*Journal of Mathematical Research with Applications*
Jan., 2017, Vol. 37, No. 1, pp. 40–46
DOI:10.3770/j.issn:2095-2651.2017.01.004
Http://jmre.dlut.edu.cn

# The Minimal Measurement Number Problem in Phase Retrieval: A Review of Recent Developments

**Zhiqiang XU**[1,2]
1. *LSEC, Inst. Comp. Math., Academy of Mathematics and System Science, Chinese Academy of Sciences, Beijing* 100091*, P. R. China;*
2. *University of Chinese Academy of Sciences, Beijing,* 100049*, P. R. China*

Dedicated to Professor Renhong WANG on the Occasion of His Eightieth Birthday

**Abstract** Phase retrieval is to recover the signals from phaseless measurements which is raised in many areas. A fundamental problem in phase retrieval is to determine the minimal measurement number $m$ so that one can recover $d$-dimensional signals from $m$ phaseless measurements. This problem attracts much attention of experts from different areas. In this paper, we review the recent development on the minimal measurement number and also raise many interesting open questions.

**Keywords** Phase retrieval; frames; measurement number; matrix recovery; bilinear form; algebraic geometry; embedding

**MR(2010) Subject Classification** 42C15; 94A12; 15A63; 15A83

## 1. Introduction

Suppose that $\mathbf{A} = (\mathbf{a}_1, \ldots, \mathbf{a}_m)^T \subset \mathbb{F}^{m\times d}$ and $\mathbf{x}_0 \in \mathbb{F}^d$ where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. We consider the linear equations $\mathbf{Ax} = \mathbf{Ax}_0$ where $\mathbf{x} \in \mathbb{F}^d$ is the unknown vector. One aim of linear algebra is to present the condition for $\mathbf{A}$ under which the solution to $\mathbf{Ax} = \mathbf{Ax}_0$ is $\mathbf{x}_0$. It is well known that the solution to the linear equations $\mathbf{Ax} = \mathbf{Ax}_0$ is $\mathbf{x}_0$ if and only if $\mathrm{rank}(\mathbf{A}) = d$. Today, the nonlinear equation $|\mathbf{Ax}| = |\mathbf{Ax}_0|$ is raised in many areas where

$$|\mathbf{Ax}| = (|\langle \mathbf{a}_1, \mathbf{x}\rangle|, \ldots, |\langle \mathbf{a}_m, \mathbf{x}\rangle|)^T \in \mathbb{F}^m.$$

Naturally, one is also interested in presenting the condition for $\mathbf{A}$ under which the solution to $|\mathbf{Ax}| = |\mathbf{Ax}_0|$ is unique. To find the solution to the nonlinear equations is called phase retrieval problem, which is raised in many practical areas, such as in X-ray imaging, crystallography, electron microscopy and coherence theory. Beyond that, phase retrieval has some fantastic connection with many pure mathematic topics, such as the dimension of algebraic variety, the nonsingular bilinear form and the embedding problem in topology [1]. Note that for any $c \in \mathbb{F}$ with $|c| = 1$ we have

$$|\mathbf{A}c\mathbf{x}_0| = |\mathbf{Ax}_0|.$$

Received November 28, 2016; Accepted December 19, 2016
Supported by the National Natural Science Foundation of China (Grant Nos. 11422113; 11331012; 91630203) and by National Basic Research Program of China (973 Program 2015CB856000).
E-mail address: xuzq@lsec.cc.ac.cn

We say the vector set $\{\mathbf{a}_1,\ldots,\mathbf{a}_m\}\subset\mathbb{R}^d$ or the matrix $\mathbf{A}$ is phase retrievable if

$$\{\mathbf{x}\in\mathbb{F}^d : |\mathbf{A}\mathbf{x}|=|\mathbf{A}\mathbf{x}_0|\}=\tilde{\mathbf{x}}_0:=\{c\mathbf{x}_0 : c\in\mathbb{F}, |c|=1\}.$$

In the context of phase retrieval, a fundamental problem is to present the minimal measurement number $m$ so that there exists $\mathbf{A}\in\mathbb{R}^{m\times d}$ which is phase retrievable. To state conveniently, we set

$$\mathfrak{m}_{\mathbb{F}}(d) := \min\{m : \text{ there exists } \mathbf{A}=(\mathbf{a}_1,\ldots,\mathbf{a}_m)^T\in\mathbb{F}^{m\times d} \text{ which is phase retrievable in } \mathbb{F}^d\}.$$

The aim of this paper is to review the recent developments about $\mathfrak{m}_{\mathbb{F}}(d)$ and also raise many open questions. The rest of the paper is organized as follows. In Section 2, we introduce the results of $\mathfrak{m}_{\mathbb{F}}(d)$ for $\mathbb{F}=\mathbb{R}$ and $\mathbb{C}$, respectively. We consider the case where $\mathbf{x}_0$ is $s$-sparse in Section 3. Finally, the results about generalized phase retrieval are introduced in Section 4.

## 2. Phase retrieval for general signals

### 2.1. Real case

The minimal measurement number problem with $\mathbb{F}=\mathbb{R}$ was investigated in [2] with presenting a condition for $\mathbf{A}$ under which $\mathbf{A}$ is phase retrievable. To this end, we set

$$\operatorname{span}(\mathbf{A}) := \operatorname{span}(\{\mathbf{a}_1,\ldots,\mathbf{a}_m\}) \quad\text{and}\quad \mathbf{A}_S := (\mathbf{a}_j : j\in S)^T$$

where $S\subset\{1,\ldots,m\}$. Then we have

**Theorem 2.1** ([2]) *Let $\mathbf{A}=(\mathbf{a}_1,\mathbf{a}_2,\ldots,\mathbf{a}_m)^T\in\mathbb{R}^{m\times d}$. The following properties are equivalent:*

*(A) $\mathbf{A}$ is phase retrievable on $\mathbb{R}^d$;*

*(B) For every subset $S\subset\{1,\ldots,m\}$, either* $\operatorname{span}(\mathbf{A}_S)=\mathbb{R}^d$ *or* $\operatorname{span}(\mathbf{A}_{S^c})=\mathbb{R}^d$.

If $m\le 2d-2$, then there exists $S_0\subset\{1,\ldots,m\}$ satisfying $\#S_0\le d-1$ and $\#S_0^c\le d-1$. Hence, $\operatorname{span}(\mathbf{A}_{S_0})\ne\mathbb{R}^d$ and $\operatorname{span}(\mathbf{A}_{S_0^c})\ne\mathbb{R}^d$. According to Theorem 2.1, if $\mathbf{A}$ is phase retrievable on $\mathbb{R}^d$, then we must have $m\ge 2d-1$. We next show that $2d-1$ is the minimal measurement number which means that there exists $\mathbf{A}\in\mathbb{R}^{(2d-1)\times d}$ satisfying (B) in Theorem 2.1. We set

$$\mathbf{A}_0=\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{d-1}\\ 1 & x_2 & x_2^2 & \cdots & x_2^{d-1}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 1 & x_{2d-1} & x_{2d-1}^2 & \cdots & x_{2d-1}^{d-1}\end{bmatrix}\in\mathbb{R}^{(2d-1)\times d}$$

where $x_1,\ldots,x_{2d-1}\in\mathbb{R}$ are distinct from each other. A simple observation is that $\mathbf{A}_0$ has the property (B) in Theorem 2.1 which implies that $\mathfrak{m}_{\mathbb{R}}(d)=2d-1$.

### 2.2. Complex case

For the case where $\mathbb{F}=\mathbb{C}$, the minimal measurement number problem remains open. In [2], it was shown that $\mathbf{A}=(\mathbf{a}_1,\ldots,\mathbf{a}_m)^T\in\mathbb{C}^{m\times d}$ is phase retrievable provided $m\ge 4d-2$ and $\mathbf{a}_1,\ldots,\mathbf{a}_m$ are $m$ generic vectors in $\mathbb{C}^d$. In [3], a matrix $\mathbf{A}\in\mathbb{C}^{(4d-4)\times d}$ was constructed and the authors also show the matrix $\mathbf{A}$ is phase retrievable. The result presents an upper bound of the minimal measurement number on $\mathbb{C}^d$, i.e., $\mathfrak{m}_{\mathbb{C}}(d) \le 4d-4$. In [4], one investigated the minimal measurement number with employing the results from algebraic geometry. Note that $|\langle \mathbf{a}_j, \mathbf{x}_0\rangle|^2 = Tr(\mathbf{a}_j\mathbf{a}_j^*\mathbf{x}_0\mathbf{x}_0^*)$ where $\mathbf{a}_j\mathbf{a}_j^*, \mathbf{x}_0\mathbf{x}_0^* \in \mathbb{C}^{d\times d}$. Hence, one can recast the phase retrieval problem as a low rank matrix recovery problem:

$$\text{find } \mathbf{X} \in \mathbb{C}^{d\times d} \quad \text{s.}t. \quad Tr(\mathbf{a}_j\mathbf{a}_j^* X) = Tr(\mathbf{a}_j\mathbf{a}_j^*\mathbf{x}_0\mathbf{x}_0^*), \text{rank}(X) \le 1, X^* = X.$$

Suppose that there exists $\mathbf{y}_0 \in \mathbb{C}^d$ with $\mathbf{y}_0 \notin \{c\mathbf{x}_0 : c \in \mathbb{F}, |c| = 1\}$ satisfying

$$|\langle \mathbf{a}_j, \mathbf{y}_0\rangle| = |\langle \mathbf{a}_j, \mathbf{x}_0\rangle|, \quad j = 1, \dots, m.$$

Then we have $Tr(\mathbf{a}_j\mathbf{a}_j^* Q) = 0, j = 1, \dots, m$, where $Q := \mathbf{x}_0\mathbf{x}_0^* - \mathbf{y}_0\mathbf{y}_0^*$. Motivated by the observation, the following conclusion was obtained in [4]:

**Proposition 2.2** ([3]) *Suppose that $\mathbf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_m)^T \in \mathbb{C}^{m\times d}$. Then $\mathbf{A}$ is not phase retrievable if and only if there exists a Hermitian matrix $Q \in \mathbb{C}^{d\times d}$ satisfying*

$$\text{rank}(Q) \le 2, \; Tr(\mathbf{a}_j\mathbf{a}_j^* Q) = 0, \quad j = 1, \dots, m.$$

Based on Proposition 2.2, [5], Conca, Edidin, Hering, and Vinzant applied the results about determinant variety to obtain the following theorem with showing $4d-4$ generic measurements are phase retrievable:

**Theorem 2.3** ([5]) *Suppose that $\mathbf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_m)^T \in \mathbb{C}^{m\times d}$.*

*(1) If $m \ge 4d-4$ and $\mathbf{a}_1, \dots, \mathbf{a}_m$ are $m$ generic vectors in $\mathbb{C}^d$, then $\mathbf{A}$ is phase retrievable.*

*(2) If $d = 2^k + 1, k \in \mathbb{Z}_+$ and $m < 4d-4$, then $\mathbf{A}$ is not phase retrievable.*

Theorem 2.3 also shows $\mathfrak{m}_{\mathbb{C}}(d) = 4d-4$ provided $d$ is in the form of $2^k+1$. In [5], it was conjectured $\mathfrak{m}_{\mathbb{C}}(d) = 4d-4$ for any $d \in \mathbb{Z}_+$. According to Theorem 2.3, the conjecture holds when $d = 2, 3, 5, 9, \dots$. In [6], Vinzant considered the case where $d = 4$ with constructing $11 < 12 = 4\times 4 - 4$ vectors $\mathbf{a}_1, \dots, \mathbf{a}_{11}$. Employing the method from computational algebraic geometry, she verifies the matrix $\mathbf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_{11})$ is phase retrievable by `maple` code and hence disprove the $4d-4$ conjecture for the case $d = 4$. We state her result as a proposition:

**Proposition 2.4** *There exists a matrix $\mathbf{A} \in \mathbb{C}^{11\times 4}$ which is phase retrievable. Hence $\mathfrak{m}_{\mathbb{C}}(4) \le 11$.*

On the other direction, one also considers the lower bound of the minimal measurement number. Usually, the lower bound is obtained by the results from the embedding of the complex projective space $\mathbb{PC}^d$ in $\mathbb{R}^m$. The first lower bound $\mathfrak{m}_{\mathbb{C}}(d) \ge 3d-2$ was presented in [10] and an alternative lower bound $\mathfrak{m}_{\mathbb{C}}(d) \ge 4d-3-2\alpha$ was presented in [8], where $\alpha$ denotes the number of 1's in the binary expansion of $d-1$. The result was improved in [1]:

**Theorem 2.5** ([1]) *Let $d > 4$. Then $\mathfrak{m}_{\mathbb{C}}(d) \ge 4d-2-2\alpha+\epsilon_\alpha$, where $\alpha = \alpha(d-1)$ denotes the* *number of* 1*'s in the binary expansion of* $d-1$,

$$\epsilon_\alpha = \begin{cases} 2 & d \text{ odd}, \alpha \equiv 3 \ (\text{mod } 4) \\ 1 & d \text{ odd}, \alpha \equiv 2 \ (\text{mod } 4) \\ 0 & \text{otherwise.} \end{cases}$$

We list the minimal measurement number $\mathfrak{m}_{\mathbb{C}}(d)$ for $d \in [2,9] \cap \mathbb{Z}$ in Table 1 which presents the exact value of $\mathfrak{m}_{\mathbb{C}}(d)$ or an interval the $\mathfrak{m}_{\mathbb{C}}(d)$ lies in. The results for $d = 2, 3$ in Table 1 were firstly obtained in [4]. For the case where $d = 4$, the lower bound was obtained by the result $\mathfrak{m}_{\mathbb{C}}(d) \geq 3d-2$ (see [7]) while the upper bound $\mathfrak{m}_{\mathbb{C}}(4) \leq 11$ follows from the example in [6]. When $d \geq 5$, the upper bound is obtained by $\mathfrak{m}_{\mathbb{C}}(d) \leq 4d-4$ while the lower bound follows from Theorem 2.5. Note that $\alpha(d-1) = 1$ and $\epsilon_\alpha = 0$ provided $d$ is in the form of $2^k+1$. Theorem 2.5 implies the lower bound $\mathfrak{m}_{\mathbb{C}}(d) \geq 4d-4$ provided $d = 2^k+1$. Combining it with the upper bound $\mathfrak{m}_{\mathbb{C}}(d) \leq 4d-4$, we recover $\mathfrak{m}_{\mathbb{C}}(d) = 4d-4$ provided $d = 2^k+1, k \geq 2$. According to Table 1, the first $d$ for which $\mathfrak{m}_{\mathbb{C}}(d)$ is unknown is 4. This leads us to consider the following open question:

**Open question 2.6** *Does there exist* 10 *vectors* $\mathbf{a}_1, \ldots, \mathbf{a}_{10}$ *so that* $\mathbf{A} = (\mathbf{a}_1, \ldots, \mathbf{a}_{10})^T \in \mathbb{C}^{10\times 4}$ *is phase retrievable on* $\mathbb{C}^4$*?*

According to the results mentioned before, we know $\mathfrak{m}_{\mathbb{C}}(d) \leq 4d-4$. We already know $\mathfrak{m}_{\mathbb{C}}(d) \neq 4d-4$ for some $d$. Hence, we are interested in the distance between $4d-4$ and $\mathfrak{m}_{\mathbb{C}}(d)$. According to the lower bound presented in Theorem 2.5, $4d-4-\mathfrak{m}_{\mathbb{C}}(d) \leq O(\log_2(d))$. We are interested in whether the bound $O(\log_2 d)$ is tight. Particularly, we would like to know whether $4d-4-\mathfrak{m}_{\mathbb{C}}(d)$ is bound. We state the question as follows:

**Open question 2.7** *Is* $\limsup_{d\to\infty}(4d - \mathfrak{m}_{\mathbb{C}}(d))$ *finite?*

**Remark 2.8** The generalized phase retrieval is to recover $\mathbf{x} \in \mathbb{F}^d$ from the measurement $\{\mathbf{x}^*\mathbf{A}_j\mathbf{x}\}_{j=1}^m$ where $\mathbf{A}_j \in \mathbb{F}^{d\times d}$ and $\mathbf{A}_j^* = \mathbf{A}_j$ which includes the phase retrieval by projection as a special case where each $\mathbf{A}_j$ satisfies $\mathbf{A}_j^2 = \mathbf{A}_j$ (see [9–11]). Here, we assume that $\mathbf{A}_j^* = \mathbf{A}_j$. The generalized phase retrieval was investigated in [1] with showing the connection among phase retrieval, nonsingular bilinear form and topology embedding (see [1] for detail).

| The dimension $d$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|
| $\mathfrak{m}_{\mathbb{C}}(d)$ | 4 | 8 | $[10,11]$ | 16 | $[19,20]$ | $[23,24]$ | $[26,28]$ | 32 |

Table 1 The minimal measurement number $\mathfrak{m}_{\mathbb{C}}(d)$

## 3. Phase retrieval for sparse signals

In practical applications, it is possible that some prior knowledge about $\mathbf{x}_0$ is known. For example, in many applications, we know that the aim signal $\mathbf{x}_0$ is sparse. We set

$$\mathbb{F}_s^d := \{\mathbf{x} \in \mathbb{F}^d : \|\mathbf{x}\|_0 \leq s\},$$

where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and $\|\mathbf{x}\|_0$ denotes the number of the nonzero entries of $\mathbf{x}$. In this section, we assume that $\mathbf{x}_0 \in \mathbb{F}_s^d$. The $\mathbf{A} \in \mathbb{F}^{m\times d}$ is said to be $k$-sparse phase retrievable if

$$\{\mathbf{x} \in \mathbb{F}^d : |\mathbf{A}\mathbf{x}| = |\mathbf{A}\mathbf{x}_0|\} \cap \mathbb{F}_s^d \;=\; \{c\mathbf{x}_0 : c \in \mathbb{F}, |c| = 1\}.$$

In fact, $\mathbf{A}$ is $k$-sparse phase retrievable if and only if the solution set to

$$\min_{\mathbf{x}} \|\mathbf{x}\|_0 \quad \text{s.t.} \quad |\mathbf{A}\mathbf{x}| = |\mathbf{A}\mathbf{x}_0| \tag{3.1}$$

is $\tilde{\mathbf{x}}_0$. In [12], Wang and Xu presented the condition for $\mathbf{A}$ under which $\mathbf{A}$ is $s$-sparse phase retrievable.

**Theorem 3.1** ([12]) *Suppose that $\mathbf{A} = (\mathbf{a}_1, \ldots, \mathbf{a}_m)^T \in \mathbb{R}^{m\times d}$. Assume that $\mathbf{A}$ is $s$-sparse phase retrievable on $\mathbb{R}^d$. Then $m \geq \min\{2s, 2d-1\}$. Furthermore, the $\mathbf{A}$ which contains $m \geq \min\{2s, 2d-1\}$ generically chosen vectors in $\mathbb{R}^d$ is $s$-sparse phase retrievable.*

For the complex case, the following result is obtained by Wang and Xu:

**Theorem 3.2** ([12]) *Suppose that $\{\mathbf{a}_1, \ldots, \mathbf{a}_{4s-2}\} \subset \mathbb{C}^d$ are $m = 4s-2$ generic vectors in $\mathbb{C}^d$. Then $\mathbf{A}$ is $s$-sparse phase retrievable on $\mathbb{C}^d$.*

We consider the convex relaxation of (3.1):

$$\min_{\mathbf{x}} \|\mathbf{x}\|_1 \quad \text{s.t.} \quad |\mathbf{A}\mathbf{x}| = |\mathbf{A}\mathbf{x}_0|. \tag{3.2}$$

Though the constraint condition in (3.2) is non-convex, one still develops many efficient algorithms to solve it [13,14]. Hence, it is interesting to present the condition for $\mathbf{A}$ under which the solution to (3.2) is $\tilde{\mathbf{x}}_0$ for any $\mathbf{x}_0 \in \mathbb{F}_s^d$. Motivated by the restricted isometry property in compressed sensing [15], the strong restricted isometry property was defined in [16]:

**Definition 3.3** ([16]) *We say the matrix $\mathbf{A} \in \mathbb{R}^{m\times d}$ satisfies the Strong Restricted Isometry Property (SRIP) of order $s$ and levels $\theta_-, \theta_+ \in (0,2)$ if*

$$\theta_-\|\mathbf{x}\|_2^2 \;\leq \min_{S\subseteq\{1,\ldots,m\},\#S\geq m/2} \|A_S\mathbf{x}\|_2^2 \;\leq \max_{S\subseteq\{1,\ldots,m\},\#S\geq m/2} \|A_S\mathbf{x}\|_2^2 \leq \theta_+\|\mathbf{x}\|_2^2$$

*holds for all $s$-sparse signals $\mathbf{x} \in \mathbb{R}^d$. Here $\mathbf{A}_S := [\mathbf{a}_j : j \in S]^T$ denotes the sub-matrix of $\mathbf{A}$ where only rows with indices in $S$ are kept.*

The following theorem shows that the solution to (3.2) is $\pm\mathbf{x}_0$ provided $\mathbf{A}$ satisfies SRIP:

**Theorem 3.4** ([16]) *Assume that $\mathbf{A} \in \mathbb{R}^{m\times d}$ satisfies the Strong RIP of order $t\cdot s$ and levels $\theta_-, \theta_+$ with $t \geq \max\{\frac{1}{2\theta_- - \theta_-^2}, \frac{1}{2\theta_+ - \theta_+^2}\}$. Then for any $s$-sparse signal $x_0 \in \mathbb{R}^d$ we have*

$$\operatorname*{argmin}_{\mathbf{x}\in\mathbb{R}^d}\{\|\mathbf{x}\|_1 : \; |\mathbf{A}\mathbf{x}| = |\mathbf{A}\mathbf{x}_0|\} = \{\pm\mathbf{x}_0\}, \tag{3.3}$$

*where $|\mathbf{A}\mathbf{x}| := [|\langle\mathbf{a}_j, \mathbf{x}\rangle| : j \in [m]]$ and $[m] := \{1, \ldots, m\}$.*

According to Theorem 3.4, it is useful to construct a matrix $\mathbf{A}$ which satisfies SRIP. It was shown in the following theorem that the Gaussian random matrix $\mathbf{A} \in \mathbb{R}^{m\times d}$ with $m = O(s\log(ed/s))$ satisfies SRIP with high probability:

**Theorem 3.5** ([16]) *Suppose that $t > 1$ and $s \in \mathbb{Z}$ satisfy $tk \leq n$. Suppose that $\mathbf{A} \in \mathbb{R}^{m\times d}$ is* *a random Gaussian matrix whose entries $a_{jk}$ are independent realizations of Gaussian random variables $a_{jk} \sim \mathcal{N}(0, 1/m)$ and that $m \geq C \cdot ts\log(ed/ts))$. Then there exist constants $\theta_-, \theta_+$ with $0 < \theta_- < \theta_+ < 2$, independent of $t$, such that $A$ satisfies SRIP of order $t \cdot s$ and levels $\theta_-, \theta_+$ with probability $1 - \exp(-cm/2)$, where $C, c > 0$ are absolute constants.*

Combining Theorems 3.4 and 3.5, we obtain (3.3) holds with high probability provided $\mathbf{A} \in \mathbb{R}^{m\times d}$ with $m = O(s\log(ed/s))$ being a Gaussian random matrix. The results in [16] are extended to the case with the noise in [17]. It is interesting to extend the results in [16] and [17] to the complex case:

**Open question 3.6** *Does there exist a matrix $\mathbf{A} \in \mathbb{C}^{m\times d}$ with $m = O(s\log(ed/s))$ so that*

$$\operatorname*{argmin}_{\mathbf{x}\in\mathbb{C}^d}\{\|\mathbf{x}\|_1 : \ |\mathbf{A}\mathbf{x}| = |\mathbf{A}\mathbf{x}_0|\} \ = \ \tilde{\mathbf{x}}_0,$$

*holds for any $\mathbf{x}_0 \in \mathbb{C}_d^s$?*

**Remark 3.7** One is interested in whether it is possible to recover $\mathbf{x}_0 \in \mathbb{R}_s^d$ from $O(s\log(ed/s))$ measurements in polynomial time. According to results above, a possible way to answer this question is to design the polynomial time algorithm to solve (3.2).

## 4. Conclusion

We review some of the recent developments on the minimal measurement number problem in phase retrieval. To obtain these results, one employs some results and methods from algebraic geometry and topology. As said before, phase retrieval can be considered as a special case of matrix recovery [18]. Hence, a generalized problem is to determine the minimal measurement number $m$ so that one can recover the matrix $Q \in \mathbb{F}^{d\times d}$ with $\mathrm{rank}(Q) \leq r$ from $m$ measurements. For the case $\mathbb{F} = \mathbb{C}$, the generalized problem was solved in [18] while it remains open for the case $\mathbb{F} = \mathbb{R}$. We believe the methods developed in phase retrieval are helpful to make some progress for the case $\mathbb{F} = \mathbb{R}$.

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