(x)\tag{11}$$

holds uniformly for all $x\geq\gamma n_2(t)$. Similarly, noting that $n^{[1-2]}(t)\to\infty$ as $t\to\infty$, for large enough $t>0$ we have

$$(1+\delta)^2n^{[1-2]}(t)\bar F_1((1+\varepsilon)x)>I_2>(1-\delta)^2n^{[1-2]}(t)\bar F_1(x)\tag{12}$$

holds uniformly for all $x\geq\gamma n^{[1-2]}(t)$. According to finite mean $0<\mu_1<\infty$, it is easy to check that, for any fixed $\gamma>0$,

$$\lim_{t\to\infty}\sup_{x\geq\gamma n_1(t)}n_1(t)\bar F_1(x)=0.$$

Note that

$$\begin{aligned}&\lim_{t\to\infty}\sup_{x\geq\gamma n_1(t)}\frac{n^{[1-2]}(t)n_2(t)\bar F_1^2((1+\varepsilon)x)}{n_1(t)\bar F_1(x)}\\&\quad=\lim_{t\to\infty}\sup_{x\geq\gamma n_1(t)}\frac{n^{[1-2]}(t)n_2(t)}{(n_1(t))^2}\cdot\frac{\bar F_1((1+\varepsilon)x)}{\bar F_1(x)}\cdot n_1(t)\bar F_1(x)=0.\end{aligned}$$

So we arrive at

$$I_1\cdot I_2=o(n_1(t)\bar F_1(x))\tag{13}$$

holds uniformly for $x\geq\gamma n_1(t)$. Here we show that

$$I_3+I_4=o(n_1(t)\bar F_1(x))\tag{14}$$

holds uniformly for $x\geq\gamma(n_1(t))^{p+1}$. Since for any fixed $\epsilon>0$, there exists a positive B such that

$$\frac{I_4}{n_1(t)\bar F_1(x)}\leq\frac{P(S_{1,n_2(t)}^{[1-2]}\leq-\varepsilon x+\mu^{[1-2]}n_2(t))}{n_2(t)\bar F_1(x)}\leq\frac{n_2(t)P(X_{11}-X_{21}-\mu^{[1-2]}\leq\frac{-\varepsilon x}{n_2(t)})}{n_2(t)\bar F_1(x)}$$

$$
\begin{aligned}
&\le \frac{P(X_{11}-X_{21}-(\mu_1-\mu_2)\le \frac{-\varepsilon x}{n_2(t)})}{P(X_{11}-X_{21}-(\mu_1-\mu_2)> \frac{\varepsilon x}{n_2(t)})}\cdot \frac{P(X_{11}>\frac{\varepsilon x}{n_2(t)}+(\mu_1-\mu_2))}{P(X_{11}>\frac{\varepsilon x}{n_2(t)})}\frac{P(X_{11}>\frac{\varepsilon x}{n_2(t)})}{\bar{F}_1(x)}\\
&\le \frac{\frac{\varepsilon x}{n_2(t)}P(X_{11}-X_{21}-(\mu_1-\mu_2)\le \frac{-\varepsilon x}{n_2(t)})}{P(X_{11}-X_{21}-(\mu_1-\mu_2)> \frac{\varepsilon x}{n_2(t)})}\cdot \frac{P(X_{11}>\frac{\varepsilon x}{n_2(t)}+(\mu_1-\mu_2))}{P(X_{11}>\frac{\varepsilon x}{n_2(t)})}\cdot B(\frac{\varepsilon}{n_2(t)})^{-p-1}x^{-1}\\
&= o(1)
\end{aligned}
$$

holds uniformly for $x \ge \gamma(n_1(t))^{p+1}$, where the fourth inequality uses Proposition 2.6, and the last step holds due to Lemma 3.2 and $F_1 \in \mathcal{C} \subset \mathcal{D}$. Similarly,

$$I_3 = o(n_1(t)\bar{F}_1(x))$$

holds uniformly for $x \ge \gamma(n_1(t))^{p+1}$. Combining (11)–(14), we arrive at

$$P\Big(S^{[1-2]}_{1,n_2(t)}+S^1_{n_2(t)+1,n_1(t)} > x+\mu^{[1-2]}n_2(t)+\mu_1\cdot n^{[1-2]}(t)\Big) \ge (1-\delta)^2 n_1(t)\bar{F}_1(x)+o(n_1(t)\bar{F}_1(x)).$$

Therefore, letting $\delta \downarrow 0$, (8) follows.

Next we will prove (9). Here we use the same ε which appeared above, there is

$$
\begin{aligned}
&P\Big(S^{[1-2]}_{1,n_2(t)}+S^1_{n_2(t)+1,n_1(t)} > x+\mu^{[1-2]}n_2(t)+\mu_1 n^{[1-2]}(t)\Big)\\
&\le P\Big(\{S^{[1-2]}_{1,n_2(t)} > (1-\varepsilon)x+\mu^{[1-2]}n_2(t)\}\bigcup\{S^1_{n_2(t)+1,n_1(t)} > (1-\varepsilon)x+\mu_1 n^{[1-2]}(t)\}\bigcup\\
&\quad \{S^{[1-2]}_{1,n_2(t)} > \varepsilon x+\mu^{[1-2]}n_2(t), S^1_{n_2(t)+1,n_1(t)} > \varepsilon x+\mu_1 n^{[1-2]}(t)\}\Big)\\
&\le P\Big(S^{[1-2]}_{1,n_2(t)} > (1-\varepsilon)x+\mu^{[1-2]}n_2(t)\Big)+P\Big(S^1_{n_2(t)+1,n_1(t)} > (1-\varepsilon)x+\mu_1 n^{[1-2]}(t)\Big)+\\
&\quad P\Big(S^{[1-2]}_{1,n_2(t)} > \varepsilon x+\mu^{[1-2]}n_2(t), S^1_{n_2(t)+1,n_1(t)} > \varepsilon x+\mu_1 n^{[1-2]}(t)\Big)\\
&\le P\Big(S^{[1-2]}_{1,n_2(t)} > (1-\varepsilon)x+\mu^{[1-2]}n_2(t)\Big)+P\Big(S^1_{n_2(t)+1,n_1(t)} > (1-\varepsilon)x+\mu_1 n^{[1-2]}(t)\Big)+\\
&\quad P\Big(S^{[1-2]}_{1,n_2(t)} > \varepsilon x+\mu^{[1-2]}n_2(t)\Big)P\Big(S^1_{n_2(t)+1,n_1(t)} > \varepsilon x+\mu_1 n^{[1-2]}(t)\Big)\\
&\le (1+\delta)n_1(t)\bar{F}_1((1-\epsilon)x)+(1+\delta)^2 n^{[1-2]}(t)n_2(t)(\bar{F}_1(\epsilon x))^2\\
&\le (1+\delta)^2 n_1(t)\bar{F}_1(x)+o\big(n_1(t)\left(\bar{F}_1(x)\right)\big).
\end{aligned}
$$

To get last formula we use the same method in the proof of the lower bound. Letting $\delta \downarrow 0$, we get (9). Then we complete the proof of the case of $0 \le a < 1$. □

At last let us deal with the case of $1 < a < \infty$. From (5), we rewrite $S^1_{n_1(t)} - S^2_{n_2(t)}$ as

$$
\begin{aligned}
S^1_{n_1(t)} - S^2_{n_2(t)} &= \sum_{j=1}^{n_1(t)} X_{1j} - \sum_{j=1}^{[a]n_1(t)} X_{2j} - \sum_{j=[a]n_1(t)+1}^{n_2(t)} X_{2j}\\
&= S^{[a]}_{1,n_1(t)} - S^2_{[a]n_1(t)+1,n_2(t)},
\end{aligned}
$$

where

$$S^{[a]}_{1,n_1(t)} = \sum_{j=1}^{n_1(t)}(X_{1j} - \sum_{k=(j-1)[a]+1}^{j[a]} X_{2k}),$$

and

$$S^2_{[a]n_1(t)+1,n_2(t)} = \sum_{j=[a]n_1(t)+1}^{n_2(t)} X_{2j}.$$

Using Lemma 3.1, we obtain $\{X_{1j} - \sum_{k=(j-1)[a]+1}^{j[a]} X_{2k}, X_{2i},\ 1 \le j \le n_1(t), [a]n_1(t)+1 \le i \in n_2(t)\}$ are NA. Referring to the discuss in the case of $0 \le a < 1$, we have

$$P(S^1_{1,n_1(t)} - S^2_{1,n_2(t)} - \mu_1 n_1(t) + \mu_2 n_2(t) > x) \sim n_1(t)\bar{F}_1(x).$$

If $a = 1$, when $n_1(t) \ge n_2(t)$, by Lemma 3.1, $\{X_1 - Y_1, \ldots, X_{n_2(t)} - Y_{n_2(t)}, X_{n_2(t)+1}, \ldots, X_{n_1(t)}\}$ are NA, this argument is similar to the case of $0 \le a < 1$; when $n_1(t) = n_2(t)$, by Lemma 3.1, $\{X_1 - Y_1, \ldots, X_{n_2(t)} - Y_{n_2(t)}\}$ are NA, they are also ND, Theorem 3.4 holds by Proposition 2.4; otherwise the argument is similar to the case of $1 < a < \infty$. Then the proof of Theorem 3.4 is completed. □

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References

[1] Lixin SONG, Jinghai FENG, Liangliang YUAN, et al. *Precise large deviations for compound renewal risk model with negative dependence claims*. J. Dalian. Univ. Technol., 2014, **54**(6): 696–701.

[2] K. W. NG, Qihe TANG, Jia-an YAN, et al. *Precise large deviations for sums of random variables with consistently varying tails*. J. Appl. Probab., 2004, **41**(1): 93–107.

[3] Li LIU. *Precise Large Deviations for Dependent random variables with heavy tails*. Statist. Prob. Lett., 2009, **79**(9): 1290–1298.

[4] Zhiqiang HUA, Lixin SONG, Jinghai FENG. *Precise large deviations for random sum of UEND and ϕ-mixing random variables*. J. Dalian Univ. Technol., 2014, **54**(3): 371–376.

[5] N. H. BINGHAM, C. M. GOLDIE, J. L. TEUGELS. *Regular Variation*. Cambridge University Press, 1987.

[6] J. D. KUMAR, P. FRANK. *Negative association of random variables with applications*. Ann. Statist., 1983, **11**(1): 286–295.

[7] Shijie WANG, Wensheng WANG. *Precise large deviations for sums of random variables with consistently varying tails in multi-risk models*. J. Appl. Probab., 2007, **44**(4): 889–900.

[8] Dawei LU. *Lower and upper bounds of large deviation for sums of subexponential claims in a multi-risk model*. Statist. Probab. Lett., 2011, **81**(12): 1911–1919.

[9] Dawei LU. *Lower bounds of large deviation for sums of long-tailed claims in a multi-risk model*. Statist. Probab. Lett., 2012, **82**(7): 1242–1250.

[10] Jinghai FENG, Panpan ZHAO, Libin JIAO. *Local precise large deviations for independent sums in multi-risk model*. J. Math. Res. Appl., 2014, **34**(2): 696–701.

[11] Qihe TANG. *The ruin probability of a discrete time risk model under constant interest rate with heavy tails*. Scand. Actuar. J., 2004, **3**: 229–240.