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A Unified Ternary Curve Subdivision Scheme

Qianqian LÜ, Jieqing TAN*, Jun SHI
School of Mathematics, Hefei University of Technology, Anhui 230009*, P. R. China*

Dedicated to Professor Renhong Wang on the Occasion of His Eightieth Birthday

Abstract A unified $m\,(m>2)$-point ternary scheme with some parameter is proposed. The continuity of subdivision scheme is analyzed based on the relationship between the subdivision scheme and difference scheme. Moreover, the proposed subdivision is extended to asymmetric multi-parameter subdivision and the asymmetric schemes in four cases are presented in detail. Some examples are given to show that the presented scheme has better approximating effect.

Keywords difference matrix; subdivision scheme; eigenvalues; asymmetric scheme

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1. Introduction

Subdivision method plays an important role in CAGD. A subdivision scheme refines the initial control polygon again and again and generates a limit curve satisfying special need. Subdivision schemes can be divided into two categories: interpolatory one and approximate one. The interpolatory subdivision scheme preserves all initial vertices and new vertices are inserted as linear combinations of old vertices. A lot of approximating schemes have been introduced in the literature. The first approximating subdivision scheme was introduced by Chaikin in [1] and later its smoothness was proved in [2]. Dyn, Levin and Micchelli added proper parameters to increase smoothness of curves and surfaces in [3]. Hassan et al. presented a ternary three point scheme in [4] and 4-point interpolatory scheme in [5]. Siddiqi and Ahmad proposed a 5-point approximating subdivision scheme with one parameter based on B-spline basis functions in [6], and it can generate curves with special values. Kashif Rehan and Muhammad Athar Sabri presented a new blending 4-point ternary scheme in [7]. Ghaffar and Mustafa introduced a family of odd point ternary approximating subdivision schemes with one parameter in the form of the Laurent polynomial in [8]. Zheng et al. designed binary convergent subdivision schemes with certain parameter based on eigenvalues of their difference matrices and the relation between the subdivision schemes and the difference scheme in [9]. Lian presented a-ary subdivision schemes with a parameter for curves designing in [10]. Shen and Huang proposed a class of binary convergent subdivision scheme with serval parameters in [11], but only discussed the designing of

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* Corresponding author
E-mail address: jieqingtan@hfut.edu.cn (Jieqing TAN)

convergent binary subdivision schemes with two or three parameters by analyzing the eigenvalues of its difference matrix corresponding to Laurent polynomials $D(z) = (a_1z + a_0)(z + 1)$ and $D(z) = (a_2z^2 + a_1z + a_0)(z + 1)$. In this paper, we present a novel ternary subdivision scheme with some parameters to design the curves more flexibly. The rest of the paper is organized as follows. In Section 2, with difference matrix of subdivision scheme we introduce the conditions of the convergence of the scheme. In Section 3 we propose a unified ternary subdivision scheme with high continuity. In Section 4, the proposed subdivision scheme is extended to asymmetric multi-parameter subdivision scheme. Four cases are discussed, the convergence and continuity are analyzed and examples are given to show the effect. Finally, we conclude the paper in Section 5.

2. Preliminaries

A ternary subdivision scheme S maps the polygon $p^k = \{p_i^k : i \in \mathbb{Z}\}$ to the refined polygon $p^{k+1} = \{p_i^{k+1} : i \in \mathbb{Z}\}$ in the following way

$$p_j^{k+1} = \sum_{i\in\mathbb{Z}} a_{j-3i}p_i^k, j \in \mathbb{Z}$$

where the set $a = \{a_j, j \in \mathbb{Z}\}$ is called the mask of the scheme. The above subdivision process can be described as $p^{k+1} = Sp^k$, where matrix S is called the subdivision matrix which satisfies $S_{3j+i,j} = a_i, i \in \mathbb{Z}, j \in \mathbb{Z}$. We call S the original subdivision scheme. The generating polynomial of the scheme can be written as $a(z) = \sum_i a_i z^i$.

Theorem 2.1 ([12]) *Let S be a convergent ternary subdivision scheme with the mask a. Then a satisfies*

$$\sum_{i\in\mathbb{Z}} a_{3i} = \sum_{i\in\mathbb{Z}} a_{3i+1} = \sum_{i\in\mathbb{Z}} a_{3i+2} = 1.$$

From Theorem 2.1, it is not difficult to find that the convergent condition implies $a(1) = 3$.

Theorem 2.2 ([14]) *Let S denote a ternary subdivision scheme with mask $a = \{a_i\}_{i\in\mathbb{Z}}$ and S_j $(j = 1, 2, \ldots, n)$ denote its jth order divided difference scheme with the mask $a^{(j)} = \{a_i^{(j)}\}_{i\in\mathbb{Z}}$ satisfying*

$$\sum_{i\in\mathbb{Z}} a_{3i}^{(j)} = \sum_{i\in\mathbb{Z}} a_{3i+1}^{(j)} = \sum_{i\in\mathbb{Z}} a_{3i+2}^{(j)} = 1.$$

We have

$$a(z) = (\frac{1 + z + z^2}{3z^2})^n a^{(n)}(z).$$

If there exists a smallest positive integer such that $\|(\frac{1}{3}S_{n+1})^L\|_\infty < 1$, then the ternary subdivision scheme S is C^n-continuous. In particular, when $L = 1$,

$$\|\frac{1}{3}S_{n+1}\|_\infty = \max \frac{1}{3}\Big\{\sum_{i\in\mathbb{Z}} |a_{3i}^{(n+1)}|, \sum_{i\in\mathbb{Z}} |a_{3i+1}^{(n+1)}|, \sum_{i\in\mathbb{Z}} |a_{3i+2}^{(n+1)}|\Big\}.$$

Denote by $\Delta p^j = \{(\Delta p^j)_i = p_i^j - p_{i-1}^j\}$ the set of the first difference of the set $p^j = \{p_i^j : i \in \mathbb{Z}\}$. So if we analyze the subdivision process of difference vectors, we will get a difference scheme which can be indicated as $\Delta p^{j+1} = D\Delta p^j$, where D is the subdivision matrix for this process, and we call it difference matrix.

Theorem 2.3 *Let S be a subdivision scheme with the generating polynomial $a(z)$ and $D(z)$ be the generating polynomial of its difference scheme, where $D(z) = \sum_i d_i z^i, d_i = D_{3j+i,j}, i \in \mathbb{Z}, j \in \mathbb{Z}$. The relation between the subdivision matrix S and the difference matrix D can be described as*

$$a(z) = (1 + z + z^2)D(z).$$

Proof Define generating function $p^j(z) = \sum_{i\in\mathbb{Z}} p_i^j z^i$. Since $p_j^{k+1} = \sum_{i\in\mathbb{Z}} a_{j-3i} p_i^k$, it follows

$$\sum_{j\in\mathbb{Z}} p_j^{k+1} z^j = \sum_{j\in\mathbb{Z}} z^j \sum_{i\in\mathbb{Z}} a_{j-3i} p_i^k = \sum_{j\in\mathbb{Z}} a_{j-3i} z^{j-3i} \sum_{i\in\mathbb{Z}} p_i^k z^{3i},$$

that is $p^{k+1}(z) = a(z)p^k(z^3)$. Since $\Delta p^{j+1} = D\Delta p^j$, we have

$$\sum_i (\Delta p^{j+1})_i z^i = \sum_i (D\Delta p^j)_i z^i.$$

The left hand side is equal to

$$\sum_i (p_i^{j+1} - p_{i-1}^{j+1}) z^i = \sum_i p_i^{j+1} z^i - z \sum_i p_{i-1}^{j+1} z^{i-1} = (1 - z)p^{j+1}(z).$$

The right hand side is equal to

$$\begin{aligned}\sum_i \sum_k d_{i-3k} (\Delta p^j)_k z^i &= \sum_i d_{i-3k} z^{i-3k} \sum_k (p_k^j - p_{k-1}^j) z^{3k} \\ &= \sum_i d_{i-3k} z^{i-3k} (\sum_k p_k^j z^{3k} - z^3 \sum_k p_{k-1}^j z^{3(k-1)}) \\ &= D(z)(1 - z^3)p^j(z^3).\end{aligned}$$

It follows $(1 - z)p^{j+1}(z) = D(z)(1 - z^3)p^j(z^3)$, namely

$$p^{j+1}(z) = D(z)(1 + z + z^2)p^j(z^3).$$

Since $\Delta p^j = \{(\Delta p^j)_i = p_i^j - p_{i-1}^j, i \in \mathbb{Z}\}$, $a(z) = D(z)(1 + z + z^2)$. The proof of Theorem 2.3 is completed. □

3. New scheme and the property of difference matrix

3.1. A unified ternary subdivision scheme

In this subsection we recall some existing schemes. The general forms of ternary odd point and even point approximating subdivision schemes were given respectively in [8] and [14], but the relation between odd point and even point schemes have not been given. Inspired by the results in [8] and [14], we present a unified ternary $m(m \geq 3)$-point subdivision scheme with high continuity. Denote the subdivision scheme and its generating polynomial by S and $a(z) = (1+z+z^2)D(z)$, respectively, where $D(z)$ is a polynomial corresponding to the difference scheme of S and

$$D(z) = (1 + z + z^2)^l (a_0 + a_1 z + a_2 z^2 + \cdots + a_q z^q).$$

Remark 3.1 If we set $l=3n$, $q=3$, $a_0=a_3=\frac{1}{3^{3n}}(\frac{1}{12}+w)$, $a_1=a_2=\frac{1}{3^{3n}}(\frac{5}{12}+w)$, then it is the subdivision scheme in [15] and it is a family of even-point scheme that can generate various curves with high continuity.

If we set $l=3n+2$, $q=2$, $a_0=\frac{1}{3^{3n+2}}(\frac{1}{12}+w)$, $a_1=a_2=\frac{1}{3^{3n+2}}(\frac{5}{6}-2w)$, then it is the scheme in [8] and it is a family of even-point scheme that can also generate curves with high continuity and good properties.

If we set $l=3$, $q=3$, $a_0=a_3=-\frac{35}{1296}$, $a_1=a_2=\frac{59}{1296}$, then it is the scheme in [15].

If we set $l=3$, $q=3$, $a_0=a_2=-\frac{4}{81}$, $a_1=\frac{11}{81}$, $a_3=0$, then it is the scheme with $\mu=\frac{1}{27}$ in [14] which is an interpolatory scheme.

So if we set l,q properly, we will get arbitrary (not less than 3) point ternary scheme.

3.2. The property of the eigenvalues of the difference matrix

To analyze the convergence, we introduce the matrix formalism to derive necessary conditions for a subdivision scheme to be C^n based on the eigenvalues of the subdivision matrices. By Theorem 2.3, we can simplify to study difference matrix D corresponding to the polynomial

$$D(z)=(1+z+z^2)^l(a_0+a_1z+a_2z^2+\cdots+a_qz^q),$$

which can be written as

$$D(z)=A_{2l+q}z^{2l+q}+A_{2l+q-1}z^{2l+q-1}+\cdots+A_1z+A_0.$$

In this subsection, we deduce a property of the eigenvalues of the difference matrix D which is corresponding to the polynomial

$$D(z)=(1+z+z^2)^l(a_0+a_1z+a_2z^2+\cdots+a_qz^q).$$

And these properties are more complex in ternary schemes than in binary schemes to design high continuity curves.

Proposition 3.2 *Denote by m the point number, then the number of the eigenvalues of the difference matrix corresponding to the polynomial $D(z)=(1+z+z^2)^l(a_0+a_1z+a_2z^2+\cdots+a_qz^q)$ can be divided into two situations:*

(i) When $m=2k+1$, the numbers of the eigenvalues are $3k$ and $3k-1$.

(ii) When $m=2k$, the numbers of the eigenvalues are $3k-1$ and $3k-2$.

Proof Since $D(z)=(1+z+z^2)^l(a_0+a_1z+a_2z^2+\cdots+a_qz^q)$ can be rewritten as

$$D(z)=A_{2l+q}z^{2l+q}+A_{2l+q-1}z^{2l+q-1}+\cdots+A_1z+A_0,$$

using the way in [11], we can compute the number of the eigenvalues of the difference matrix D. Let w denote the width of the mask of the difference scheme with $D(z)$ as its generating polynomial. For the ternary difference scheme determined by the mask, the orders of the corresponding matrices are $N=\lceil\frac{w}{a-1}\rceil$ and $N-1$.

When point number m is odd, say, $m=2k+1$, the two corresponding difference matrices for $D(z)$ are as follows

$$Q=\begin{pmatrix} A_{2l+q-1} & A_{2l+q-4} & A_{2l+q-7} & \cdots & 0 & 0 & 0 \\ A_{2l+q} & A_{2l+q-3} & A_{2l+q-6} & \cdots & 0 & 0 & 0 \\ 0 & A_{2l+q-2} & A_{2l+q-5} & \cdots & 0 & 0 & 0 \\ 0 & A_{2l+q-1} & A_{2l+q-4} & \cdots & 0 & 0 & 0 \\ 0 & A_{2l+q} & A_{2l+q-3} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_6 & A_3 & A_0 \\ 0 & 0 & 0 & \cdots & A_7 & A_4 & A_1 \end{pmatrix}_{(3k)\times(3k)}, \tag{1}$$

$$\bar{Q}=\begin{pmatrix} A_{2l+q-2} & A_{2l+q-5} & A_{2l+q-8} & \cdots & 0 & 0 & 0 \\ A_{2l+q-1} & A_{2l+q-4} & A_{2l+q-7} & \cdots & 0 & 0 & 0 \\ A_{2l+q} & A_{2l+q-3} & A_{2l+q-6} & \cdots & 0 & 0 & 0 \\ 0 & A_{2l+q-2} & A_{2l+q-5} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_7 & A_4 & A_1 \\ 0 & 0 & 0 & \cdots & A_8 & A_5 & A_2 \end{pmatrix}_{(3k-1)\times(3k-1)} \tag{2}$$

and the numbers of the eigenvalues of the difference matrices Q and $\bar{Q}$ are $\lceil \frac{w}{a-1} \rceil = \lceil \frac{3(2k+1)-2}{2} \rceil = 3k$ and $3k-1$, respectively.

So this scheme has two difference matrices with $3k$ and $3k-1$ eigenvalues respectively. Similarly when point number m is even, say, $m=2k$, we obtain the matrices as follows

$$L=\begin{pmatrix} A_{2l+q} & A_{2l+q-3} & A_{2l+q-6} & \cdots & 0 & 0 & 0 \\ 0 & A_{2l+q-2} & A_{2l+q-5} & \cdots & 0 & 0 & 0 \\ 0 & A_{2l+q-1} & A_{2l+q-4} & \cdots & 0 & 0 & 0 \\ 0 & A_{2l+q} & A_{2l+q-3} & \cdots & 0 & 0 & 0 \\ 0 & 0 & A_{2l+q-2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_6 & A_3 & A_0 \\ 0 & 0 & 0 & \cdots & A_7 & A_4 & A_1 \end{pmatrix}_{(3k-1)\times(3k-1)}, \tag{3}$$

$$\bar{L}=\begin{pmatrix} A_{2l+q-1} & A_{2l+q-4} & A_{2l+q-7} & \cdots & 0 & 0 & 0 \\ A_{2l+q} & A_{2l+q-3} & A_{2l+q-6} & \cdots & 0 & 0 & 0 \\ 0 & A_{2l+q-2} & A_{2l+q-5} & \cdots & 0 & 0 & 0 \\ 0 & A_{2l+q-1} & A_{2l+q-4} & \cdots & 0 & 0 & 0 \\ 0 & A_{2l+q} & A_{2l+q-3} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_7 & A_4 & A_1 \\ 0 & 0 & 0 & \cdots & A_8 & A_5 & A_2 \end{pmatrix}_{(3k-2)\times(3k-2)}. \tag{4}$$

This scheme has two difference matrices with $3k-1$ and $3k-2$ eigenvalues, respectively.

3.3. The conditions of the convergence of the scheme

In this subsection, we firstly give some conditions to construct a subdivision scheme, then give two theorems concerning the convergence of the scheme and the smoothness of the generated limiting curve.

Condition 3.3 The sums of the coefficients of the $3k$, $3k+1$ and $3k+2$ powers in the polynomial $D(z)=(1+z+z^2)^l(a_0+a_1z+a_2z^2+\cdots+a_qz^q)$ are all equal to $\frac{1}{3}$.

Condition 3.4 The spectral radius of difference matrix D which is corresponding to the polynomial $D(z)$ is less than 1.

Theorem 3.5 ([16]) *Let the mask of the subdivision scheme S satisfy Theorem 2.1. The difference matrix is D. If there exist $k\in N^+$ and $\alpha\in(0,1)$ such that $\|D^k\|\le\alpha$, then the subdivision scheme S is uniformly convergent.*

Theorem 3.6 ([16]) *If the subdivision scheme S with the generating polynomial $a(z)$ is convergent, then the subdivision scheme with the generating polynomial $(\frac{1+z+z^2}{3})^k a(z)$ generates a curve of C^k continuity.*

4. New asymmetric scheme

In this section, we propose a novel scheme which contains all four cases according to different values of l,q.

Case 1 If $2l+q=6k+2$, $k=1,2,\ldots$, we can obtain the first type of the schemes. For example, setting $l=3k, q=2$, if $k=1, a_0+a_1+a_2=\frac{1}{27}$, we have

$$D(z)=(1+z+z^2)^3(a_0+a_1z+a_2z^2).$$

The scheme can be written as

$$\begin{cases} p_{3i}^{k+1}=(10a_0+4a_1+a_2)p_{i-1}^k+(16a_0+19a_1+16a_2)p_i^k+(a_0+4a_1+10a_2)p_{i+1}^k,\\ p_{3i+1}^{k+1}=(4a_0+a_1)p_{i-1}^k+(19a_0+16a_1+10a_2)p_i^k+(4a_0+10a_1+16a_2)p_{i+1}^k+a_2p_{i+2}^k,\\ p_{3i+2}^{k+1}=a_0p_{i-1}^k+(16a_0+10a_1+4a_2)p_i^k+(10a_0+16a_1+19a_2)p_{i+1}^k+(a_1+4a_2)p_{i+2}^k\end{cases}\tag{5}$$

and

$$D(z)=\{a_0,3a_0+a_1,6a_0+3a_1+a_2,7a_0+6a_1+3a_2,6a_0+7a_1+6a_2,3a_0+6a_1+7a_2,\\ a_0+3a_1+6a_2,a_1+3a_2,a_2\}.$$

Thus, according to Proposition 3.2, the corresponding difference matrix L has 5 eigenvalues: $9(a_0+a_1+a_2)$, $3(a_0+a_1+a_2)$, $a_0+a_1+a_2$, a_0, a_2 while the difference matrix $\bar{L}$ has 4 eigenvalues: $9(a_0+a_1+a_2)$, $3(a_0+a_1+a_2)$, $a_0+a_1+a_2$, a_1.

If $a_0+a_1+a_2=\frac{1}{27}$, $a_0<\frac{1}{3}$, $a_1<\frac{1}{3}$, $a_2<\frac{1}{3}$, then the spectral radii of both difference matrices L and $\bar{L}$ are less than 1, so Condition 3.4 is satisfied.

In addition, according to Theorem 2.2, if the scheme satisfies

$$\|\frac{1}{3}S_3\|_\infty = \frac{1}{3}\max 27\{|a_0| + |a_1 + a_2|, |a_0 + a_1| + |a_2|, |a_0 + a_1 + a_2|\} < 1,$$

then this scheme is C^2-continuous.

Taking advantage of this scheme, one can design both approximating and interpolatory curve subdivision schemes.

Example 1 (approximating scheme) Setting $a_0 = \frac{1}{108} - \frac{v}{9}$, $a_1 = \frac{1}{54} + \frac{2v}{9}$, $a_2 = \frac{1}{108} - \frac{v}{9}$, we have

$$\begin{cases} p_{3i}^{k+1} = (\frac{19}{108} - \frac{1}{9}v)p_{i-1}^k + (\frac{70}{108} + \frac{2}{9}v)p_i^k + (\frac{19}{108} - \frac{1}{9}v)p_{i+1}^k, \\ p_{3i+1}^{k+1} = (\frac{1}{18} - \frac{2}{27}v)p_{i-1}^k + (\frac{61}{108} + \frac{1}{9}v)p_i^k + (\frac{10}{27})p_{i+1}^k + (\frac{1}{108} - \frac{1}{27}v)p_{i+2}^k, \\ p_{3i+2}^{k+1} = (\frac{1}{108} - \frac{1}{27}v)p_{i-1}^k + (\frac{10}{27})p_i^k + (\frac{61}{108} + \frac{1}{9}v)p_{i+1}^k + (\frac{1}{18} - \frac{2}{27}v)p_{i+2}^k. \end{cases} \tag{6}$$

If $v \in (-\frac{7}{4}, \frac{5}{4})$, $a^{(3)} = \{\frac{1}{108} - \frac{1}{27}v, \frac{3}{108} + \frac{1}{27}v, \frac{1}{27}, \frac{3}{108} + \frac{1}{27}v, \frac{1}{108} - \frac{1}{27}v\}$. From

$$\|\frac{1}{3}S_3\|_\infty = 9\max\{|\frac{1}{108} - \frac{1}{27}v| + |\frac{3}{108} + \frac{1}{27}v|, |\frac{1}{27}|\} < 1,$$

it follows that the scheme is of C^2-continuity. If $v = 0$, the scheme will be of C^3-continuity.

Example 2 (interpolatory scheme) Setting $a_0 = a_2 = -\frac{1}{84}$, $a_1 = \frac{11}{81}$, $a_3 = 0$ yields the scheme

$$\begin{cases} p_{3i}^{k+1} = p_i^k, \\ p_{3i+1}^{k+1} = -\frac{5}{81}p_{i-1}^k + \frac{20}{27}p_i^k + \frac{10}{27}p_{i+1}^k - \frac{4}{81}p_{i+2}^k, \\ p_{3i+2}^{k+1} = -\frac{4}{81}p_{i-1}^k + \frac{10}{27}p_i^k + \frac{20}{27}p_{i+1}^k - \frac{5}{81}p_{i+2}^k, \end{cases} \tag{7}$$

which is the scheme of C^2-continuity introduced in [4].

Case 2 If $2l + q = 6k + 1$, $k = 1, 2, \ldots$, then we get the second type of schemes. For example, setting $l = 3k - 1$, $k = 1$, $q = 3$ and choosing

$$a(z) = (1 + z + z^2)D(z) = (1 + z + z^2)^3(a_0 + a_1z + a_2z^2 + a_3z^3)$$

leads to the following subdivision scheme

$$\begin{cases} p_{3i}^{k+1} = a_0p_{i-1}^k + (7a_0 + 6a_1 + 3a_2 + a_3)p_i^k + (a_0 + 3a_1 + 6a_2 + 7a_3)p_{i+1}^k + a_3p_{i+2}^k, \\ p_{3i+1}^{k+1} = (6a_0 + 3a_1 + a_2)p_i^k + (3a_0 + 6a_1 + 7a_2 + 6a_3)p_{i+1}^k + (a_2 + 3a_3)p_{i+2}^k, \\ p_{3i+2}^{k+1} = (3a_0 + a_1)p_i^k + (6a_0 + 7a_1 + 6a_2 + 3a_3)p_{i+1}^k + (a_1 + 3a_2 + 6a_3)p_{i+2}^k, \end{cases} \tag{8}$$

where

$$\begin{aligned} D(z) = \{&a_0, 2a_0 + a_1, 3a_0 + 2a_1 + a_2, 2a_0 + 3a_1 + 2a_2 + a_3, a_0 + 2a_1 + 3a_2 + 2a_3, \\ &a_1 + 2a_2 + 3a_3, a_2 + 2a_3, a_3\}. \end{aligned}$$

Using the way in [11], we get the numbers of the eigenvalues of the difference matrices are 5 and 4, and the eigenvalues of the matrices are $3(a_0 + a_1 + a_2 + a_3)$, $a_0 + a_1 + a_2 + a_3$, a_1, a_3, 0 and $3(a_0 + a_1 + a_2 + a_3)$, $a_0 + a_1 + a_2 + a_3$, a_2, 0, respectively.

If $a_0 + a_1 + a_2 + a_3 = \frac{1}{9}$, $a_1 < \frac{1}{3}$, $a_2 < \frac{1}{3}$, $a_3 < \frac{1}{3}$, then the spectral radius of each difference matrix D is less than 1 and Condition 3.4 is satisfied.

Besides, it follows from Theorem 2.2 that if $|a_0|+|a_3|<\frac{1}{9}$, $|a_1|<\frac{1}{9}$ and $|a_2|<\frac{1}{9}$, then this scheme is C^2-continuous.

Similarly, we show this family of schemes by an example.

Example 3 Setting $a_0=a_3=\frac{1}{54}-\frac{v}{9}$, $a_1=a_2=\frac{1}{27}+\frac{v}{9}$, we have

$$\begin{cases} p_{3i}^{k+1}=(\frac{1}{54}-\frac{1}{9}v)p_{i-1}^k+(\frac{13}{27}+\frac{1}{9}v)p_i^k+(\frac{13}{27}+\frac{1}{9}v)p_{i+1}^k+(\frac{1}{54}-\frac{1}{9}v)p_{i+2}^k,\\ p_{3i+1}^{k+1}=(\frac{14}{54}-\frac{2}{9}v)p_i^k+(\frac{35}{54}+\frac{4}{9}v)p_{i+1}^k+(\frac{5}{54}-\frac{2}{9}v)p_{i+2}^k,\\ p_{3i+2}^{k+1}=(\frac{5}{54}-\frac{2}{9}v)p_i^k+(\frac{35}{54}+\frac{4}{9}v)p_{i+1}^k+(\frac{14}{54}-\frac{2}{9}v)p_{i+2}^k,\end{cases} \tag{9}$$

which is C^2-continuous when $v\in(-\frac{1}{3},\frac{2}{3})$. In particular, when v is equal to 0, the scheme is C^3-continuous.

Case 3 If the point number is odd, one can set $2l+q=6k+5$, $k=1,2,\ldots$, and obtain a new family of schemes. For instance, setting $l=3k+1$, $q=3$, $k=1$, we have

$$D(z)=(1+z+z^2)^4(a_0+a_1z+a_2z^2+a_3z^3).$$

It is not difficult to get the corresponding subdivision scheme as follows

$$\begin{cases} p_{3i}^{k+1}=(a_2+5a_3)p_{i-2}^k+(5a_0+15a_1+30a_2+45a_3)p_{i-1}^k+(45a_0+51a_1+45a_2+30a_3)p_i^k+\\ \qquad (30a_0+15a_1+5a_2+a_3)p_{i+1}^k+a_0p_{i+2}^k,\\ p_{3i+1}^{k+1}=a_3p_{i-2}^k+(a_0+5a_1+15a_2+30a_3)p_{i-1}^k+(30a_0+45a_1+51a_2+45a_3)p_i^k+\\ \qquad (45a_0+30a_1+15a_2+5a_3)p_{i+1}^k+(5a_0+a_1)p_{i+2}^k,\\ p_{3i+2}^{k+1}=(a_1+5a_2+15a_3)p_{i-1}^k+(15a_0+30a_1+45a_2+51a_3)p_i^k+\\ \qquad (51a_0+45a_1+30a_2+15a_3)p_{i+1}^k+(15a_0+5a_1+a_2)p_{i+2}^k.\end{cases} \tag{10}$$

And $D(z)$ can be written in detail as

$$\begin{aligned} D(z)=\{&a_3,a_2+4a_3,a_1+4a_2+10a_3,a_0+4a_1+10a_2+16a_3,4a_0+10a_1+16a_2+19a_3,\\ &10a_0+16a_1+19a_2+16a_3,16a_0+19a_1+16a_2+10a_3,19a_0+16a_1+10a_2+4a_3,\\ &16a_0+10a_1+4a_2+a_3,10a_0+4a_1+a_2,4a_0+a_1,a_0\}.\end{aligned}$$

According to Proposition 3.2, the difference matrix Q has 6 eigenvalues: $27(a_0+a_1+a_2+a_3)$, $9(a_0+a_1+a_2+a_3)$, $3(a_0+a_1+a_2+a_3)$, $a_0+a_1+a_2+a_3$, a_1,a_3 while the difference matrix $\bar{Q}$ has 5 eigenvalues: $27(a_0+a_1+a_2+a_3)$, $9(a_0+a_1+a_2+a_3)$, $3(a_0+a_1+a_2+a_3)$, $a_0+a_1+a_2+a_3$, a_2.

If $a_0+a_1+a_2=\frac{1}{81}$, $a_1<\frac{1}{3}$, $a_2<\frac{1}{3}$, $a_3<\frac{1}{3}$, then the spectral radii of both difference matrices are less than 1 and Condition 3.4 is satisfied. Therefore, by Theorem 2.2, if $|a_0|+|a_3|<\frac{1}{81}$, $|a_1|<\frac{1}{81}$, $|a_2|<\frac{1}{81}$, then this scheme is C^4-continuous.

Example 4 Setting $a_0 = a_3 = \frac{1}{486} - \frac{v}{81}, a_1 = a_2 = \frac{1}{243} + \frac{v}{81}$, we have

$$\begin{cases} p_{3i}^{k+1} = (\frac{7}{486} - \frac{4}{81}v)p_{i-2}^k + (\frac{70}{243} - \frac{5}{81}v)p_{i-1}^k + (\frac{267}{486} + \frac{21}{27}v)p_i^k + \\ \qquad (\frac{71}{486} - \frac{11}{81}v)p_{i+1}^k + (\frac{1}{486} - \frac{1}{81}v)p_{i+2}^k, \\ p_{3i+1}^{k+1} = (\frac{1}{486} - \frac{1}{81}v)p_{i-2}^k + (\frac{71}{486} - \frac{11}{81}v)p_{i-1}^k + (\frac{267}{486} + \frac{21}{81}v)p_i^k + \\ \qquad (\frac{70}{243} - \frac{5}{81}v)p_{i+1}^k + (\frac{7}{486} - \frac{4}{81}v)p_{i+2}^k, \\ p_{3i+2}^{k+1} = (\frac{27}{486} - \frac{1}{9}v)p_{i-1}^k + (\frac{216}{486} + \frac{1}{9}v)p_i^k + \\ \qquad (\frac{216}{486} + \frac{1}{9}v)p_{i+1}^k + (\frac{27}{486} - \frac{1}{9}v)p_{i+2}^k, \end{cases} \tag{11}$$

which is C^4-continuous. Especially, when v is equal to 0, the scheme is C^5-continuous. In addition, setting the scheme of the multi-parameters

$$D(z) = (1 + z + z^2)^{3k}(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5),$$

one can also design a class of ternary subdivision schemes.

Case 4 Setting $2l + q = 6k + 4,\ k = 1, 2, \ldots$. By the rule of difference scheme, we can get subdivision scheme. For example, setting $l = 3k + 1,\ q = 2,\ k = 1$ with the point number being 5, we have

$$a(z) = (1 + z + z^2)D(z) = (1 + z + z^2)^5(a_0 + a_1 z + a_2 z^2).$$

In this case the subdivision scheme becomes

$$\begin{cases} p_{3i}^{k+1} = (a_1 + 5a_2)p_{i-2}^k + (15a_0 + 30a_1 + 45a_2)p_{i-1}^k + (51a_0 + 45a_1 + 30a_2)p_i^k + \\ \qquad (15a_0 + 5a_1 + a_2)p_{i+1}^k, \\ p_{3i+1}^{k+1} = a_2 p_{i-2}^k + (5a_0 + 15a_1 + 30a_2)p_{i-1}^k + (45a_0 + 51a_1 + 45a_2)p_i^k + \\ \qquad (30a_0 + 15a_1 + 5a_2)p_{i+1}^k + a_0 p_{i+2}^k, \\ p_{3i+2}^{k+1} = (a_0 + 5a_1 + 15a_2)p_{i-1}^k + (30a_0 + 45a_1 + 51a_2)p_i^k + (45a_0 + 30a_1 + 15a_2)p_{i+1}^k + \\ \qquad (5a_0 + a_1)p_{i+2}^k. \end{cases} \tag{12}$$

It is not difficult to get

$$\begin{aligned} D(z) = \{&a_2, a_1 + 4a_2, a_0 + 4a_1 + 10a_2, 4a_0 + 10a_1 + 16a_2, 10a_0 + 16a_1 + 19a_2, \\ &16a_0 + 19a_1 + 16a_2, 19a_0 + 16a_1 + 10a_2, 16a_0 + 10a_1 + 4a_2, 10a_0 + 4a_1 + a_2, \\ &4a_0 + a_1, a_0\}. \end{aligned}$$

Using the way in [11], we can compute the numbers of the eigenvalues of the difference matrices are 6 and 5, and the corresponding eigenvalues of the matrices are $27(a_0+a_1+a_2+a_3)$, $9(a_0+a_1+a_2+a_3)$, $3(a_0+a_1+a_2+a_3)$, $a_0+a_1+a_2+a_3$, a_1, 0, and $27(a_0+a_1+a_2+a_3)$, $9(a_0+a_1+a_2+a_3)$, $3(a_0 + a_1 + a_2 + a_3)$, $a_0 + a_1 + a_2 + a_3$, 0, respectively.

If $a_0 + a_1 + a_2 + a_3 = \frac{1}{81}$, $a_1 < \frac{1}{3}$, then the spectral radii of difference matrices are less than 1 and Condition 3.4 is satisfied.

In addition, if

$$\sum_{i\in\mathbb{Z}} a_{3i}^{(4)} = \sum_{i\in\mathbb{Z}} a_{3i+1}^{(4)} = \sum_{i\in\mathbb{Z}} a_{3i+2}^{(4)} = 1,$$

and $|a_0| < \frac{1}{81}$, $|a_1| < \frac{1}{81}$, $|a_2| < \frac{1}{81}$, then

$$\|\frac{1}{3}S_5\|_\infty = \frac{1}{3}\max 243\{|a_0|, |a_1|, |a_2|\} < 1.$$

Therefore, this scheme is C^4-continuous.

Example 5 Setting $a_0 = a_2 = \frac{1}{324} - \frac{v}{81}$, $a_1 = \frac{1}{162} + \frac{2v}{81}$, we have

$$\begin{cases} p_{3i}^{k+1} = (\frac{7}{324} - \frac{3}{81}v)p_{i-2}^k + (\frac{120}{324})p_{i-1}^k + (\frac{171}{324} + \frac{1}{9}v)p_i^k + (\frac{26}{324} - \frac{6}{81}v)p_{i+1}^k, \\ p_{3i+1}^{k+1} = (\frac{1}{324} - \frac{1}{81}v)p_{i-2}^k + (\frac{65}{324} - \frac{5}{81}v)p_{i-1}^k + (\frac{324}{192} + \frac{12}{81}v)p_i^k + \\ \qquad (\frac{65}{324} - \frac{5}{81}v)p_{i+1}^k + (\frac{1}{324} - \frac{1}{81}v)p_{i+2}^k, \\ p_{3i+2}^{k+1} = (\frac{26}{324} - \frac{6}{81}v)p_{i-1}^k + (\frac{171}{324} + \frac{1}{9}v)p_i^k + (\frac{120}{324})p_{i+1}^k + (\frac{7}{324} - \frac{3}{81}v)p_{i+2}^k, \end{cases} \tag{13}$$

which is of C^4-continuity. In particular, when $v = 0$, the scheme is of C^5-continuity. Performance of the proposed subdivision scheme is demonstrated and compared with some existing schemes in terms of support, continuity and visual inspection in the following table.

Scheme	Type	Continuity	Support
Ternary 4-point [4]	Interpolatory	C^2	5
Ternary 4-point [15]	Approximating	C^2	5.5
Proposed scheme (Example 1)	Approximating	C^3	5
Proposed scheme (Example 3)	Approximating	C^3	4.5
Proposed scheme (Example 4)	Approximating	C^5	6.5
Proposed scheme (Example 5)	Approximating	C^5	6

Table 1 Comparison of the subdivision schemes

Figure 1 shows some continuous curves generated by new schemes. In Figure 1, the blue dashed line indicates initial control polygon, and the blue solid, the green dashed, the red solid and the magenta dashed curves shown in (a), (b), (c) and (d) are the curves generated by the interpolatory scheme in Example 2, the subdivision scheme in Example 1 whose limit curve is of C^2, the subdivision scheme in Example 4 by setting parameter $v = \frac{6}{5}$ whose limit curve is of C^4 and the scheme in Example 5 by setting parameter $v = 1.5$ whose limit curve is of C^4, respectively. Shown in Figure 2 are the four cases of the limiting curves generated by subdivision schemes, which are corresponding to Examples 1, 3, 4 and 5, from left to right, respectively, where the tension parameters are set to be $\frac{5}{4}$, $\frac{13}{24}$, $\frac{2}{3}$, 2, respectively.

5. Conclusion

In this paper, we have presented a unified ternary approximating subdivision scheme with several parameters. It can be used to construct arbitrary point ternary scheme which has high continuity and good visual effect. Some numerical examples are presented to show the visual performance of our scheme.

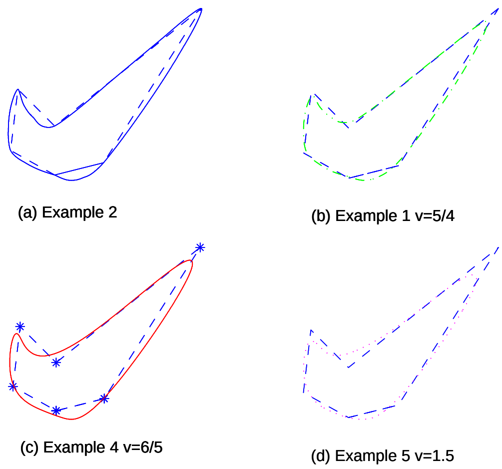

Figure 1 Some continuous curves generated by new schemes

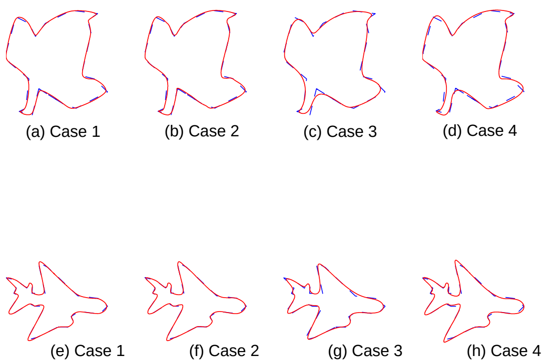

Figure 2 The four cases of the subdivision scheme that generates limiting curves

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