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Identification of Planar Sextic Pythagorean-Hodograph Curves

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Dedicated to Professor Renhong WANG on the Occasion of His Eightieth Birthday

Abstract Pythagorean-hodograph (PH) curves offer computational advantages in Computer Aided Geometric Design, Computer Aided Design, Computer Graphics, Computer Numerical Control machining and similar applications. In this paper, three methods are utilized to construct the identifications of planar regular sextic PH curves. The first exhibits purely the control polygon legs' constraints in the complex form. Such reconstruction of a PH sextic can be elaborated by C^1 Hermite data and another one condition. The second uses polar representation in two cases. One of them can produce a family of convex sextic PH curves related with a quintic PH curve, and the other one may naturally degenerate a sextic PH curve to a quintic PH curve. In the third identification, we use some odd PH curves to construct a family of sextic PH curves with convexity-preserving property.

Keywords Pythagorean-hodograph sextic curves; control polygon; degree elevation; geometric characteristic

MR(2010) Subject Classification 65D17; 68U07

1. Introduction

The researches on Pythagorean-hodograph (PH) curves can track back to the work of Farouki and Sakkalis [1]. With the advantages that the arc-length function is polynomial with curve parameter and the offsets are rational polynomial curves, PH curves are widely applied in Computer Aided Geometric Design (CAGD), Computer Aided Design (CAD), Computer Graphics, Computer Numerical Control (CNC) machining and so on.

There are a mount of researches on PH curves of odd degrees. Farouki et al. [1–4] studied intensively on PH curves, among which there are much more identification methods of cubic and quintic PH curves. PH curves of a little higher odd degrees were also studied, such as Yang and Wang [5] presented a method for the identification of septic PH curves in three cases, and Zheng et al. [6] presented the necessary and sufficient geometric constraints for the PH septic. Even

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Hermite interpolations by the PH curves of degree seven and nine were also discussed [7–11]. We refer to [12–18] for other researches on the interpolation by PH curves.

Although some even PH curves may have singularities, the study on PH curves of even degrees is still interesting for us. Identification of PH quartic refers to [1,19,20]. Farouki and Sakkalis [1] deduced an obscure condition between cusp and control polygon legs of quartic PH curves. Wang and Fang [19] provided the geometric properties on control polygon of quartic PH curves in two cases comprised of cusp or not. Li et al. [20] constructed a family of quartic PH curves from the cubic PH curves with adding extra points.

Besides quartic PH curves mentioned above, other PH curves of even degree were not studied intensively as the odd ones. For the mathematics, it is interesting to study on the theory and the geometric properties of PH curves of even degrees. For applications, PH curves of even degrees are suitable for Hermite interpolation. For example, Fang and Wang [21] presented the C^1 Hermite interpolation method by sextic PH curves and Wang et al. [22] solved the G^2 Hermite interpolation problem by sextic PH curve.

We aim to propose the identification of the PH sextic. In this paper, we present three methods to construct the identifications of planar regular sextic PH curves. Compared with methods by cubic, quartic and quintic PH curves [3,19,20], our method is more complicated to give the identification. We analyze the algebraic constraints by control-polygon legs, frame properties and construction of the pre-image polynomials, and we find that sextic PH curves preserve the good shape flexibility and characteristics.

2. Preliminaries

A planar parameter curve $\mathbf{r}(t) = (x(t), y(t))$ of degree n can be represented in the complex form $\mathbf{r}(t) = x(t) + i\,y(t)$. Its Bézier form is

$$\mathbf{r}(t) = \sum_{j=0}^{n} \mathbf{P}_j B_j^n(t), \tag{2.1}$$

and hodograph is

$$\mathbf{r}'(t) = n \sum_{j=0}^{n-1} \Delta \mathbf{P}_j B_j^{n-1}(t), \quad t \in [0,1], \tag{2.2}$$

where $B_j^n(t) = \frac{n!}{j!(n-j)!}(1-t)^{n-j}t^j$ $(j = 0,\ldots,n)$ are Bernstein basis functions, $\mathbf{P}_j \in \mathbb{R}^2$ $(j = 0,\ldots,n)$ are control points, and $\Delta\mathbf{P}_j$ $(j = 0,\ldots,n-1)$ are the first forward differences of control points.

Definition 2.1 ([1]) *A planar curve $\mathbf{r}(t) = x(t) + i\,y(t)$ of degree n is a PH curve if there exists a polynomial $\sigma(t)$ such that $x'^2(t) + y'^2(t) = \sigma^2(t)$, that is*

$$|\mathbf{r}'(t)|^2 = \sigma^2(t), \quad \sigma(t) = n \sum_{k=0}^{n-1} \sigma_k B_k^{n-1}(t), \quad t \in [0,1]. \tag{2.3}$$

In fact $\sigma_0, \ldots, \sigma_{n-1}$ satisfy the following equation [4]

$$\sum_{j=\max(0,k-n+1)}^{\min(n-1,k)} \frac{\binom{n-1}{j}\binom{n-1}{k-j}(\Delta\mathbf{P}_j \cdot \Delta\mathbf{P}_{k-j} - \sigma_j\sigma_{k-j})}{\binom{2n-2}{k}} = 0, \quad k = 0, \ldots, 2n-2, \tag{2.4}$$

and the arc length is Length $= \frac{\sigma_1+\sigma_2+\sigma_3+\sigma_4+\sigma_5}{6}$.

Theorem 2.2 ([1]) *A planar curve* $\mathbf{r}(t) = x(t) + i\,y(t)$ *is a PH curve if and only if* $\mathbf{r}'(t) = x'(t) + i\,y'(t) = W(t)\boldsymbol{Q}^2(t)$, *where* $x'(t) = W(t)[u^2(t) - v^2(t)], y'(t) = 2W(t)u(t)v(t)$ *and* $\boldsymbol{Q}(t) = u(t) + i\,v(t)$ *for some real polynomials* $W(t)$, $u(t)$, $v(t), t \in [0,1]$.

Due to above analysis, the hodograph of a planar PH curve $\mathbf{r}(t)$ of degree 6 can be represented as

$$\mathbf{r}'(t) = 5\sum_{j=0}^{5} \Delta\mathbf{P}_j B_j^5(t) = W(t)\mathbf{Q}^2(t), \quad t \in [0,1]. \tag{2.5}$$

3. Identification by control polygons

For a sextic PH curve $\mathbf{r}(t)$ with seven control points $\mathbf{P}_i$ $(i = 0, \ldots, 6)$, we denote the control polygon legs $\mathbf{L}_j = \mathbf{P}_{j+1} - \mathbf{P}_j$ and $l_j = |\mathbf{L}_j|$ $(j = 0, \ldots, 5)$. From (2.3) and (2.4), we have

Theorem 3.1 *A planar sextic Bézier curve with control polygon legs* $\mathbf{L}_j$ $(j = 0, \ldots, 5)$ *is a PH curve if and only if it satisfies*

$$\begin{aligned}
&5\,l_0^5\,\boldsymbol{L}_{45}^2 + 25\,l_5^3\,l_{01}\,\boldsymbol{L}_{01}^2 = 4\,l_0^4\,l_5^2\,(l_5\,\boldsymbol{L}_0 - l_0\,\boldsymbol{L}_5)\,\boldsymbol{L}_3 + 20\,l_0^2\,l_5^3\,\boldsymbol{L}_{01}\,\boldsymbol{L}_{02},\\
&5\,l_5^5\,\boldsymbol{L}_{01}^2 + 25\,l_0^3\,l_{45}\,\boldsymbol{L}_{45}^2 = 4l_5^4l_0^2\,(l_0\,\boldsymbol{L}_5 - l_5\,\boldsymbol{L}_0)\boldsymbol{L}_2 + 20\,l_5^2\,l_0^3\,\boldsymbol{L}_{35}\,\boldsymbol{L}_{45},\\
&5\left(4\,l_{02}\,l_0^2 + 5\,\boldsymbol{L}_{01}^2\right)^2 l_5^3 = 80\,l_0^6 l_5^3\left(l_2^2 + l_{13}\right) + 8\,l_0^6 l_5^2\,(l_{04}\,l_5 - l_0 l_{45}) - 20\,l_{01}\,l_0^5\left(4\,l_{35}\,l_5^2 + 5\,\boldsymbol{L}_{45}^2\right),\\
&5\left(4\,l_{35}\,l_5^2 + 5\,\boldsymbol{L}_{45}^2\right)^2 l_0^3 = 80\,l_0^3 l_5^6\left(l_3^2 + l_{24}\right) - 8\,l_0^2 l_5^6\,(l_{01}\,l_5 - l_{15}\,l_0) - 20\,l_{45}\,l_5^5\left(4\,l_{02}\,l_0^2 + 5\,\boldsymbol{L}_{01}^2\right),\\
&25\left(4\,l_{02}\,l_0^2 + 5\,\boldsymbol{L}_{01}^2\right)\left(4\,l_{35}\,l_5^2 + 5\,\boldsymbol{L}_{45}^2\right) = 4\,l_0^3 l_5^3\,(l_{05} + 25\,l_{14} + 100\,l_{23}) - 4\,l_0^2 l_5^2\left(l_0^2 l_5^2 + 25\,l_{01}\,l_{45}\right),
\end{aligned} \tag{3.1}$$

where $l_{ij} = \boldsymbol{L}_i \cdot \boldsymbol{L}_j$, $\boldsymbol{L}_{ij} = \boldsymbol{L}_i \times \boldsymbol{L}_j$ $(i,j = 0, \ldots, 5, i \neq j)$.

Proof From (2.4), for $n = 6$, $k = 0, \ldots, 10$, we can deduce eleven equations as follows

$$\begin{aligned}
&l_0^2 = \sigma_0^2, \quad l_{01} = \sigma_0\sigma_1,\\
&4\,l_{02} + 5\,l_1^2 = 4\,\sigma_0\sigma_2 + 5\,\sigma_1^2,\\
&l_{03} + 5\,l_{12} = \sigma_0\sigma_3 + 5\,\sigma_1\sigma_2,\\
&l_{04} + 10\,l_{13} + 10\,l_2^2 = \sigma_0\sigma_4 + 10\,\sigma_1\sigma_3 + 10\,\sigma_2^2,\\
&l_{05} + 25\,l_{14} + 100\,l_{23} = \sigma_0\sigma_5 + 25\,\sigma_1\sigma_4 + 100\,\sigma_2\sigma_3,\\
&l_{15} + 10\,l_{24} + 10\,l_3^2 = \sigma_1\sigma_5 + 10\,\sigma_2\sigma_4 + 10\,\sigma_3^2,\\
&l_{25} + 5\,l_{34} = \sigma_2\sigma_5 + 5\,\sigma_3\sigma_4,\\
&4\,l_{35} + 5\,l_4^2 = 4\,\sigma_3\sigma_5 + 5\,\sigma_4^2,\\
&l_{45} = \sigma_4\sigma_5, \quad l_5^2 = \sigma_5^2,
\end{aligned}$$

where σ_0, $\sigma_5 > 0$ for a regular curve [4]. By the first and the last three equations of those eleven equations, we obtain

$$\sigma_0 = l_0, \quad \sigma_1 = \frac{l_{01}}{l_0}, \quad \sigma_2 = \frac{l_{02}}{l_0} + \frac{5\,|\mathbf{L}_{01}|^2}{4\,l_0^3},$$
$$\sigma_5 = l_5, \quad \sigma_4 = \frac{l_{45}}{l_5}, \quad \sigma_3 = \frac{l_{35}}{l_5} + \frac{5\,|\mathbf{L}_{45}|^2}{4\,l_5^3}.$$

By substituting $\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$ into the middle five of those eleven equations, we get (3.1).

Theorem 3.1 gives us the identification of sextic PH curves, which is a little more complicated than the methods by cubic and quintic [4]. This identification can help us to solve the C^1 Hermite interpolation of sextic PH curves.

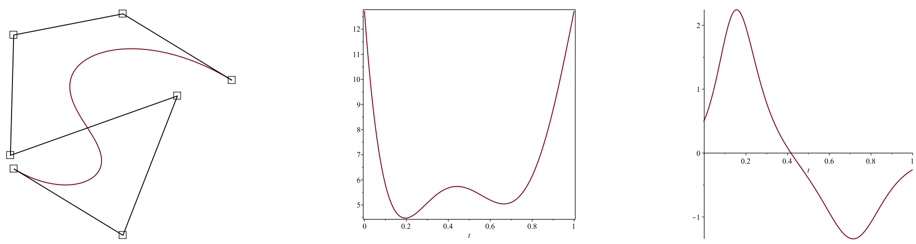
The C^1 Hermite interpolation of a sextic PH curve is to find a sextic PH curve $\mathbf{r}(t) = \sum_{j=0}^{6} \mathbf{P}_j B_j^6(t)$, $t \in [0,1]$, satisfying $\mathbf{r}(0) = \mathbf{P}_0$, $\mathbf{r}(1) = \mathbf{P}_6$, $\mathbf{r}'(0) = 6\,\Delta\mathbf{P}_0$, $\mathbf{r}'(1) = 6\,\Delta\mathbf{P}_5$.

Generally, there are three selection criteria of PH Hermite interpolation [3]: arc-length $S = \int \mathrm{d}s = \int_0^1 \sqrt{x'^2(t) + y'^2(t)}\mathrm{d}t$, bending energy $E = \int k^2 \mathrm{d}s = \int_0^1 k^2 |\mathbf{r}'(t)| \mathrm{d}t$ and absolute rotation number $R_{abs} = \frac{1}{2\pi} \int |k| \mathrm{d}s = \frac{1}{2\pi} \int k^2 |\mathbf{r}'(t)| \mathrm{d}t$, where k is curvature.

Example 3.2 Let the end control points be in complex form as $\mathbf{P}_0 = 1 + i$, $\mathbf{P}_1 = 2.5 - 0.5\,i$, $\mathbf{P}_5 = 2.5 + 4.5\,i$, $\mathbf{P}_6 = 4 + 3\,i$ ([4]). There are six unknowns for the sextic PH curve. Since Theorem 3.1 has five equations, an additional condition is needed. We set the real part of $\mathbf{P}_2$ to be 3.25 in this example. By (3.1), the remaining points can be solved by HOM4PS-2.0 (see [23]). One feasible solution is

$$\mathbf{P}_2 = 3.25 + 2.64656776861715\,i,$$
$$\mathbf{P}_3 = 0.952268110575585 + 1.30508325492550\,i,$$
$$\mathbf{P}_4 = 0.997313616609088 + 4.02170326476971\,i,$$

with tolerance 10^{-13}. The resulting sextic PH curve, variation of the parametric speed $\sigma(t)$ and its curvature are shown in Figure 1. We choose 10^5 as the subdivision unit to carry out the numerical integration about the arc-length S, bending energy E and absolute rotation number R. In this example, we have $S = 6.336845417$, $E = 0.1621214283$ and $R = 0.02593188064$.



(a) Sextic PH curve (b) $\sigma(t)$ (c) Curvature

Figure 1 Sextic PH curve, the parametric speed and curvature

This identification scheme of sextic PH curves presented above is a general method, which can deal with any shape of sextic PH curves, no matter whether it is convex or nonconvex. But for most cases, convex curves are more convenient for applications. So in the next part, we not only give an another general identification but also study on the geometric characteristic of convex sextic PH curves.

4. Identification by geometric characteristic

Without loss of generality, we assume $\mathbf{P}_0$ is the origin and $\mathbf{P}_1$ is on the positive x-axis, such that $\mathbf{r}'(0)$ is along the x axis direction, then $\mathbf{L}_0 = l_0 = 1$. Let $\mathbf{L}_j$ be in polar form. That is $\mathbf{L}_j = l_j\, e^{i\,\theta_j}$ and $\mathbf{L}_5 = l_5\, e^{i\,2\,\theta_5}$, where $\theta_j \in (0,\pi)$ with $j = 1,\ldots,5$ are the rotary angles from positive x-axis direction to $\mathbf{L}_j$, $j = 1,\ldots,5$.

For a sextic PH curve $\mathbf{r}(t)$ based on equation (2.5) and the degree relationship [1] between $W(t)$ and $\mathbf{Q}(t)$, $\mathbf{r}'(t) = W(t)\mathbf{Q}^2(t)$ can be classified into two cases: $\deg(W(t)) = 3$, $\deg(\mathbf{Q}(t)) = 1$, and $\deg(W(t)) = 1$, $\deg(\mathbf{Q}(t)) = 2$. These two types of PH sextics will be discussed respectively as follows.

4.1. $(\deg(W(t)), \deg(\mathbf{Q}(t))) = (3,1)$

In this case, suppose the hodograph of the sextic PH curve $\mathbf{r}(t)$, $t \in [0,1]$, is

$$\mathbf{r}'(t) = [w_0 B_0^3(t) + w_1 B_1^3(t) + w_2 B_2^3(t) + B_3^3(t)][\mathbf{u}_0 B_0^1(t) + \mathbf{u}_1 B_1^1(t)]^2, \tag{4.1}$$

where w_i $(i = 0,1,2)$ and $\mathbf{u}_j$ $(j = 0,1)$ are uncertain real and complex coefficients, respectively.

From (2.5), the relationships between control points and coefficients are

$$6(\mathbf{P}_1 - \mathbf{P}_0) = 6\mathbf{L}_0 = w_0\mathbf{u}_0^2, \tag{4.2 a}$$

$$30(\mathbf{P}_2 - \mathbf{P}_1) = 30\mathbf{L}_1 = 3\mathbf{u}_0^2 w_1 + 2\mathbf{u}_0\mathbf{u}_1 w_0, \tag{4.2 b}$$

$$60(\mathbf{P}_3 - \mathbf{P}_2) = 60\mathbf{L}_2 = 3\mathbf{u}_0^2 w_2 + 6\mathbf{u}_0\mathbf{u}_1 w_1 + \mathbf{u}_1^2 w_0, \tag{4.2 c}$$

$$60(\mathbf{P}_4 - \mathbf{P}_3) = 60\mathbf{L}_3 = 6\mathbf{u}_0\mathbf{u}_1 w_2 + 3\mathbf{u}_1^2 w_1 + \mathbf{u}_0^2, \tag{4.2 d}$$

$$30(\mathbf{P}_5 - \mathbf{P}_4) = 30\mathbf{L}_4 = 3\mathbf{u}_1^2 w_2 + 2\mathbf{u}_0\mathbf{u}_1, \tag{4.2 e}$$

$$6(\mathbf{P}_6 - \mathbf{P}_5) = 6\mathbf{L}_5 = \mathbf{u}_1^2. \tag{4.2 f}$$

Set $w_0 > 0$, $w_1 > 0$, $w_2 > 0$, which avoids the existence of singular points of sextic PH curve. Similarly to the analysis of quartic PH curves [19], from (4.2 a), (4.2 b), (4.2 e), (4.2 f), we can easily get $\mathbf{u}_0 = u_0 = \frac{2l_0 A \sin\theta_5}{5l_1 \sin\theta_1} = \frac{15 l_4 \sin(2\theta_5 - \theta_4)}{A\sin\theta_5}$, $\mathbf{u}_1 = A(\cos\theta_5 + i\sin\theta_5)$, $w_0 = \frac{6l_0 w_1 \sin\theta_5}{10 l_1 \sin(\theta_5 - \theta_1)} = \frac{25 l_1^2 \sin^2\theta_1}{4 l_0 l_5 \sin^2\theta_5}$, $w_1 = \frac{125 l_1^3 \sin(\theta_5-\theta_1)\sin^2\theta_1}{12 l_0^2 l_5 \sin^3\theta_5}$ and $w_2 = \frac{5l_4\sin(\theta_4-\theta_5)}{3l_5\sin\theta_5}$, where $A = \pm\sqrt{6l_5}$ and $\theta_5 - \theta_1, \theta_4 - \theta_5, 2\theta_5 - \theta_4 \in (0,\pi)$. Then we get an important equation

$$4 l_0 l_5 \sin^2\theta_5 = 25 l_1 l_4 \sin(2\theta_5 - \theta_4)\sin\theta_1. \tag{4.3}$$

Substituting the coefficients above into the real and imaginary parts of (4.2 c) and (4.2 d), some equations are derived as

$$8l_0l_2l_5^2\sin^3\theta_5\cos\theta_2 = 5l_1^2l_5^2\sin^2\theta_1\cos(2\theta_5)\sin\theta_5 + 20l_1^2l_5^2\sin(\theta_5-\theta_1)\sin\theta_1\sin\theta_5\cos\theta_5 + 25l_0l_4^3\sin^2(2\theta_5-\theta_4)\sin(\theta_4-\theta_5), \tag{4.4 a}$$

$$8l_0l_2l_5^2\sin^3\theta_5\sin\theta_2 = 5l_1^2l_5^2\sin^2\theta_1\sin(2\theta_5)\sin\theta_5 + 20l_1^2l_5^2\sin(\theta_5-\theta_1)\sin\theta_1\sin^2\theta_5 + 25l_0l_4^3\sin^2(2\theta_5-\theta_4)\sin(\theta_4-\theta_5), \tag{4.4 b}$$

$$8l_0^2l_3l_5\sin^3\theta_5\cos\theta_3 = 5l_0^2l_4^2\sin^2(2\theta_5-\theta_4)\sin\theta_5 + 20l_0^2l_4^2\sin(\theta_4-\theta_5)\sin(2\theta_5-\theta_4)\sin\theta_5\cos\theta_5 + 25l_5l_1^3\sin^2\theta_1\sin(\theta_5-\theta_1)\cos(2\theta_5), \tag{4.4 c}$$

$$8l_0^2l_3l_5\sin^3\theta_5\sin\theta_3 = 5l_0^2l_4^2\sin^2(2\theta_5-\theta_4)\sin\theta_5 + 20l_0^2l_4^2\sin(\theta_4-\theta_5)\sin(2\theta_5-\theta_4)\sin^2\theta_5 + 25l_5l_1^3\sin^2\theta_1\sin(\theta_5-\theta_1)\sin(2\theta_5). \tag{4.4 d}$$

Since (4.3), (4.4 a), (4.4 b), (4.4 c), (4.4 d) are equivalent to (4.2 a)–(4.2 f), we have following result.

Theorem 4.1 *A planar sextic Bézier curve* $\mathbf{r}(t)$ *is a PH curve, whose hodograph can be represented as (4.1), if and only if (4.3), (4.4 a), (4.4 b), (4.4 c), (4.4 d) are satisfied.*

4.1.1. Convex sextic PH curves

From above analysis, $\theta_5-\theta_1, \theta_4-\theta_5, 2\theta_5-\theta_4 \in (0,\pi)$. If we let $\theta_4 > \theta_3 > \theta_5 > \theta_2 > \theta_1$, there must exist a line passing through $\mathbf{P}_3$, which intersects x-axis and line $\mathbf{P}_6\mathbf{P}_5$ at $\mathbf{Q}_1$ and $\mathbf{Q}_2$, such that $\angle\mathbf{P}_1\mathbf{Q}_1\mathbf{P}_3 = \angle\mathbf{P}_3\mathbf{Q}_2\mathbf{P}_5$. Figure 2 shows the control polygon of a sextic Bézier curve. Suppose line $\mathbf{P}_1\mathbf{P}_2$ and $\mathbf{P}_5\mathbf{P}_4$ intersect line $\mathbf{Q}_1\mathbf{Q}_2$ at $\mathbf{Q}_3$ and $\mathbf{Q}_4$, respectively. We set $R_0 = |\mathbf{P}_1\mathbf{Q}_1|$, $R_1 = |\mathbf{Q}_1\mathbf{Q}_3|$, $R_2 = |\mathbf{Q}_3\mathbf{P}_3|$, $R_3 = |\mathbf{P}_3\mathbf{Q}_4|$, $R_4 = |\mathbf{Q}_4\mathbf{Q}_2|$, $R_5 = |\mathbf{Q}_2\mathbf{P}_5|$, $S_1 = |\mathbf{P}_2\mathbf{Q}_3|$, $S_2 = |\mathbf{Q}_4\mathbf{P}_4|$, and $T_1 = |\mathbf{P}_1\mathbf{Q}_3|$, $T_2 = |\mathbf{P}_5\mathbf{Q}_4|$, $T_3 = |\mathbf{Q}_3\mathbf{Q}_4|$.

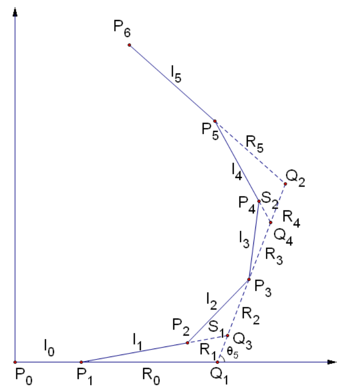


Figure 2 The control polygon of a sextic Bézier curve

In $\triangle\mathbf{P}_1\mathbf{Q}_1\mathbf{Q}_3$, $\triangle\mathbf{P}_5\mathbf{Q}_2\mathbf{Q}_4$, $\triangle\mathbf{P}_2\mathbf{P}_3\mathbf{Q}_3$, $\triangle\mathbf{P}_3\mathbf{P}_4\mathbf{Q}_4$, we can deduce the relationships between edges and corners by the law of sines

$$\frac{R_0}{\sin(\theta_5-\theta_1)}=\frac{T_1}{\sin\theta_5}=\frac{R_1}{\sin\theta_1},\quad \frac{R_5}{\sin(\theta_4-\theta_5)}=\frac{T_2}{\sin\theta_5}=\frac{R_4}{\sin(2\theta_5-\theta_4)},$$
$$\frac{S_1}{\sin(\theta_5-\theta_2)}=\frac{R_2}{\sin(\theta_2-\theta_1)}=\frac{l_2}{\sin(\theta_5-\theta_1)},\quad \frac{S_2}{\sin(\theta_3-\theta_5)}=\frac{R_3}{\sin(\theta_4-\theta_3)}=\frac{l_3}{\sin(\theta_4-\theta_5)}.$$

Further more, (4.3) becomes

$$4l_0l_5T_1T_2=25l_1l_4R_1R_4, \tag{4.5}$$

and we obtain

$$\frac{S_2R_5}{l_3T_2}=\frac{B}{C}\sin\theta_3-\cos\theta_3,\quad \frac{S_1R_0}{l_2T_1}=\cos\theta_2-\frac{B}{C}\sin\theta_2, \tag{4.6}$$

where $B=T_1^2-R_0^2-R_1^2$, $C=\sqrt{2R_0^2R_1^2+2T_1^2R_0^2+2T_1^2R_1^2-R_0^4-R_1^4-T_1^4}$ and $\cos\theta_5=\frac{B}{2R_0R_1}$, $\sin\theta_5=\frac{C}{2R_0R_1}$ by the law of cosine.

By the equations (4.4 a), (4.4 b), (4.4 c), (4.4 d), (4.5), and (4.6), we have

Lemma 4.2 *A planar sextic Bézier curve* $\mathbf{r}(t)$ *is a PH curve, whose hodograph can be represented as (4.1) and control polygon is shown in Figure 2, if and only if*

$$5\,R_1^2T_2l_1^2\,(25\,R_0R_5l_1l_4-T_1T_2l_0l_5)=8\,R_0T_1^2T_2^2l_0^2l_5\,(T_1-l_1)+40\,R_5^2T_1^3l_0^2l_4\,(T_2-l_4)\,. \tag{4.7}$$

Let $R_0=\lambda_1\,l_0$, $R_5=\mu_1\,l_5$, $l_1=\lambda_2\,T_1$, $l_4=\mu_2\,T_2$, where $\lambda_1,\mu_1>0$ and $\lambda_2,\mu_2\in(0,1)$. Then (4.5) leads to

$$4R_0R_5=25\lambda_1\lambda_2\mu_1\mu_2R_1R_4. \tag{4.8}$$

Theorem 4.3 *A planar sextic Bézier curve* $\mathbf{r}(t)$ *is a PH curve, whose hodograph can be represented as (4.1) and control polygon shown in Figure 2, if and only if*

$$5\,R_1^2\lambda_2^2(25\,\lambda_1\,\lambda_2\,\mu_1\,\mu_2-1)=8\,l_0^2\lambda_1\,(1-\lambda_2)+40\,l_0l_5\mu_1^2\mu_2\,(1-\mu_2)\,. \tag{4.9}$$

4.1.2. Convex PH sextics degenerate to convex PH quintics

According to the geometric characteristic of the quintic PH curves in [3], the curve

$$\hat{\mathbf{r}}(t)=\sum_{i=0}^{5}\mathbf{P}'_iB_i^5(t),\quad t\in[0,1]$$

where $(\mathbf{P}'_0,\mathbf{P}'_1,\mathbf{P}'_2,\mathbf{P}'_3,\mathbf{P}'_4,\mathbf{P}'_5)=(\mathbf{P}_0,\mathbf{P}_1,\mathbf{Q}_3,\mathbf{Q}_4,\mathbf{P}_5,\mathbf{P}_6)$, is a PH curve if and only if the following four conditions hold

$$\frac{T_1}{T_2}=\sqrt{\frac{l_0}{l_5}}, \tag{4.10 a}$$
$$\theta_1+\theta_5=\theta_4, \tag{4.10 b}$$
$$3\,l_0\,T_1\,T_3\,\cos\theta_5=l_0^2\,T_2\,\cos(2\theta_5)+2\,T_1^3\,\cos\theta_1, \tag{4.10 c}$$
$$3\,l_0\,T_1\,T_3\,\sin\theta_5=l_0^2\,T_2\,\sin(2\theta_5)+2\,T_1^3\,\sin\theta_1. \tag{4.10 d}$$

Thus we have the following results.

Lemma 4.4 $\hat{\mathbf{r}}(t)$ *is a quintic PH curve if and only if*

$$\frac{l_1}{l_0} = \frac{K\,\lambda_2}{2\,\lambda_1}, \tag{4.11}$$

where $K = \frac{5\sqrt{10}\sqrt{\lambda_1(1-\lambda_2)}\lambda_1\lambda_2\mu_2}{2\sqrt{3\,\lambda_2{}^2+8\,\mu_2{}^2-8\,\mu_2}}$.

Proof By (4.10 b), $\triangle\mathbf{Q}_4\mathbf{Q}_2\mathbf{P}_5 \sim \triangle\mathbf{P}_1\mathbf{Q}_1\mathbf{Q}_3$, which means $\frac{R_5}{R_1} = \frac{R_4}{R_0} = \frac{T_2}{T_1}$. Then (4.8) becomes $25\,\lambda_1\lambda_2\mu_1\mu_2 = 4$. If we let $\frac{T_2}{T_1} = K > 0$, after substituting it and (4.10 a) into (4.9), we have $K = \frac{2\sqrt{10}\sqrt{\lambda_1(1-\lambda_2)}}{5\sqrt{25\,\lambda_1\lambda_2{}^3\mu_1\mu_2-\lambda_2{}^2+8\,\mu_2{}^2-8\,\mu_2}\,\mu_1}$, that is $K = \frac{5\sqrt{10}\sqrt{\lambda_1(1-\lambda_2)}\lambda_1\lambda_2\mu_2}{2\sqrt{3\,\lambda_2{}^2+8\,\mu_2{}^2-8\,\mu_2}}$. The equations (4.10 c) and (4.10 d) lead to two different representations of T_3, which implies $2\,\lambda_1 T_1 = l_0 K$. Then we have (4.11), if T_1 is substituted by $\frac{l_1}{\lambda_2}$.

Theorem 4.5 *The sextic PH curve* $\mathbf{r}(t)$ *degenerates to a quintic PH curve* $\hat{\mathbf{r}}(t)$ *if and only if*

$$3\,l_5^2\left(3\,\lambda_2^2+8\,\mu_2^2-8\,\mu_2\right)\mu_1^2 = 500\,l_0 l_5\lambda_1^3\left(1-\lambda_2\right)\mu_1^2\mu_2^3\left(1-\mu_2\right)+100\,l_0^2\lambda_1^4\left(1-\lambda_2\right)^2\mu_2^2. \tag{4.12}$$

Proof From Equations (4.5) and (4.11), we get $R_1 = \frac{4\,l_5}{25\,K\lambda_1\lambda_2\mu_2}$. If R_1 and $25\,\lambda_1\lambda_2\mu_1\mu_2 = 4$ are substituted into (4.9), we have

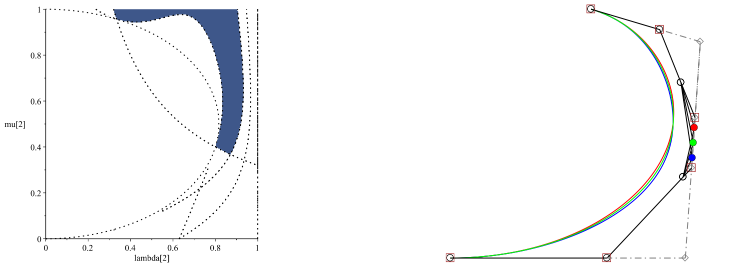
$$6\,l_5^2 = 625\,K^2 l_0 l_5\lambda_1^2\mu_1^2\mu_2^3\left(1-\mu_2\right)+125\,K^2 l_0^2\lambda_1^3\mu_2^2\left(1-\lambda_2\right). \tag{4.13}$$

Obviously, (4.13) is equivalent to (4.12) from $K = \frac{5\sqrt{10}\sqrt{\lambda_1(1-\lambda_2)}\lambda_1\lambda_2\mu_2}{2\sqrt{3\,\lambda_2{}^2+8\,\mu_2{}^2-8\,\mu_2}}$.

Since (4.7) is independent on $\mathbf{L}_2$ and $\mathbf{L}_3$, the sextic PH curve $\mathbf{r}(t)$ is independent of $\mathbf{P}_3$. That implies the following result.

Corollary 4.6 *For a quintic PH curve with control points* $\boldsymbol{P}_0$, $\boldsymbol{P}_1$, $\boldsymbol{Q}_3$, $\boldsymbol{Q}_4$, $\boldsymbol{P}_5$, $\boldsymbol{P}_6$ *as shown in Figure 2, where* $\boldsymbol{P}_6 - \boldsymbol{P}_5 = \boldsymbol{L}_5 = l_5\,e^{2\,\theta_5\,i}$ *and the polar angle of* $\boldsymbol{Q}_4 - \boldsymbol{Q}_3$ *is* θ_5. *Then* $\boldsymbol{P}_0$, $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, $\boldsymbol{P}_3$, $\boldsymbol{P}_4$, $\boldsymbol{P}_5$, $\boldsymbol{P}_6$ *form the control polygon for a sextic PH curve if* (4.12) *is satisfied, where* $\boldsymbol{P}_2$, $\boldsymbol{P}_4$ *are chosen based on* λ_2, μ_2, *and* $\boldsymbol{P}_3$ *is an arbitrary point on the segment* $\overline{\boldsymbol{Q}_3\boldsymbol{Q}_4}$.

Example 4.7 Set $l_0 = 1$, $\lambda_1 = \frac{1}{2}$, and we need estimate the possible range of λ_2, μ_2 before choosing them. From $K > 0$, we find the geometric feature of $\triangle\mathbf{P}_1\mathbf{Q}_1\mathbf{Q}_3$, $\triangle\mathbf{P}_5\mathbf{Q}_2\mathbf{Q}_4$ are $T_1 > R_0$, $T_1 > R_1$, $R_0 + R_1 > T_1$, $R_0 + T_1 > R_1$, $R_1 + T_1 > R_0$, $T_2 > R_4$, $T_2 > R_5$, $R_4 + R_5 > T_2$, $R_4 + T_2 > R_5$, $R_5 + T_2 > R_4$. We can get possible values of λ_2 and μ_2 in the shadow part of Figure 3 (a) We choose $\lambda_2 = \frac{9}{10}$, $\mu_2 = \frac{6}{10}$, and substitute them into Equations (4.9), (4.11) and (4.12). We can get all concrete values shown in the Figure 2. As we know, Corollary 4.6 shows the sextic PH curves are independent of $\mathbf{P}_3$. If we set ρ to be the proportion of $|\mathbf{P}_3 - \mathbf{Q}_3|$ in $|\mathbf{Q}_4 - \mathbf{Q}_3|$, then different ρ results in different sextic PH curves. For $\rho = 0.2, 0.5, 0.8$, the resulting curves are colored by red, green, blue in Figure 3 (b).



(a) Possible values of λ_2 and μ_2 (b) Three different sextic PH curves while $\rho = 0.2,\ 0.5,\ 0.8$.

Figure 3 The ranges of λ_2 and μ_2 and three first kinds of PH sextics

4.2. $(\deg(W(t)),\ \deg(\mathbf{Q}(t))) = (1, 2)$

Similarly, for $t \in [0,1]$, the hodograph of the sextic PH curve $\mathbf{r}(t)$ is

$$\mathbf{r}'(t) = [w_0 B_0^1(t) + B_1^1(t)][\mathbf{u}_0 B_0^2(t) + \mathbf{u}_1 B_1^2(t) + \mathbf{u}_2 B_2^2(t)]^2, \tag{4.14}$$

where w_0 and $\mathbf{u}_i$ $(i = 0, 1, 2)$ are uncertain real coefficient and complex coefficients, respectively.

The relationships between control points and coefficients are

$$6(\mathbf{P}_1 - \mathbf{P}_0) = 6\mathbf{L}_0 = w_0 \mathbf{u}_0^2, \tag{4.15 a}$$

$$30(\mathbf{P}_2 - \mathbf{P}_1) = 30\mathbf{L}_1 = \mathbf{u}_0^2 + 4\mathbf{u}_0\mathbf{u}_1 w_0, \tag{4.15 b}$$

$$30(\mathbf{P}_3 - \mathbf{P}_2) = 30\mathbf{L}_2 = 2\mathbf{u}_0\mathbf{u}_1 + \mathbf{u}_0\mathbf{u}_2 w_0 + 2\mathbf{u}_1^2 w_0, \tag{4.15 c}$$

$$30(\mathbf{P}_4 - \mathbf{P}_3) = 30\mathbf{L}_3 = 2\mathbf{u}_1\mathbf{u}_2 w_0 + \mathbf{u}_0\mathbf{u}_2 + 2\mathbf{u}_1^2, \tag{4.15 d}$$

$$30(\mathbf{P}_5 - \mathbf{P}_4) = 30\mathbf{L}_4 = \mathbf{u}_2^2 w_0 + 4\mathbf{u}_1\mathbf{u}_2, \tag{4.15 e}$$

$$6(\mathbf{P}_6 - \mathbf{P}_5) = 6\mathbf{L}_5 = \mathbf{u}_2^2. \tag{4.15 f}$$

Note $w_0 > 0$ makes the sextic PH curve $\mathbf{r}(t)$ regular. Suppose the control polygon is the same as in Figure 2, and $\mathbf{u}_0 = u_0$.

From (4.15 a) and (4.15 f), we have $u_0^2 = \frac{6l_0}{w_0}$, and $\mathbf{u}_2^2 = 6\mathbf{L}_5$. By substituting them to (4.15 b) and (4.15 e), we get $u_0\mathbf{u}_1 = \frac{15w_0\mathbf{L}_1 - 3l_0}{2w_0^2}$, $\mathbf{u}_1\mathbf{u}_2 = \frac{15\mathbf{L}_4 - 3w_0\mathbf{L}_5}{2}$. Then $\mathbf{u}_0\mathbf{u}_2 = \frac{u_0\mathbf{u}_1}{\mathbf{u}_1\mathbf{u}_2}\mathbf{u}_2^2 = \frac{30w_0\mathbf{L}_1\mathbf{L}_5 - 6l_0\mathbf{L}_5}{5w_0^2\mathbf{L}_4 - w_0^3\mathbf{L}_5}$. By (4.15 c) and (4.15 d), we get the following result

Theorem 4.8 *A planar sextic Bézier curve $\mathbf{r}(t)$ is a PH curve, whose hodograph can be represented as (4.14), if and only if*

$$A_0 + A_1 w_0 + 5A_2 w_0^2 - 5A_3 w_0^3 - 10A_4 w_0^4 + A_5 w_0^5 = 0, \tag{4.16}$$

where $A_0 = 5\,\boldsymbol{L}_0 \cdot \boldsymbol{L}_4 + 2\,\boldsymbol{L}_0 \cdot \boldsymbol{L}_5$, $A_1 = 3\,\boldsymbol{L}_0 \cdot \boldsymbol{L}_5 + 25\,\boldsymbol{L}_1 \cdot \boldsymbol{L}_4 + 10\,\boldsymbol{L}_1 \cdot \boldsymbol{L}_5$, $A_2 = 3\,\boldsymbol{L}_1 \cdot \boldsymbol{L}_5 + 10\,\boldsymbol{L}_2 \cdot \boldsymbol{L}_4 - 10\,\boldsymbol{L}_3 \cdot \boldsymbol{L}_4$, $A_3 = 2\,\boldsymbol{L}_2 \cdot \boldsymbol{L}_5 - 2\,\boldsymbol{L}_3 \cdot \boldsymbol{L}_5 - 5\,\boldsymbol{L}_4{}^2$, $A_4 = -10\,\boldsymbol{L}_4 \cdot \boldsymbol{L}_5$, and $A_5 = \boldsymbol{L}_5{}^2$.

When $w_0 = 1$, the sextic PH curve $\mathbf{r}(t)$ degenerates to a quintic PH curve $\tilde{\mathbf{r}}'(t) = [\mathbf{u}_0 B_0^2(t) + \mathbf{u}_1 B_1^2(t) + \mathbf{u}_2 B_2^2(t)]^2$, $t \in [0,1]$, with the control points $\mathbf{Q}_i$, $i = 0, \ldots, 5$. According to the geometric characteristic of quintic PH curves [3], we can get that

$$\begin{aligned}
&\mathbf{Q}_0 = \mathbf{P}_0,\\
&\mathbf{Q}_1 = \frac{6}{5}\mathbf{P}_1 - \frac{1}{5}\mathbf{P}_0,\\
&\mathbf{Q}_2 = \frac{3}{2}\mathbf{P}_2 - \frac{3}{5}\mathbf{P}_1 + \frac{1}{10}\mathbf{P}_0,\\
&\mathbf{Q}_3 = 2\mathbf{P}_3 - \frac{3}{2}\mathbf{P}_2 + \frac{3}{5}\mathbf{P}_1 - \frac{1}{10}\mathbf{P}_0 = \frac{3}{2}\mathbf{P}_4 - \frac{3}{5}\mathbf{P}_5 + \frac{1}{10}\mathbf{P}_6,\\
&\mathbf{Q}_4 = \frac{6}{5}\mathbf{P}_5 - \frac{1}{5}\mathbf{P}_6,\\
&\mathbf{Q}_5 = \mathbf{P}_6.
\end{aligned}$$

For instance, Figure 4 shows the degenerated quintic PH curve.

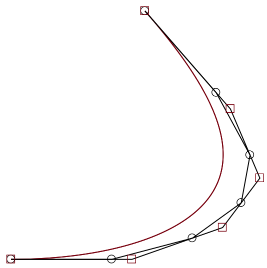


Figure 4 Degenerated PH quintic for the second kind of PH sextic

5. Identification by PH cubics and quintics

5.1. Identification by PH cubics

For the first kind of sextic PH curve $\mathbf{r}(t)$, whose hodograph is represented as (4.1), let

$$\mathbf{r}_1'(t) = \mathbf{G}^2(t) = [\mathbf{u}_0 B_0^1(t) + \mathbf{u}_1 B_1^1(t)]^2, \quad t \in [0,1].$$

This means that $\mathbf{r}_1(t)$ is a cubic PH curve with control points $\mathbf{Q}_0$, $\mathbf{Q}_1$, $\mathbf{Q}_2$, $\mathbf{Q}_3$ and control legs $\Delta\mathbf{Q}_0 = \widehat{\mathbf{L}}_0 = \frac{1}{3}\,\mathbf{u}_0^2$, $\Delta\mathbf{Q}_1 = \widehat{\mathbf{L}}_1 = \frac{1}{3}\,\mathbf{u}_0\,\mathbf{u}_1$ and $\Delta\mathbf{Q}_2 = \widehat{\mathbf{L}}_2 = \frac{1}{3}\,\mathbf{u}_1^2$. Similarly to the construction of quartic PH curve [20], we add extra eight points $\mathbf{E}$, $\mathbf{F}$, $\mathbf{H}$, $\mathbf{G}$, $\mathbf{K}$, $\mathbf{J}$, $\mathbf{M}$, $\mathbf{N}$ satisfying (see Figure 5)

$$\begin{aligned}
&\mathbf{L}_0 = \frac{w_0}{2}\,\widehat{\mathbf{L}}_0 = \overrightarrow{\mathbf{P}_0\mathbf{P}_1},\\
&\mathbf{L}_1 = \frac{3\,w_1}{10}\,\widehat{\mathbf{L}}_0 + \frac{w_0}{5}\,\widehat{\mathbf{L}}_1 = \overrightarrow{\mathbf{P}_1\mathbf{E}} + \overrightarrow{\mathbf{E}\mathbf{P}_2},\\
&\mathbf{L}_2 = \frac{3\,w_2}{20}\,\widehat{\mathbf{L}}_0 + \frac{3\,w_1}{10}\,\widehat{\mathbf{L}}_1 + \frac{w_0}{20}\,\widehat{\mathbf{L}}_2 = \overrightarrow{\mathbf{P}_2\mathbf{F}} + \overrightarrow{\mathbf{F}\mathbf{H}} + \overrightarrow{\mathbf{H}\mathbf{P}_3},\\
&\mathbf{L}_3 = \frac{1}{20}\,\widehat{\mathbf{L}}_0 + \frac{3\,w_2}{10}\,\widehat{\mathbf{L}}_1 + \frac{3\,w_1}{20}\,\widehat{\mathbf{L}}_2 = \overrightarrow{\mathbf{P}_3\mathbf{G}} + \overrightarrow{\mathbf{G}\mathbf{K}} + \overrightarrow{\mathbf{K}\mathbf{P}_4},\\
&\mathbf{L}_4 = \frac{1}{5}\,\widehat{\mathbf{L}}_1 + \frac{3\,w_2}{10}\,\widehat{\mathbf{L}}_2 = \overrightarrow{\mathbf{P}_4\mathbf{J}} + \overrightarrow{\mathbf{J}\mathbf{P}_5},\\
&\mathbf{L}_5 = \frac{1}{2}\,\widehat{\mathbf{L}}_2 = \overrightarrow{\mathbf{P}_5\mathbf{P}_6},
\end{aligned}$$

and

$$\overrightarrow{\mathbf{P}_0\mathbf{P}_1} = a\,\overrightarrow{\mathbf{P}_0\mathbf{M}},\ \overrightarrow{\mathbf{E}\mathbf{P}_2} = b\,\overrightarrow{\mathbf{M}\mathbf{N}},\ \overrightarrow{\mathbf{H}\mathbf{P}_3} = c\,\overrightarrow{\mathbf{N}\mathbf{P}_6},\ \overrightarrow{\mathbf{K}\mathbf{P}_4} = d\,\overrightarrow{\mathbf{N}\mathbf{P}_6},\ \overrightarrow{\mathbf{J}\mathbf{P}_5} = e\,\overrightarrow{\mathbf{N}\mathbf{P}_6}, \tag{5.1}$$

where $a = \frac{10\,w_0}{9\,w_0+6\,w_1+3\,w_2}$, $b = \frac{2\,w_0}{2\,w_0+3\,w_1+3\,w_2+2+w_0\,\cos\theta_5+\cos\theta_5}$, $c = \frac{w_0}{3\,w_1+9+6\,w_2}$, $c : d : e = w_0 : 3\,w_1 : 6\,w_2$. Note conditions $3\,w_1 > w_0 > 0$, $3\,w_2 > 1$ make $0 < a,\ b,\ c,\ d,\ e,\ f < 1$, and

$$\pi - \angle\mathbf{N} = \theta_5,\ |\overrightarrow{\mathbf{M}\mathbf{N}}|^2 = \frac{8\,ac}{5\,b^2}|\overrightarrow{\mathbf{P}_0\mathbf{M}}| \cdot |\overrightarrow{\mathbf{N}\mathbf{P}_6}|. \tag{5.2}$$

It is easy to get following results.

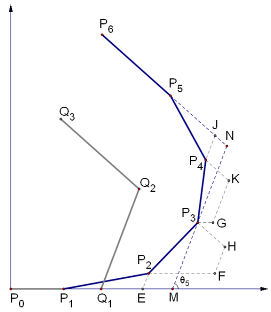
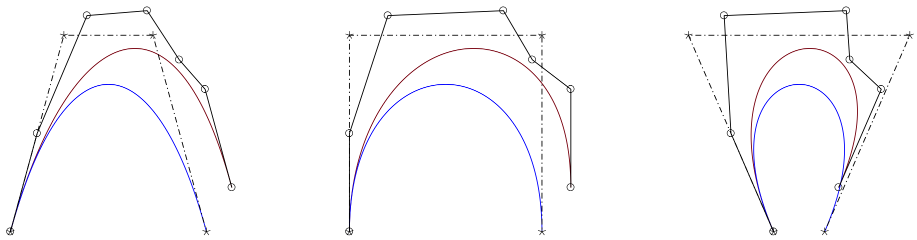
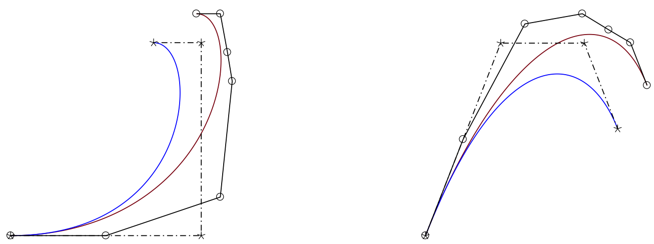


Figure 5 The first kind of control polygon of PH sextic by PH cubic



(a) Symmetric (b) Symmetric (c) Symmetric



(d) Asymmetric (e) Asymmetric

Figure 6 PH cubics and the resulting PH sextics

Proposition 5.1 *A planar sextic Bézier curve* $\mathbf{r}(t)$ *with control points* $\boldsymbol{P}_i$ $(i = 0, \ldots, 6)$ *is a PH curve, whose hodograph is defined in (4.1) with* $3\,w_1 > w_0 > 0$ *and* $3\,w_2 > 1$*, if and only if relations (5.1) and (5.2) are satisfied.*

As we know, PH cubics have no real inflection points and they must be convex [1]. Then another identification of convex PH sextics for the first kind is

Corollary 5.2 *A planar sextic Bézier curve* $\mathbf{r}(t)$, *which is constructed by a PH cubic* $\mathbf{r}_1(t)$ *and satisfies (5.1) and (5.2), is a strictly convex PH curve if* $3\,w_1 > w_0 > 0$ *and* $3\,w_2 > 1$.

Example 5.3 We choose five PH cubics with symmetric and asymmetric Bézier control polygons from [1], and relevant constructed PH sixtics are shown in Figure 6.

5.2. Identification by PH quintic

For the second kind of sextic PH curve $\mathbf{r}(t)$, whose hodograph is represented as (4.14), let

$$\mathbf{r}_2'(t) = \mathbf{G}^2(t) = [\mathbf{u}_0 B_0^2(t) + \mathbf{u}_1 B_1^2(t) + \mathbf{u}_2 B_2^2(t)]^2, \quad t \in [0,1],$$

which means that $\mathbf{r}_2(t)$ is a quintic PH curve with control points $\mathbf{Q}_0$, $\mathbf{Q}_1$, $\mathbf{Q}_2$, $\mathbf{Q}_3$, $\mathbf{Q}_4$, $\mathbf{Q}_5$ and control legs $\Delta\mathbf{Q}_0 = \widehat{\mathbf{L}}_0 = \frac{1}{5}\,\mathbf{u}_0^2$, $\Delta\mathbf{Q}_1 = \widehat{\mathbf{L}}_1 = \frac{1}{5}\,\mathbf{u}_0\,\mathbf{u}_1$, $\Delta\mathbf{Q}_2 = \widehat{\mathbf{L}}_2 = \frac{1}{15}\,(\mathbf{u}_0\,\mathbf{u}_2 + 2\,\mathbf{u}_1^2)$, $\Delta\mathbf{Q}_3 = \widehat{\mathbf{L}}_3 = \frac{1}{5}\,\mathbf{u}_1\,\mathbf{u}_2$, $\Delta\mathbf{Q}_4 = \widehat{\mathbf{L}}_4 = \frac{1}{5}\,\mathbf{u}_2^2$. Similarly, we add extra four points $\mathbf{E}$, $\mathbf{F}$, $\mathbf{G}$, $\mathbf{K}$ satisfying (see Figure 7)

$$\begin{aligned}
\mathbf{L}_0 &= \frac{5\,w_0}{6}\,\widehat{\mathbf{L}}_0 = \overrightarrow{\mathbf{P}_0\mathbf{P}_1},\\
\mathbf{L}_1 &= \frac{1}{6}\,\widehat{\mathbf{L}}_0 + \frac{2\,w_0}{3}\,\widehat{\mathbf{L}}_1 = \overrightarrow{\mathbf{P}_1\mathbf{E}} + \overrightarrow{\mathbf{E}\mathbf{P}_2},\\
\mathbf{L}_2 &= \frac{1}{3}\,\widehat{\mathbf{L}}_1 + \frac{w_0}{2}\,\widehat{\mathbf{L}}_2 = \overrightarrow{\mathbf{P}_2\mathbf{F}} + \overrightarrow{\mathbf{F}\mathbf{P}_3},\\
\mathbf{L}_3 &= \frac{1}{2}\,\widehat{\mathbf{L}}_2 + \frac{w_0}{3}\,\widehat{\mathbf{L}}_3 = \overrightarrow{\mathbf{P}_3\mathbf{G}} + \overrightarrow{\mathbf{G}\mathbf{P}_4},\\
\mathbf{L}_4 &= \frac{2}{3}\,\widehat{\mathbf{L}}_3 + \frac{w_0}{6}\,\widehat{\mathbf{L}}_4 = \overrightarrow{\mathbf{P}_4\mathbf{K}} + \overrightarrow{\mathbf{K}\mathbf{P}_5},\\
\mathbf{L}_5 &= \frac{5}{6}\,\widehat{\mathbf{L}}_4 = \overrightarrow{\mathbf{P}_5\mathbf{P}_6},
\end{aligned}$$

such that

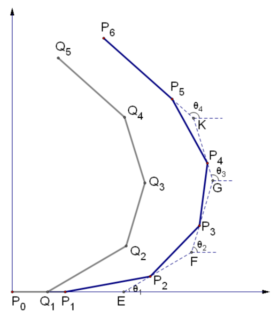


Figure 7 The second kind of control polygon of PH sextic by PH quintic

$$\overrightarrow{\mathbf{P}_0\mathbf{P}_1} = a\,\overrightarrow{\mathbf{P}_0\mathbf{E}},\ \ \overrightarrow{\mathbf{E}\mathbf{P}_2} = b\,\overrightarrow{\mathbf{E}\mathbf{F}},\ \ \overrightarrow{\mathbf{F}\mathbf{P}_3} = c\,\overrightarrow{\mathbf{F}\mathbf{G}},\ \overrightarrow{\mathbf{G}\mathbf{P}_4} = d\,\overrightarrow{\mathbf{G}\mathbf{K}},\ \overrightarrow{\mathbf{K}\mathbf{P}_5} = e\,\overrightarrow{\mathbf{K}\mathbf{P}_6}, \tag{5.3}$$

where $a = \frac{5\,w_0}{5\,w_0+1}$, $b = \frac{2\,w_0}{2\,w_0+1}$, $c = \frac{w_0}{w_0+1}$, $d = \frac{w_0}{w_0+2}$, and $e = \frac{w_0}{w_0+5}$. And $w_0 > 0$ makes $0 < a,\ b,\ c,\ d,\ e,\ f < 1$, and

$$\theta_1 + \theta_4 = \theta_2 + \theta_3,\ \frac{|\overrightarrow{\mathbf{EF}}|^2}{|\overrightarrow{\mathbf{GK}}|^2} = \frac{4\,ad^2}{5\,b^2 e}\,\frac{|\overrightarrow{\mathbf{P}_0\mathbf{E}}|}{|\overrightarrow{\mathbf{KP}_6}|}, \tag{5.4 a}$$

$$40\,abc|\overrightarrow{\mathbf{P}_0\mathbf{E}}||\overrightarrow{\mathbf{EF}}||\overrightarrow{\mathbf{FG}}|\cos(\theta_2) = 16\,a^2 d|\overrightarrow{\mathbf{P}_0\mathbf{E}}|^2|\overrightarrow{\mathbf{GK}}|\cos(\theta_4) + 25\,b^3|\overrightarrow{\mathbf{EF}}|^3\cos(\theta_1), \tag{5.4 b}$$

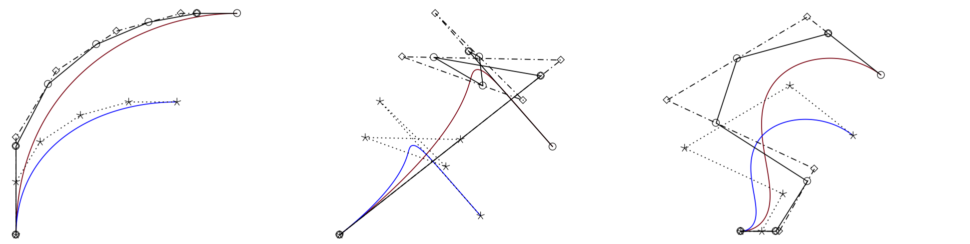
$$40\,abc|\overrightarrow{\mathbf{P}_0\mathbf{E}}||\overrightarrow{\mathbf{EF}}||\overrightarrow{\mathbf{FG}}|\sin(\theta_2) = 16\,a^2 d|\overrightarrow{\mathbf{P}_0\mathbf{E}}|^2|\overrightarrow{\mathbf{GK}}|\sin(\theta_4) + 25\,b^3|\overrightarrow{\mathbf{EF}}|^3\sin(\theta_1). \tag{5.4 c}$$

Proposition 5.4 *A planar sextic Bézier curve* $\mathbf{r}(t)$ *with control points* $\boldsymbol{P}_i$ $(i = 0,\ldots,6)$ *is a PH curve, whose hodograph is defined in (4.14) with* $w_0 > 0$*, if and only if relations (5.3), (5.4 a), (5.4 b) and (5.4 c) are satisfied.*

Since PH quintics may have inflection points [1], another identification of PH sextics for the second kind is represented

Corollary 5.5 *A planar sextic PH curve* $\mathbf{r}(t)$*, which is constructed by a quintic PH curve* $\mathbf{r}_2(t)$ *and satisfies (5.3), (5.4 a), (5.4 b) and (5.4 c), preserves the convexity with* $\mathbf{r}_2(t)$ *if* $w_0 > 0$*.*

Remark 5.6 We find $\frac{|\overrightarrow{\mathbf{EF}}|^2}{|\overrightarrow{\mathbf{GK}}|^2} = \frac{|\overrightarrow{\mathbf{P}_0\mathbf{E}}|}{|\overrightarrow{\mathbf{KP}_6}|}$ if and only if $w_0 = 1$. This coincides with the condition shown in Figure 4, which means that the sextic PH curve degenerates to a quintic PH curve.



(a) Convex PH quintic (b) Nonconvex PH quintic (c) Nonconvex PH quintic

Figure 8 PH quintics and constructed PH sextics

Example 5.7 We choose three quintic PH curves with convex and nonconvex Bézier control polygons in this example. The first two of which are from [4], and the other one is from [1]. The sextic PH curves constructed from PH quintics are illustrated in Figure 8.

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