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Reconstruction of the Linear Ordinary Differential System Based on Discrete Points

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Dedicated to Professor Renhong WANG on the Occasion of His Eightieth Birthday

Abstract In this paper, we discuss an inverse problem, i.e., the reconstruction of a linear differential dynamic system from the given discrete data of the solution. We propose a model and a corresponding algorithm to recover the coefficient matrix of the differential system based on the normal vectors from the given discrete points, in order to avoid the problem of parameterization in curve fitting and approximation. We also give some theoretical analysis on our algorithm. When the data points are taken from the solution curve and the set composed of these data points is not degenerate, the coefficient matrix A reconstructed by our algorithm is unique from the given discrete and noisefree data. We discuss the error bounds for the approximate coefficient matrix and the solution which are reconstructed by our algorithm. Numerical examples demonstrate the effectiveness of the algorithm.

Keywords differential system; discrete data; normal vector method; least square method; parameterization

MR(2010) Subject Classification 65D10; 65J22

1. Introduction

The theory of differential equations is an important research field in modern mathematics. The dynamical system is a system in which a function describes the time dependence of a point in a geometrical space. Differential equations and dynamical system theory have a very wide range of applications, such as in physics, mechanics, chemistry, biology, medicine, engineering, meteorology, statistics and other disciplines [1–4]. There are many classical differential models in practice including population model, epidemic model, population ecological model [5] and Lorenz equation.

The theory of differential equations has been studied many years, since differential equations and dynamic systems have a large applied background. Usually the research on the differential equations is about the behavior of the solution of the differential equation [6–8]. But there is also a kind of inverse problem: how to describe a dynamical system from the given discrete points of a solution in practical applications.

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In geometric design, there are many effective methods to approximate a curve from the discrete data, such as the polynomial curve, Bézier curve, B-spline curve, NURBS curve [9–11], unit quaternion curve [12] and so on. However, these methods cannot directly represent the kinetic properties of the curves or surfaces from the discrete geometric data. Some methods have been proposed to construct the surfaces according to their physical constraints such as hydrodynamic and aerodynamic constrains. Ye et al. [13] developed a method for geometric design of functional surfaces directly from their underlying physical constraints which involve normal vectors of surfaces.

There are some methods of differential system reconstruction from the given discrete points in the natural sciences. Wang and Lin [14] proposed a new K-value estimation method for Logistic curve by analyzing the characteristics of the Logistic equation. The number of susceptible, infected and removed in Beijing city has been calculated using susceptible-infected-removed (SIR) model, all parameters with epidemiological meaning including transmission rate, removal ratio and threshold value have been estimated by the difference method in [15].

In this paper, we also consider the reconstruction problem of the differential system from the given discrete points. The simplest way to calculate the derivatives is by the difference method. However, the difference method needs the parameterization of the observed data and the fitting accuracy is largely affected by the parameterization. In order to avoid the problem of parameterization, we discuss the reconstruction method of differential system based on normal vectors from the given discrete points.

The organization of our paper is as follows: In Section 2, we propose a model and devise an algorithm to reconstruct the differential system based on normal vectors from the given discrete points. We deduce some theoretical analysis on our algorithm in Section 3. We point out that when the points are taken from the solution curve and the set composed of these data points is not degenerate, the coefficient matrix A reconstructed by our algorithm is unique from the given discrete and noisefree data. And we discuss the error bound between the coefficient matrix which is reconstructed by our algorithm and the coefficient matrix of the original differential system. The error bound between the solution of the original differential system and the solution of the the reconstructed differential system is also discussed. In Section 4, the numerical experiments show the effectiveness of our algorithm.

2. The reconstruction of homogeneous linear ordinary differential system

For the given set of discrete points on an exponential curve, $X_i \in \mathbb{R}^3$, $i = 0, 1, 2, \ldots, n$, the problem is to reconstruct the curve $X(t)$ as a solution of a homogeneous linear differential system(LDS) with an initial condition, i.e.,

$$\begin{cases} \frac{\mathrm{d}X(t)}{\mathrm{d}t} = AX(t), \\ X(0) = X_0, \end{cases} \tag{1}$$

where $A \in \mathbb{R}^{3\times 3}$ is an unknown matrix to be determined.

When A is a constant matrix, Eq. (1) is the simplest differential dynamic system, and the solution can be represented explicitly by

$$X(t) = e^{At} X_0, \tag{2}$$

where the exponential of the matrix At is defined by the Taylor's series

$$e^{At} = I + tA + \frac{t^2}{2!} A^2 + \cdots + \frac{t^k}{k!} A^k + \cdots . \tag{3}$$

If the discrete points are parameterized by $X_i = X(t_i)$, $i = 0, 1, \ldots, n$, then the reconstruction problem can be expressed as the fitting problem

$$P_0 \ \text{ model}: \min_{A \in \mathbb{R}^{3\times 3}} \sum_{i=0}^{n} \|X(t_i) - X_i\|^2 = \min_{A \in \mathbb{R}^{3\times 3}} \sum_{i=0}^{n} \|e^{At_i} X_0 - X_i\|^2. \tag{4}$$

However, the above optimization problem cannot be solved directly, since the unknown matrix A is involved in the Taylor's series of the exponential function.

Considering the differential system, we can use the numerical difference method to reconstruct this system from discrete points, such as by two points scheme,

$$\frac{\mathrm{d}X(t_i)}{\mathrm{d}t} \simeq \frac{X_i - X_{i-1}}{t_i - t_{i-1}}.$$

Therefore, we can consider the following optimization model

$$P_1 \ \text{ model}: \min_{A \in \mathbb{R}^{3\times 3}} \sum_{i=1}^{n} \|X_i - X_{i-1} - (t_i - t_{i-1}) A X_i\|^2.$$

This method of parameterization is from the observed data. Usually, we use the method of accumulative chord lengths. However, the method of parameterization directly affects the fitting accuracy, and the fitting result is sensitive to the parameterization. As shown by the blue line in Figures 1–3, fitting accuracy is poor by the difference method P_1 model.

Since the tangent vector is perpendicular to the normal vectors, we consider the following optimization model

$$\min_{A \in \mathbb{R}^{3\times 3}} \sum_{i=1}^{n-1} \{(1-\omega)(N_i^T A X_i)^2 + \omega(\tilde{N}_i^T A X_i)^2\}, \tag{5}$$

where N_i and $\tilde{N}_i$,$i = 1, 2, \ldots, n-1$ represent the principal normal vectors and binormal vectors of the spatial curve, respectively, and ω is a constant in $[0, 1]$. It is clear that the zero matrix is a global optimal solution of (5). In order to avoid zero solution of the optimization problem, we need a constraint condition of A.

In fact, let $\tilde{A} = cA$, where c is a nonzero constant. The two solutions for the two differential systems $\frac{\mathrm{d}X(t)}{\mathrm{d}t} = AX(t)$ and $\frac{\mathrm{d}X(\tau)}{\mathrm{d}\tau} = \tilde{A}X(\tau)$ are $X_1(t) = e^{At}$ and $X_2(\tau) = e^{\tilde{A}\tau}$, with corresponding parameters t and τ satisfying $t = c\tau$. It means that $X_1(t)$ and $X_2(\tau)$ can represent the same solution curve. So we can introduce a condition by $\|A\|_F^2 = 1$. Therefore, we can consider the following optimization model with constraints

$$\begin{aligned} P_2 \ \text{ model}: \ & \min\nolimits_{A \in \mathbb{R}^{3\times 3}} f(A) = \min\nolimits_{A \in \mathbb{R}^{3\times 3}} \textstyle\sum_{i=1}^{n-1} \{(1-\omega)(N_i^T A X_i)^2 + \omega(\tilde{N}_i^T A X_i)^2\}, \\ & \text{s.t. } \|A\|_F^2 = 1. \end{aligned} \tag{6}$$

If we denote the matrix $A=\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}$ as a vector $Y=[a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7\ a_8\ a_9]^T$, and denote $N_i=[n_{i,1},n_{i,2},n_{i,3}]^T$, $\tilde{N}_i=[\tilde{n}_{i,1},\tilde{n}_{i,2},\tilde{n}_{i,3}]^T$. Then the optimization model (6) can be transformed to the following model

$$\begin{aligned}&\min_{Y\in\mathbb{R}^9} g(Y)=\min_{Y\in\mathbb{R}^9}\textstyle\sum_{i=1}^{n-1} Y^T\{(1-\omega)B_i^TB_i+\omega C_i^TC_i\}Y=\min_{Y\in\mathbb{R}^9} Y^TFY,\\ &\text{s.t.}\|Y\|_2^2=1,\end{aligned} \tag{7}$$

where $B_i=[n_{i,1}X_i^T,n_{i,2}X_i^T,n_{i,3}X_i^T]=N_i^T\otimes X_i^T$ and $C_i=[\tilde{n}_{i,1}X_i^T,\tilde{n}_{i,2}X_i^T,\tilde{n}_{i,3}X_i^T]=\tilde{N}_i^T\otimes X_i^T$ are two row vectors in $\mathbb{R}^9$, and $F=\sum_{i=1}^{n-1}\{(1-\omega)B_i^TB_i+\omega C_i^TC_i\}$, is a 9×9 positive semidefinite matrix. Here $\otimes$ denotes the Kronecker product.

Let us denote the lagrangian function of the optimization model (7) as

$$L(Y,\mu)=g(Y)+\mu(1-\|Y\|_2^2)=Y^TFY+\mu(1-Y^TY). \tag{8}$$

The partial derivatives of $L(Y,\mu)$ are

$$\begin{cases} \frac{\partial}{\partial Y}(L(Y,\mu))=FY-\mu Y,\\ \frac{\partial}{\partial \mu}(L(Y,\mu))=1-Y^TY. \end{cases} \tag{9}$$

Let $\begin{cases} \frac{\partial}{\partial Y}(L(Y,\mu))=0\\ \frac{\partial}{\partial \mu}(L(Y,\mu))=0 \end{cases}$. We have $\begin{cases} FY=\mu YX\\ Y^TY=1 \end{cases}$, i.e., μ should be an eigenvalue of F and Y is the corresponding eigenvector satisfying $\|Y\|_2^2=1$. So $\min g(Y)=\min Y^TFY=\min Y^T(\mu Y)=\min\mu$, i.e., μ is the smallest eigenvalue of F.

From the given data, we propose an algorithm for reconstructing the differential system as follows:

Algorithm Input: given the discrete points $X_i, i=1,2,\ldots,n$;

(i) Compute the principal normal vectors $N_i=[n_{i,1},n_{i,2},n_{i,3}]^T$, and binormal vectors $\tilde{N}_i=[\tilde{n}_{i,1},\tilde{n}_{i,2},\tilde{n}_{i,3}]^T$ of the discrete points $X_i, i=1,2,\ldots,n$;

(ii) Compute F by $F=(1-\omega)B_i^TB_i+\omega C_i^TC_i$, B_i and C_i are defined in the optimization model (7);

(iii) Compute the unit eigenvector $Y=[a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7\ a_8\ a_9]^T$ corresponding to the smallest eigenvalue of F.

Output: $A=\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}$.

3. Algorithm analysis

In Section 2, we have shown that when $\tilde{A}=cA$, $X_1(t)=e^{At}$ and $X_2(\tau)=e^{\tilde{A}\tau}$ can represent the same curve under appropriate parameters, i.e., $t=c\tau$. So in the optimization model (6), we assume $\|A\|_F=1$. In this section, we also consider the case $\|A\|_F=1$.

If we denote by $\tilde{A}$ the matrix computed by our algorithm, then the reconstructed differential system can be represented as

$$\begin{cases} \frac{\mathrm{d}X(\tau)}{\mathrm{d}\tau} = \tilde{A}X(\tau), \\ X(0) = X_0. \end{cases} \tag{10}$$

In this section, we analyze the error between A and $\tilde{A}$, and the error between the solutions of Eqs. (1) and (10).

3.1. The analysis of A

In Section 2, we have known that the model (6) and the model (7) is equivalent. We analyze the model (6) in this section.

3.1.1. The uniqueness of A obtained by the model (6)

Definition 3.1 *We call A and B are equivalent, if there is a nonzero constant c such that $A = cB$ and the solutions of the differential systems determined by A and B can represent the same curve.*

In this paper, A is said to be unique in the sense of equivalence.

Definition 3.2 *The set composed of discrete data points $X_i \in \mathbb{R}^3$, $i = 0, 1, 2, \dots, n$, is called degenerate if these discrete data points satisfy one of the following cases.*

(i) X_i lies on no more than three straight lines or a plane which passes through the origin;

(ii) X_i lies on a straight line and a plane which passes through the origin;

(iii) X_i lies on a straight line which does not pass through the origin;

(iv) X_i lies on a plane which does not pass through the origin and tangent vectors of these points lie on the same plane with $X_i - X_0$.

Theorem 3.3 *Suppose that all data points $X_i \in \mathbb{R}^3$, $i = 0, 1, 2, \dots, n$, are taken from an exponential curve without singularities, $N_i \in \mathbb{R}^3$ and $\tilde{N}_i \in \mathbb{R}^3$ are exact and data points $X_i \in \mathbb{R}^3$ are different from each other, and the set composed of discrete data points $X_i \in \mathbb{R}^3$, $i = 0, 1, 2, \dots, n$, is not degenerate, then the $A \in \mathbb{R}^{3\times 3}$ obtained by the model (6) is unique.*

Proof Assume that there is a nonzero matrix B which is obtained by the model (6) and $B \neq A$, $f(B) = 0$. We have

$$\begin{aligned} N_i^T A X_i = 0, \tilde{N}_i^T A X_i = 0, \\ N_i^T B X_i = 0, \tilde{N}_i^T B X_i = 0, \end{aligned} \tag{11}$$

for all $i = 0, 1, 2, \dots, n$. Therefore,

$$AX_i \in \mathrm{span}(N_i, \tilde{N}_i)^{\perp}, \tag{12}$$

$$BX_i \in \mathrm{span}(N_i, \tilde{N}_i)^{\perp}. \tag{13}$$

From (12) and (13), $AX_i = \lambda_i B X_i$, i.e.,

$$(A - \lambda_i B)X_i = 0. \tag{14}$$

Clearly, Eq. (14) has the non-zero solution if and only if $\det(A - \lambda_i B) = 0$. Denote $p(\lambda) = \det(A - \lambda B)$.

(i) If $p(\lambda)=0$ and $\det(A)\neq 0$, then $p(\lambda)$ is a cubic polynomial at most. So there are at most three real roots $\tilde{\lambda}_1,\tilde{\lambda}_2$ and $\tilde{\lambda}_3$ for $p(\lambda)=0$.

(a) When $\tilde{\lambda}_1=\tilde{\lambda}_2=\tilde{\lambda}_3=\lambda$

• If rank $(A-\lambda B)=0$, we have $A=\lambda B$, i.e., A is unique.

• If rank $(A-\lambda B)=1$ and $(A-\lambda B)X_i=0$, X_i lies in a plane which passes through the origin, i.e., the degenerate case (i). This contradicts the fact that $\{X_i\}$ is not degenerate.

• If rank $(A-\lambda B)=2$ and $(A-\lambda B)X_i=0$, X_i lies on a straight line which passes through the origin, i.e., the degenerate case (i). This contradicts the fact that $\{X_i\}$ is not degenerate.

(b) When $\tilde{\lambda}_1=\tilde{\lambda}_2\neq\tilde{\lambda}_3$

• If rank $(A-\tilde{\lambda}_1B)=1$ and rank $(A-\tilde{\lambda}_3B)=2$, we have $(A-\tilde{\lambda}_1B)X_j=0$ for j such that $\lambda_j=\tilde{\lambda}_1$ and $(A-\tilde{\lambda}_3B)X_k=0$ for k such that $\lambda_k=\tilde{\lambda}_3$. So X_i lies on a straight line and a plane intersecting at the orign, i.e., the degenerate case (ii). This contradicts the fact that $\{X_i\}$ is not degenerate.

• If rank $(A-\tilde{\lambda}_1B)=2$ and rank $(A-\tilde{\lambda}_3B)=2$, we have $(A-\tilde{\lambda}_1B)X_j=0$ for j such that $\lambda_j=\tilde{\lambda}_1$ and $(A-\tilde{\lambda}_3B)X_k=0$ for k such that $\lambda_k=\tilde{\lambda}_3$. So X_i lies on two straight lines intersecting at the orign, i.e., the degenerate case (i). This contradicts the fact that $\{X_i\}$ is not degenerate.

(c) When $\tilde{\lambda}_1\neq\tilde{\lambda}_2\neq\tilde{\lambda}_3$, $\mathrm{rank}(A-\tilde{\lambda}_1B)=\mathrm{rank}(A-\tilde{\lambda}_2B)=\mathrm{rank}(A-\tilde{\lambda}_3B)=2$, we have $(A-\tilde{\lambda}_1B)X_j=0$ for j such that $\lambda_j=\tilde{\lambda}_1$, $(A-\tilde{\lambda}_2B)X_k=0$ for k such that $\lambda_k=\tilde{\lambda}_2$ and $(A-\tilde{\lambda}_3B)X_m=0$ for m such that $\lambda_m=\tilde{\lambda}_3$. So X_i lies on three straight lines intersecting at the origin, i.e., the degenerate case (i). This contradicts the fact that $\{X_i\}$ is not degenerate.

(ii) If $\det(A)=0$, there are two cases based on the rank of A.

(a) If $\mathrm{rank}(A)=1$, then $X(t)=e^{At}X_0$ is a straight line. In fact, if $\mathrm{rank}(A)=1$, there are two column vectors $\mathbf{a}$, $\mathbf{b}$ such that $A=\mathbf{a}\mathbf{b}^T$. For each parameterization t, the tangent vector $AX(t)=\mathbf{a}\mathbf{b}^TX(t)=\mathbf{b}^TX(t)\mathbf{a}=\phi(t)\mathbf{a}$, where $\phi(t)=\mathbf{b}^TX(t)$, i.e., the tangent vectors $AX(t)$ are collinear. For arbitrary t_i, from $\frac{\mathrm{d}X(t)}{\mathrm{d}t}=AX(t)$, we have $X_i-X_0=\int_0^{t_i}AX(\tau)\mathrm{d}\tau=\int_0^{t_i}\phi(\tau)\mathrm{d}\tau\mathbf{a}$, i.e., the discrete data point X_i is collinear. Since the exponential curve without singularities, X_0 is a nonzero vector. This is the degenerate case (iii). It contradicts the fact that $\{X_i\}$ is not degenerate.

(b) If $\mathrm{rank}(A)=2$, denote $T_{i,j,k}=[AX(t_i)\quad AX(t_j)\quad AX(t_k)]$ for arbitrary t_i,t_j,t_k. Since $\det(A)=0$, we have $\det(T_{i,j,k})=\det(A)\cdot\det([X(t_i)\quad X(t_j)\quad X(t_k)])=0$. It means that the tangent vectors of $X(t)$ for arbitrary t are coplanar. It will show that the plane S_1 in which X_i-X_0 lie is coplanar with the plane S_2 in which AX_i lie. Suppose that there are two linearly independent vectors $\mathbf{a}$,$\mathbf{b}$ on the plane S_2. Then there are $\varphi(t)$ and $\psi(t)$ such that the tangent vector $AX(t)=\varphi(t)\mathbf{a}+\psi(t)\mathbf{b}$. For arbitrary t_i, from $\frac{dX(t)}{dt}=AX$ we have $X_i-X_0=\int_0^{t_i}AX(\tau)\mathrm{d}\tau=\int_0^{t_i}\varphi(\tau)\mathrm{d}\tau\mathbf{a}+\int_0^{t_i}\psi(\tau)\mathrm{d}\tau\mathbf{b}$, which means that X_i-X_0 is also on the plane S_2, i.e., the degenerate case (iv). This contradicts the fact that $\{X_i\}$ is not degenerate.

To sum up, the proposition has been proved.

For the inverse case of Theorem 3.3, we present the following remark.

Remark 3.4 Suppose that all data points $X_i\in\mathbb{R}^3$, $i=0,1,2,\ldots,n$, are taken from an exponential curve without singularities, $N_i \in \mathbb{R}^3$ and $\tilde{N}_i \in \mathbb{R}^3$ are exact and data points $X_i \in \mathbb{R}^3$ are different from each other. We can prove that the matrix A is not unique by the model (7) in the following case.

In fact, the vector Y is composed of all the entries of A, which is an eigenvector corresponding to the zero eigenvalue of the matrix $F \in \mathbb{R}^{9\times 9}$ in the model (7). If the rank$(F) < 8$, then the eigenvector Y is not unique, i.e., A is not unique.

(i) If X_i lies on a straight line which passes through the origin, then A is not unique.

In fact, assume X_i lies on the straight line $X(t) = t\mathbf{a}$ and $X_i = t_i\mathbf{a}$, where $\mathbf{a}$ is a vector. Denote the principal normal vector and binormal vector of this straight line as N_1, N_2, respectively.

$$\begin{aligned} B_i =& N_1^T \otimes X_i^T, C_i = N_2^T \otimes X_i^T, \\ B_i^T B_i =& (N_1^T \otimes X_i^T)^T (N_1^T \otimes X_i^T) = (N_1 N_1^T) \otimes (X_i X_i^T) \\ =& (N_1 N_1^T) \otimes [(t_i\mathbf{a})(t_i\mathbf{a})^T = t_i^2 (N_1 N_1^T) \otimes (\mathbf{a}\mathbf{a}^T) \end{aligned}$$

and

$$\sum_{i=0}^{n} B_i^T B_i = \zeta (N_1 N_1^T) \otimes (\mathbf{a}\mathbf{a}^T),$$

where $\zeta = \sum_{i=0}^{n} t_i^2$.

In the same way, we obtain $\sum_{i=0}^{n} C_i^T C_i = \zeta (N_2 N_2^T) \otimes (\mathbf{a}\mathbf{a}^T)$. Then

$$F = (1-\omega)\sum_{i=0}^{n} B_i^T B_i + \omega \sum_{i=0}^{n} C_i^T C_i = \zeta((1-\omega) N_1 N_1^T + \omega N_2 N_2^T) \otimes (\mathbf{a}\mathbf{a}^T).$$

Since rank$(N_1 N_1^T) = 1$, rank$(N_2 N_2^T) = 1$ and rank$(\mathbf{a}\mathbf{a}^T) = 1$, then rank$(F) =$ rank$((1-\omega) N_1 N_1^T + \omega N_2 N_2^T) \times$ rank$(\mathbf{a}\mathbf{a}^T) \le 2 < 8$. Thus A is not unique.

(ii) If X_i lies on a straight line which does not pass through the origin, then A is not unique.

In fact, assume X_i lies on the straight line $X(t) = X_0 + t\mathbf{b}$ and $X_i = X_0 + t_i\mathbf{b}$, where $\mathbf{b}$ is a vector. Denote the principal normal vector and binormal vector of this straight line as N_1, N_2, respectively.

$$\begin{aligned} B_i =& N_1^T \otimes X_i^T, C_i = N_2^T \otimes X_i^T, \\ B_i^T B_i =& (N_1^T \otimes X_i^T)^T (N_1^T \otimes X_i^T) = (N_1 N_1^T) \otimes (X_i X_i^T) \\ =& (N_1 N_1^T) \otimes [(X_0 + t_i\mathbf{b})(X_0 + t_i\mathbf{b})^T] \\ =& (N_1 N_1^T) \otimes (X_0 X_0^T) + t_i (N_1 N_1^T) \otimes (\mathbf{b} X_0^T) + t_i (N_1 N_1^T) \otimes (X_0 \mathbf{b}^T) + t_i^2 (N_1 N_1^T) \otimes (\mathbf{b}\mathbf{b}^T), \end{aligned}$$

and

$$\sum_{i=0}^{n} B_i^T B_i = (n+1)(N_1 N_1^T)\otimes(X_0 X_0^T) + \eta (N_1 N_1^T)\otimes(\mathbf{b} X_0^T) + \eta (N_1 N_1^T)\otimes(X_0 \mathbf{b}^T) + \zeta (N_1 N_1^T)\otimes(\mathbf{b}\mathbf{b}^T),$$

where $\eta = \sum_{i=0}^{n} t_i$ and $\zeta = \sum_{i=0}^{n} t_i^2$.

In the same way, we obtain

$$\sum_{i=0}^{n} C_i^T C_i = (n+1)(N_2 N_2^T)\otimes(X_0 X_0^T) + \eta (N_2 N_2^T)\otimes(\mathbf{b} X_0^T) + \eta (N_2 N_2^T)\otimes(X_0 \mathbf{b}^T) + \zeta (N_2 N_2^T)\otimes(\mathbf{b}\mathbf{b}^T).$$

Then

$$\begin{aligned}F &= (1-\omega)\sum_{i=0}^{n} B_i^T B_i + \omega \sum_{i=0}^{n} C_i^T C_i\\ &= ((1-\omega)N_1N_1^T + \omega N_2 N_2^T) \otimes ((n+1)(X_0X_0^T)) + \eta(\mathbf{b}X_0^T) + \eta(X_0\mathbf{b}^T) + \zeta(\mathbf{b}\mathbf{b}^T))\\ &= ((1-\omega)N_1N_1^T + \omega N_2 N_2^T) \otimes \left([X_0 \ \ \mathbf{b}] \begin{bmatrix} n+1 & \eta \\ \eta & \zeta \end{bmatrix} \begin{bmatrix} X_0^T \\ \mathbf{b}^T \end{bmatrix} \right).\end{aligned}$$

Since $\mathrm{rank}(N_1N_1^T) = 1$, $\mathrm{rank}(N_2N_2^T) = 1$ and $\mathrm{rank}([X_0 \ \ \mathbf{b}] \begin{bmatrix} n+1 & \eta \\ \eta & \zeta \end{bmatrix} \begin{bmatrix} X_0^T \\ \mathbf{b}^T \end{bmatrix}) \le 2$, then $\mathrm{rank}(F) \le 4 < 8$. Thus A is not unique.

(iii) If X_i lies on two straight lines which pass through the origin, then A is not unique.

In fact, assume $X_j, j \in I_1$, lies on the straight line $X^1(t) = t\mathbf{a}$ and $X_j = t_j\mathbf{a}$, $X_k, k \in I_2$, lies on the straight line $X^2(s) = s\mathbf{b}$ and $X_k = s_k\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are two vectors, $I_1 \cup I_2 = \{0, 1, 2, \dots, n\}$. Denote the principal normal vector and binormal vector of the straight line $X^1(t)$ as N_1, $\tilde{N}_1$, respectively, and the principal normal vector and binormal vector of the straight line $X^2(s)$ as N_2, $\tilde{N}_2$, respectively.

$$\begin{aligned} B_j &= N_1^T \otimes X_j^T, C_j = \tilde{N}_1^T \otimes X_j^T,\\ B_k &= N_2^T \otimes X_k^T, C_k = \tilde{N}_2^T \otimes X_k^T,\\ B_j^T B_j &= (N_1^T \otimes X_j^T)^T (N_1^T \otimes X_j^T) = (N_1N_1^T) \otimes (X_jX_j^T)\\ &= (N_1N_1^T) \otimes [(t_j\mathbf{a})(t_j\mathbf{a})^T] = t_j^2 (N_1N_1^T) \otimes (\mathbf{a}\mathbf{a}^T), \end{aligned}$$

and

$$\sum_{j\in I_1} B_j^T B_j = \xi(N_1N_1^T) \otimes (\mathbf{a}\mathbf{a}^T),$$

where $\xi = \sum_{j\in I_1} t_j^2$.

In the same way, we obtain

$$\sum_{j\in I_1} C_j^T C_j = \xi(\tilde{N}_1\tilde{N}_1^T)\otimes(\mathbf{a}\mathbf{a}^T),\ \sum_{k\in I_2} B_k^T B_k = \varepsilon(N_2N_2^T)\otimes(\mathbf{b}\mathbf{b}^T),\ \sum_{k\in I_2} C_k^T C_k = \varepsilon(\tilde{N}_2\tilde{N}_2^T)\otimes(\mathbf{b}\mathbf{b}^T),$$

where $\varepsilon = \sum_{k\in I_2} s_k^2$. Then

$$\begin{aligned} F =&(1-\omega)\sum_{j\in I_1} B_j^T B_j + \omega \sum_{j\in I_1} C_j^T C_j + (1-\omega)\sum_{k\in I_2} B_k^T B_k + \omega \sum_{k\in I_2} C_k^T C_k\\ =&\xi((1-\omega)N_1N_1^T + \omega\tilde{N}_1\tilde{N}_1^T) \otimes (\mathbf{a}\mathbf{a}^T) + \varepsilon((1-\omega)N_2N_2^T + \omega\tilde{N}_2\tilde{N}_2^T) \otimes (\mathbf{b}\mathbf{b}^T). \end{aligned}$$

Since $\mathrm{rank}(N_1N_1^T) = 1$, $\mathrm{rank}(N_2N_2^T) = 1$, $\mathrm{rank}(\tilde{N}_1\tilde{N}_1^T) = 1$, $\mathrm{rank}(\tilde{N}_2\tilde{N}_2^T) = 1$, $\mathrm{rank}(\mathbf{a}\mathbf{a}^T) = 1$ and $\mathrm{rank}(\mathbf{b}\mathbf{b}^T) = 1$, $\mathrm{rank}(F) \le 4 < 8$. Thus A is not unique.

(iv) If X_i lies on three straight lines which pass through the origin, then A is not unique.

In fact, assume $X_j, j \in I_1$, lies on the straight line $X^1(t) = t\mathbf{a}$ and $X_j = t_j\mathbf{a}$, $X_k, k \in I_2$, lies on the straight line $X^2(s) = s\mathbf{b}$ and $X_k = s_k\mathbf{b}$, and $X_m, m \in I_3$, lies on the straight line $X^3(v) = v\mathbf{c}$ and $X_m = v_m\mathbf{c}$, where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three vectors and $I_1 \cup I_2 \cup I_3 = \{0, 1, 2, \dots, n\}$. Denote the principal normal vector and binormal vector of the straight line $X^1(t)$ as N_1, $\tilde{N}_1$, respectively, the principal normal vector and binormal vector of the straight line $X^2(s)$ as N_2, $\tilde{N}_2$, respectively, and the principal normal vector and binormal vector of the straight line $X^3(v)$ as N_3, $\tilde{N}_3$, respectively.

$$
\begin{aligned}
B_j &= N_1^T \otimes X_j^T, C_j = \tilde{N}_1^T \otimes X_j^T, \\
B_k &= N_2^T \otimes X_k^T, C_k = \tilde{N}_2^T \otimes X_k^T, \\
B_m &= N_3^T \otimes X_m^T, C_m = \tilde{N}_3^T \otimes X_m^T, \\
B_j^T B_j &= (N_1^T \otimes X_j^T)^T (N_1^T \otimes X_j^T) = (N_1 N_1^T) \otimes (X_j X_j^T) = \\
&(N_1 N_1^T) \otimes [(t_j \mathbf{a})(t_j \mathbf{a})^T] = t_j^2 (N_1 N_1^T) \otimes (\mathbf{a}\mathbf{a}^T),
\end{aligned}
$$

and

$$
\sum_{j \in I_1} B_j^T B_j = \varpi (N_1 N_1^T) \otimes (\mathbf{a}\mathbf{a}^T),
$$

where $\varpi = \sum_{j \in I_1} t_j^2$.

In the same way, we obtain

$$
\begin{aligned}
&\sum_{j \in I_1} C_j^T C_j = \varpi (\tilde{N}_1 \tilde{N}_1^T) \otimes (\mathbf{a}\mathbf{a}^T), \ \sum_{k \in I_2} B_k^T B_k = \varrho (N_2 N_2^T) \otimes (\mathbf{b}\mathbf{b}^T), \\
&\sum_{k \in I_2} C_k^T C_k = \varrho (\tilde{N}_2 \tilde{N}_2^T) \otimes (\mathbf{b}\mathbf{b}^T), \ \sum_{k \in I_3} B_m^T B_m = \sigma (N_3 N_3^T) \otimes (\mathbf{c}\mathbf{c}^T), \\
&\sum_{m \in I_3} C_m^T C_m = \sigma (\tilde{N}_3 \tilde{N}_3^T) \otimes (\mathbf{c}\mathbf{c}^T),
\end{aligned}
$$

where $\varrho = \sum_{k \in I_2} s_k^2$ and $\sigma = \sum_{m \in I_3} v_m^2$. Then

$$
\begin{aligned}
F =& (1-\omega) \sum_{j \in I_1} B_j^T B_j + \omega \sum_{j \in I_1} C_j^T C_j + (1-\omega) \sum_{k \in I_2} B_k^T B_k + \\
& \omega \sum_{k \in I_2} C_k^T C_k + (1-\omega) \sum_{m \in I_3} B_m^T B_m + \omega \sum_{m \in I_3} C_m^T C_m \\
=& \varpi ((1-\omega) N_1 N_1^T + \omega \tilde{N}_1 \tilde{N}_1^T) \otimes (\mathbf{a}\mathbf{a}^T) + \varrho ((1-\omega) N_2 N_2^T + \omega \tilde{N}_2 \tilde{N}_2^T) \otimes (\mathbf{b}\mathbf{b}^T) + \\
& \sigma ((1-\omega) N_3 N_3^T + \omega \tilde{N}_3 \tilde{N}_3^T) \otimes (\mathbf{c}\mathbf{c}^T).
\end{aligned}
$$

Since $\mathrm{rank}(N_1 N_1^T) = 1$, $\mathrm{rank}(N_2 N_2^T) = 1$, $\mathrm{rank}(N_3 N_3^T) = 1$, $\mathrm{rank}(\tilde{N}_1 \tilde{N}_1^T) = 1$, $\mathrm{rank}(\tilde{N}_2 \tilde{N}_2^T) = 1$, $\mathrm{rank}(\tilde{N}_3 \tilde{N}_3^T) = 1$, $\mathrm{rank}(\mathbf{a}\mathbf{a}^T) = 1$, $\mathrm{rank}(\mathbf{b}\mathbf{b}^T) = 1$ and $\mathrm{rank}(\mathbf{c}\mathbf{c}^T) = 1$, $\mathrm{rank}(F) \le 6 < 8$. Thus A is not unique.

(v) If X_i lies on a plane which passes through the origin, then A is not unique.

In fact, assume X_i lies on the plane S. For arbitrary i_1, i_2, i_3, since X_i is coplanar, i.e., $\det([X_{i_1} \quad X_{i_2} \quad X_{i_3}]) = 0$, $[T_{i_1} \quad T_{i_2} \quad T_{i_3}] = A[X_{i_1} \quad X_{i_2} \quad X_{i_3}] = 0$, i.e., T_i are coplanar. Assume T_i lies on the plane S_1 (may be not S) and the intersection line of the plane S and the plane S_1 is l, and the unit tangent vector of l is $\mathbf{b}$. There is a vector $\mathbf{a}$ on S such that $\mathbf{a}$ is perpendicular to $\mathbf{b}$, and there is a vector $\mathbf{c}$ on S_1 such that $\mathbf{c}$ is perpendicular to $\mathbf{b}$. Then there are two real numbers α_i and θ_i for each X_i such that $X_i = \alpha_i \mathbf{a} + \theta_i \mathbf{b}$, and there are two real numbers β_i and γ_i for each T_i such that $T_i = \beta_i \mathbf{b} + \gamma_i \mathbf{c}$. Assume that the normal vector of the plane S_1

is $\tilde{N} = \mathbf{b} \times \mathbf{c}$, i.e., the binormal vectors of all X_i are the same. The principal vector of X_i is $N_i = -T_i \times \tilde{N} = -(\beta_i \mathbf{b} + \gamma_i \mathbf{c}) \times \tilde{N} = \beta_i \mathbf{c} - \gamma_i \mathbf{b}$.

From $C_i = \tilde{N}^T \otimes X_i^T$ and

$$C_i^T C_i = (\tilde{N}^T \otimes X_i^T)^T (\tilde{N}^T \otimes X_i^T) = (\tilde{N}\tilde{N}^T) \otimes (X_i X_i^T) = (\tilde{N}\tilde{N}^T) \otimes \left([\mathbf{a} \ \ \mathbf{b}] \begin{bmatrix} \alpha_i \\ \theta_i \end{bmatrix} [\alpha_i \ \ \theta_i] \begin{bmatrix} \mathbf{a}^T \\ \mathbf{b}^T \end{bmatrix} \right),$$

we obtain $\sum_{i=0}^n C_i^T C_i = (\tilde{N}\tilde{N}^T) \otimes ([\mathbf{a} \ \ \mathbf{b}] \begin{bmatrix} \alpha & \theta \\ \theta & \tilde{\theta} \end{bmatrix} \begin{bmatrix} \mathbf{a}^T \\ \mathbf{b}^T \end{bmatrix})$, where $\begin{bmatrix} \alpha & \theta \\ \theta & \tilde{\theta} \end{bmatrix} = \sum_{i=0}^n \begin{bmatrix} \alpha_i \\ \theta_i \end{bmatrix} [\alpha_i \ \ \theta_i]$.

From $B_i = N_i^T \otimes X_i^T$ and

$$\begin{aligned} B_i^T B_i =& (N_i^T \otimes X_i^T)^T (N_i^T \otimes X_i^T) = ((\beta_i \mathbf{c} - \gamma_i \mathbf{b}) \otimes (\alpha_i \mathbf{a} + \theta_i \mathbf{b}))((\beta_i \mathbf{c} - \gamma_i \mathbf{b})^T \otimes (\alpha_i \mathbf{a} + \theta_i \mathbf{b})^T) \\ =& \beta_i^2 \alpha_i^2 (\mathbf{c} \otimes \mathbf{a})(\mathbf{c} \otimes \mathbf{a})^T + \beta_i^2 \alpha_i \theta_i (\mathbf{c} \otimes \mathbf{a})(\mathbf{c} \otimes \mathbf{b})^T - \beta_i \gamma_i \alpha_i^2 (\mathbf{c} \otimes \mathbf{a})(\mathbf{b} \otimes \mathbf{a})^T - \\ & \beta_i \gamma_i \alpha_i \theta_i (\mathbf{c} \otimes \mathbf{a})(\mathbf{b} \otimes \mathbf{b})^T + \beta_i^2 \theta_i \alpha_i (\mathbf{c} \otimes \mathbf{b})(\mathbf{c} \otimes \mathbf{a})^T + \beta_i^2 \theta_i^2 (\mathbf{c} \otimes \mathbf{b})(\mathbf{c} \otimes \mathbf{b})^T - \\ & \beta_i \gamma_i \alpha_i \theta_i (\mathbf{c} \otimes \mathbf{b})(\mathbf{b} \otimes \mathbf{a})^T - \beta_i \gamma_i \theta_i^2 (\mathbf{c} \otimes \mathbf{b})(\mathbf{b} \otimes \mathbf{b})^T - \beta_i \gamma_i \alpha_i^2 (\mathbf{b} \otimes \mathbf{a})(\mathbf{c} \otimes \mathbf{a})^T - \\ & \beta_i \gamma_i \alpha_i \theta_i (\mathbf{b} \otimes \mathbf{a})(\mathbf{c} \otimes \mathbf{b})^T + \gamma_i^2 \alpha_i^2 (\mathbf{b} \otimes \mathbf{a})(\mathbf{b} \otimes \mathbf{a})^T + \gamma_i^2 \alpha_i \theta_i (\mathbf{b} \otimes \mathbf{a})(\mathbf{b} \otimes \mathbf{b})^T - \\ & \beta_i \gamma_i \alpha_i \theta_i (\mathbf{b} \otimes \mathbf{b})(\mathbf{c} \otimes \mathbf{a})^T - \beta_i \gamma_i \theta_i^2 (\mathbf{b} \otimes \mathbf{b})(\mathbf{c} \otimes \mathbf{b})^T + \gamma_i^2 \alpha_i \theta_i (\mathbf{b} \otimes \mathbf{b})(\mathbf{b} \otimes \mathbf{a})^T + \\ & \gamma_i^2 \theta_i^2 (\mathbf{b} \otimes \mathbf{b})(\mathbf{b} \otimes \mathbf{b})^T, \end{aligned}$$

it follows

$$\begin{aligned} \sum_{i=0}^n B_i^T B_i =& \delta_1 (\mathbf{c} \otimes \mathbf{a})(\mathbf{c} \otimes \mathbf{a})^T + \delta_5 (\mathbf{c} \otimes \mathbf{a})(\mathbf{c} \otimes \mathbf{b})^T + \delta_6 (\mathbf{c} \otimes \mathbf{a})(\mathbf{b} \otimes \mathbf{a})^T + \\ & \delta_7 (\mathbf{c} \otimes \mathbf{a})(\mathbf{b} \otimes \mathbf{b})^T + \delta_5 (\mathbf{c} \otimes \mathbf{b})(\mathbf{c} \otimes \mathbf{a})^T + \delta_2 (\mathbf{c} \otimes \mathbf{b})(\mathbf{c} \otimes \mathbf{b})^T + \\ & \delta_8 (\mathbf{c} \otimes \mathbf{b})(\mathbf{b} \otimes \mathbf{a})^T + \delta_9 (\mathbf{c} \otimes \mathbf{b})(\mathbf{b} \otimes \mathbf{b})^T + \delta_6 (\mathbf{b} \otimes \mathbf{a})(\mathbf{c} \otimes \mathbf{a})^T + \\ & \delta_8 (\mathbf{b} \otimes \mathbf{a})(\mathbf{c} \otimes \mathbf{b})^T + \delta_3 (\mathbf{b} \otimes \mathbf{a})(\mathbf{b} \otimes \mathbf{a})^T + \delta_{10} (\mathbf{b} \otimes \mathbf{a})(\mathbf{b} \otimes \mathbf{b})^T + \\ & \delta_7 (\mathbf{b} \otimes \mathbf{b})(\mathbf{c} \otimes \mathbf{a})^T + \delta_9 (\mathbf{b} \otimes \mathbf{b})(\mathbf{c} \otimes \mathbf{b})^T + \delta_{10} (\mathbf{b} \otimes \mathbf{b})(\mathbf{b} \otimes \mathbf{a})^T + \\ & \delta_4 (\mathbf{b} \otimes \mathbf{b})(\mathbf{b} \otimes \mathbf{b})^T \\ =& (\mathbf{c}\mathbf{c}^T) \otimes (\delta_1 \mathbf{a}\mathbf{a}^T + \delta_5 \mathbf{a}\mathbf{b}^T + \delta_5 \mathbf{b}\mathbf{a}^T + \delta_2 \mathbf{b}\mathbf{b}^T) + \\ & (\mathbf{c}\mathbf{b}^T) \otimes (\delta_6 \mathbf{a}\mathbf{a}^T + \delta_7 \mathbf{a}\mathbf{b}^T + \delta_8 \mathbf{b}\mathbf{a}^T + \delta_9 \mathbf{b}\mathbf{b}^T) + \\ & (\mathbf{b}\mathbf{c}^T) \otimes (\delta_6 \mathbf{a}\mathbf{a}^T + \delta_8 \mathbf{a}\mathbf{b}^T + \delta_7 \mathbf{b}\mathbf{a}^T + \delta_9 \mathbf{b}\mathbf{b}^T) + \\ & (\mathbf{b}\mathbf{b}^T) \otimes (\delta_3 \mathbf{a}\mathbf{a}^T + \delta_{10} \mathbf{a}\mathbf{b}^T + \delta_{10} \mathbf{b}\mathbf{a}^T + \delta_4 \mathbf{b}\mathbf{b}^T) \end{aligned}$$

$$= (\mathbf{c}\mathbf{c}^T) \otimes \left([\mathbf{a} \ \ \mathbf{b}] \begin{bmatrix} \delta_1 & \delta_5 \\ \delta_5 & \delta_2 \end{bmatrix} \begin{bmatrix} \mathbf{a}^T \\ \mathbf{b}^T \end{bmatrix} \right) + (\mathbf{c}\mathbf{b}^T) \otimes \left([\mathbf{a} \ \ \mathbf{b}] \begin{bmatrix} \delta_6 & \delta_7 \\ \delta_7 & \delta_9 \end{bmatrix} \begin{bmatrix} \mathbf{a}^T \\ \mathbf{b}^T \end{bmatrix} \right) +$$

$$(\mathbf{b}\mathbf{c}^T) \otimes \left([\mathbf{a} \ \ \mathbf{b}] \begin{bmatrix} \delta_6 & \delta_7 \\ \delta_7 & \delta_9 \end{bmatrix} \begin{bmatrix} \mathbf{a}^T \\ \mathbf{b}^T \end{bmatrix} \right) + (\mathbf{b}\mathbf{b}^T) \otimes \left([\mathbf{a} \ \ \mathbf{b}] \begin{bmatrix} \delta_3 & \delta_{10} \\ \delta_{10} & \delta_9 \end{bmatrix} \begin{bmatrix} \mathbf{a}^T \\ \mathbf{b}^T \end{bmatrix} \right)$$

$$= (\mathbf{c} \otimes [\mathbf{a} \ \ \mathbf{b}]) \left\{ \mathbf{c}^T \otimes \left(\begin{bmatrix} \delta_1 & \delta_5 \\ \delta_5 & \delta_2 \end{bmatrix} \begin{bmatrix} \mathbf{a}^T \\ \mathbf{b}^T \end{bmatrix} \right) + \mathbf{b}^T \otimes \left(\begin{bmatrix} \delta_6 & \delta_7 \\ \delta_7 & \delta_9 \end{bmatrix} \begin{bmatrix} \mathbf{a}^T \\ \mathbf{b}^T \end{bmatrix} \right) \right\} +$$

$$(\mathbf{b}\otimes[\mathbf{a}\ \ \mathbf{b}])\left\{\mathbf{c}^T\otimes\left(\begin{bmatrix}\delta_6 & \delta_7\\ \delta_7 & \delta_9\end{bmatrix}\begin{bmatrix}\mathbf{a}^T\\ \mathbf{b}^T\end{bmatrix}\right)+\mathbf{b}^T\otimes\left(\begin{bmatrix}\delta_3 & \delta_{10}\\ \delta_{10} & \delta_9\end{bmatrix}\begin{bmatrix}\mathbf{a}^T\\ \mathbf{b}^T\end{bmatrix}\right)\right\},$$

where $\delta_1=\sum_{i=0}^n\beta_i^2\alpha_i^2$, $\delta_2=\sum_{i=0}^n\beta_i^2\theta_i^2$, $\delta_3=\sum_{i=0}^n\gamma_i^2\alpha_i^2$, $\delta_4=\sum_{i=0}^n\gamma_i^2\theta_i^2$, $\delta_5=\sum_{i=0}^n\beta_i^2\alpha_i\theta_i$, $\delta_6=-\sum_{i=0}^n\beta_i\gamma_i\alpha_i^2$, $\delta_7=-\sum_{i=0}^n\beta_i\gamma_i\alpha_i\theta_i$, $\delta_8=\delta_7$, $\delta_9=-\sum_{i=0}^n\beta_i\gamma_i\theta_i^2$, $\delta_{10}=-\sum_{i=0}^n\gamma_i^2\alpha_i\theta_i$. Since $F=(1-\omega)\sum_{i=0}^n B_i^TB_i+\omega\sum_{i=0}^n C_i^TC_i$, $\mathrm{rank}(\mathbf{c}\otimes[\mathbf{a}\ \ \mathbf{b}])\le 2$, $\mathrm{rank}(\mathbf{b}\otimes[\mathbf{a}\ \ \mathbf{b}])\le 2$, $\mathrm{rank}([\mathbf{a}\ \ \mathbf{b}]\begin{bmatrix}\alpha & \theta\\ \theta & \tilde{\theta}\end{bmatrix}\begin{bmatrix}\mathbf{a}^T\\ \mathbf{b}^T\end{bmatrix})\le 2$, $\mathrm{rank}(F)\le 6<8$. Thus A is not unique.

Remark 3.5 In the 2-dimensional case, the set composed of discrete data points $X_i\in\mathbb{R}^2$, $i=0,1,2,\ldots,n$, is called degenerate if these discrete data points satisfy one of the following cases:

(i) X_i lies on no more than two straight lines which pass through the origin;

(ii) X_i lies on a straight line which does not pass through the origin.

Remark 3.6 If $X_i\in\mathbb{R}^2$, $i=0,1,2,\ldots,n$, Theorem 3.3 is also true.

So in this paper, we consider that the set of discrete data points is not degenerate, in order to guarantee the uniqueness of A.

3.1.2. The discussion of the error of A

In this section, we analyze the error between A and $\tilde{A}$. The error $\|\tilde{A}-A\|_F$ can be transformed to the distance between eigenvectors corresponding to the smallest eigenvalues F and F_ε, where F_ε is computed from data points containing noise $X_i^\eta=X_i+\eta_i$, $N_i^\delta=N_i+\delta_i$, $\tilde{N}_i^\xi=\tilde{N}_i+\xi_i$ based on the definitions in the model (7).

Denote $\Delta F=F_\varepsilon-F$. Since F is a symmetric positive semidefinite matrix, there exists an invertible matrix T such that F can be diagonalized, i.e., $T^{-1}FT=D$. We denote the invariant subspace of F corresponding to the smallest eigenvalue as $\mathfrak{T}_1$ and the invariant subspace of F corresponding to rest of eigenvalues as $\mathfrak{T}_2$. Let the columns of T_1 form a basis for $\mathfrak{T}_1$ and the columns of T_2 form a basis for $\mathfrak{T}_2$. Then, let $(S_1\ \ S_2)^H=(T_1\ \ T_2)^{-1}$. We have

$$(S_1\ \ S_2)^HF(T_1\ \ T_2)=\begin{pmatrix}D_1 & 0\\ 0 & D_2\end{pmatrix},\tag{15}$$

where D_1 is a diagonal matrix the diagonal entries of which are the smallest eigenvalues of F, D_2 is a diagonal matrix the diagonal entries of which are the rest eigenvalues of F. Then we can get the basis T_1^ε and T_2^ε of invariant subspaces $\mathfrak{T}_1^\varepsilon$ and $\mathfrak{T}_2^\varepsilon$ of F_ε from T_1 and T_2, where $\mathfrak{T}_1^\varepsilon$ and $\mathfrak{T}_2^\varepsilon$ are respectively invariant subspaces of F_ε.

Theorem 3.7 ([16]) *Let F have the spectral resolution (15) and set*

$$(S_1\ \ S_2)^H\Delta F(T_1\ \ T_2)=\begin{pmatrix}M_{1,1} & M_{1,2}\\ M_{2,1} & M_{2,2}\end{pmatrix}.$$

Let $\|\cdot\|$ represent a consistent family of norms, and set

$$\tilde{\gamma}=\|M_{2,1}\|,\ \tilde{\eta}=\|M_{1,2}\|,\ \tilde{\delta}=\mathrm{sep}(D_1,D_2)-\|M_{1,1}\|-\|M_{2,2}\|.$$

If $\tilde{\delta} > 0$ and

$$\frac{\tilde{\gamma}\tilde{\eta}}{\tilde{\delta}^2} < \frac{1}{4}, \tag{16}$$

there is a unique matrix P satisfying

$$\|P\| \le \frac{2\tilde{\gamma}}{\tilde{\delta} + \sqrt{\tilde{\delta}^2 - 4\tilde{\gamma}\tilde{\eta}}} < 2\frac{\tilde{\gamma}}{\tilde{\delta}},$$

such that the columns of $T_1^{\varepsilon} = T_1 + T_2P$ and $S_2^{\varepsilon} = S_2 - S_1P^H$ form bases for simple right and left invariant subspace of $F_{\varepsilon} = F + \Delta F$. The representation of F_{ε} with respect to T_1^{ε} is $D_1^{\varepsilon} = D_1 + M_{1,1} + M_{1,2}P$ and the representation of F_{ε} with respect to S_2^{ε} is $D_2^{\varepsilon} = D_2 + M_{2,2} + PM_{1,2}$.

Remark 3.8 In Theorem 3.3,

$$\mathrm{sep}(D_1, D_2) \stackrel{\mathrm{def}}{=} \inf_{\|P\|=1} \|\mathbb{T}(P)\| > 0$$

and $\mathbb{T} := P \mapsto PD_1 - D_2P$.

For Algorithm 1, we want to obtain the unit eigenvector Y corresponding to the smallest eigenvalue λ of the 9×9 matrix F. When the assumptions of Theorem 3.3 are satisfied, the smallest eigenvalue $\lambda = 0$ is a simple root. That is, the submatrix D_1 in Eq. (15) is $D_1 = \lambda = 0$ and D_2 is a 8×8 submatrix. $T_1 = Y$ is the corresponding unit eigenvector. In this case, the operator T in Remark 3.7 becomes the matrix $T = -D_2$. Further, if the condition (16) is also satisfied, there is a unique matrix P, s.t. $Y^{\varepsilon} = Y + T_2P$. Since the necessary and sufficient condition that T_1^{ε} and S_2^{ε} are the invariant subspaces respectively is $(S_2^{\varepsilon})^H(F + \Delta F)T_1^{\varepsilon} = 0$, i.e., $D_2P + M_{2,1} + M_{2,2}P - PM_{1,1} - PM_{1,2}P = 0$. The matrix $PM_{1,1}$, $M_{2,2}P$ are of order $\|\Delta F\|^2$ and $PM_{1,2}P$ is of order $\|\Delta F\|^3$, we have

$$P \cong D_2^{-1}S_2^H\Delta FT_1, \quad Y^{\varepsilon} \cong Y + S_2D_2^{-1}S_2^H\Delta FY.$$

Therefore, $\|Y^{\varepsilon} - Y\|_2 \lesssim \|S_2D_2^{-1}S_2^H\|_2\|\Delta F\|_2$. Hence, $\|\tilde{A} - A\|_F \lesssim \|S_2D_2^{-1}S_2^H\|_2\|\Delta F\|_2$.

3.2. The analysis of solution $X(t)$

In this section, we consider the case $\|A\|_F = 1$. The absolute value of the eigenvalue of A is no bigger than 1. In fact, $\|A\|_F^2 = \mathrm{trace}(A^TA) = \sum_{i=1}^d |\lambda_i|^2=1$, where λ_i is eigenvalue of A, and d is the row number of A, so that $|\lambda_i| \le 1$.

Theorem 3.9 *Assume that$\|A\|_F = 1$ and $E = \tilde{A} - A$, the solution of differential system (1) is $X(t)$ and the solution of differential system (10) is $\tilde{X}(t), t \in [0, t_1]$, then $\|\tilde{X}(t) - X(t)\| \le c\|E\|t_1e^{c\|E\|(t-t_0)}$.*

Proof Assume that the basic solution matrix of differential system (1) is $\Phi(t) = e^{At}$. Then

$$X(t) = \Phi(t)\Phi^{-1}(0)X_0 = e^{At}X_0$$

is the solution of the differential system (1).

Assume

$$\tilde{X}(t) = \Phi(t)c(t) \tag{17}$$

is the solution of differential system (10).

Now applying Eq. (16) to Eq. (10) gives

$$\frac{\mathrm{d}\Phi(t)}{\mathrm{d}t}c(t) + \Phi(t)\frac{\mathrm{d}c(t)}{\mathrm{d}t} = \tilde{A}\Phi(t)c(t) = (A+E)\Phi(t)c(t). \tag{18}$$

Since $\Phi(t)$ is the basic solution matrix of differential system (1), we have

$$\frac{\mathrm{d}\Phi(t)}{\mathrm{d}t} = A\Phi(t). \tag{19}$$

Applying Eq. (19) to Eq. (18) yields

$$\Phi(t)\frac{\mathrm{d}c(t)}{\mathrm{d}t} = E\Phi(t)c(t). \tag{20}$$

From (20), we have

$$c(t) = \int_0^t \Phi(s)^{-1}E\Phi(s)c(s)\mathrm{d}s + c(0). \tag{21}$$

Due to initial condition,

$$c(0) = \Phi(0)^{-1}X_0. \tag{22}$$

From Eqs.(17), (21) and (22), the solution of differential system (10) can be expressed as

$$\tilde{X}(t) = \Phi(t)\Phi^{-1}(0)X_0 + \int_0^t \Phi(t)\Phi^{-1}(s)E\tilde{X}(s)\mathrm{d}s.$$

Then

$$\begin{aligned}
\|\tilde{X}(t) - X(t)\| =& \| \int_0^t \Phi(t)\Phi^{-1}(s)E\tilde{X}(s)\mathrm{d}s\| \\
=& \| \int_0^t \Phi(t)\Phi^{-1}(s)E(\tilde{X}(s) - X(s))\mathrm{d}s + \int_0^t \Phi(t)\Phi^{-1}(s)EX(s)\mathrm{d}s\| \\
\leq& \int_0^t \|\Phi(t)\Phi^{-1}(s)E\|\|\tilde{X}(s) - X(s)\|\mathrm{d}s + \int_0^t \|\Phi(t)\Phi^{-1}(s)E\|\|X(s)\|\mathrm{d}s \\
\leq& \|E\| \int_0^t \|\Phi(t)\Phi^{-1}(s)\|(\|\tilde{X}(s) - X(s)\| + \|X(s)\|)\mathrm{d}s \\
\leq& \|E\| \int_0^t \|e^{A(t-s)}\|(\|\tilde{X}(s) - X(s)\| + \|e^{As}X_0\|)\mathrm{d}s \\
\leq& c\|E\| \int_0^t (\|\tilde{X}(s) - X(s)\| + cX_0\|)\mathrm{d}s
\end{aligned}$$

where $c = e^{\mathrm{Re}(\lambda(A))_{\max}t_1}$ and $\mathrm{Re}(\lambda(A))_{\max} \leq 1$. Applying the Gronwall inequality to the last inequality, we have $\|\tilde{X}(t) - X(t)\| \leq c^2\|E\|\|X_0\|t_1e^{c\|E\|t}$.

4. Numerical experiment

In order to demonstrate the effectiveness of the Algorithm 1, we give some numerical examples as follows. In each example, we take points from the differential system, then use our algorithm and the difference method to reconstruct the curves. In Figures, we show the reconstructed curves by our algorithm with accurate normal vectors and discrete normal vectors and the difference method. We also give the errors of our algorithm and difference method from noise-free data and noisy data in Tables. In the following tables, h is the step size, δ_X is the noise level of noise data, the error(A) represents $\|\tilde{A} - A\|_F$, error($X(t)$) represents $\max_{t\in\mathbb{R}} \|\tilde{X}(t) - X(t)\|$.

Examples 4.1 Given the homogeneous linear differential system(LDS) with an initial condition as follows

$$\begin{cases} \frac{\mathrm{d}X(t)}{\mathrm{d}t} = AX(t), \\ X(0) = X_0, \end{cases}$$

where $A = \begin{pmatrix} 2.84 & -5.01 \\ 11.88 & -4.72 \end{pmatrix}$, $X_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

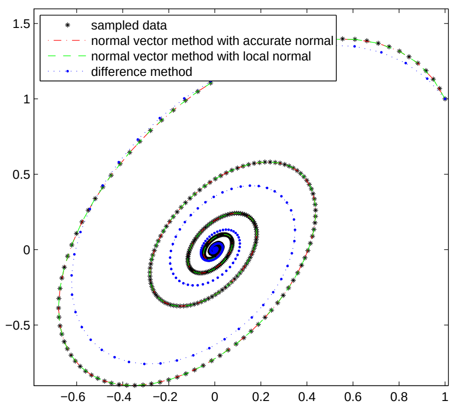

Figure 1 Reconstructions of the curve in Example 4.1 by our algorithm and difference method with $h = 0.01$

h	exact normal vectors		approximate normal vectors		difference method	
	error(A)	error($X(t)$)	error(A)	error($X(t)$)	error(A)	error($X(t)$)
0.05	$5.27E(-15)$	$1.74E(-15)$	$5.12E(-2)$	$2.37E(-2)$	$2.24E(0)$	$4.88E(-1)$
0.01	$7.11E(-15)$	$3.05E(-15)$	$2.02E(-3)$	$9.39E(-4)$	$1.07E(0)$	$1.74E(-1)$
0.005	$1.13E(-14)$	$3.21E(-15)$	$5.09E(-4)$	$2.35E(-4)$	$9.71E(-1)$	$1.16E(-1)$

Table 1 The errors of our algorithm and difference method in Example 4.1 from noisefree data

δ_X	normal vectors method		difference method	
	error(A)	error($X(t)$)	error(A)	error($X(t)$)
$1.0E(-2)$	$7.04E(-1)$	$9.72E(-2)$	$7.80E(-1)$	$1.64E(-1)$
$1.0E(-3)$	$9.02E(-3)$	$3.61E(-3)$	$9.08E(-1)$	$6.66E(-2)$
$1.0E(-4)$	$2.56E(-4)$	$1.60E(-4)$	$9.07E(-1)$	$6.54E(-2)$

Table 2 The errors of our algorithm and difference method in Example 4.1 from noisy data

Examples 4.2 Given the homogeneous linear differential system(LDS) with an initial condition as follows

$$\begin{cases} \frac{\mathrm{d}X(t)}{\mathrm{d}t} = AX(t), \\ X(0) = X_0, \end{cases}$$

where $A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0.05 \end{pmatrix}$, $X_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

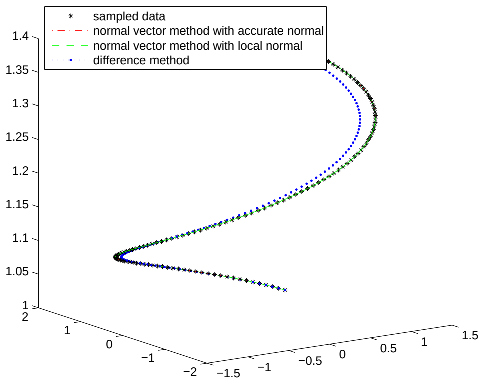

Figure 2 Reconstructions of the curve in Example 4.2 by our algorithm and difference method with $h = 0.05$

h	exact normal vectors		approximate normal vectors		difference method	
	error(A)	error$(X(t))$	error(A)	error$(X(t))$	error(A)	error$(X(t))$
0.1	$4.41E(-16)$	$5.83E(-15)$	$8.36E(-5)$	$1.60E(-3)$	$7.08E(-2)$	$9.38E(-1)$
0.05	$9.01E(-16)$	$1.18E(-14)$	$2.09E(-5)$	$4.02E(-4)$	$3.54E(-2)$	$4.99E(-1)$
0.01	$4.28E(-16)$	$8.80E(-15)$	$8.35E(-7)$	$1.61E(-5)$	$7.10E(-3)$	$1.05E(-1)$

Table 3 The errors of our algorithm and difference method in Example 4.2 from noisefree data

δ_X	normal vectors method		difference method	
	error(A)	error$(X(t))$	error(A)	error$(X(t))$
$1.0E(-3)$	$6.29E(-2)$	$1.85E(-1)$	$8.5E(-3)$	$4.96E(-2)$
$1.0E(-4)$	$6.29E(-4)$	$2.50E(-3)$	$7.73E(-4)$	$1.03E(-2)$
$1.0E(-5)$	$5.66E(-6)$	$5.59E(-5)$	$7.69E(-4)$	$1.06E(-2)$

Table 4 The errors of our algorithm and difference method in Example 4.2 from noisy data

Examples 4.3 Given the homogeneous linear differential system(LDS) with an initial condition as follows

$$\begin{cases} \frac{\mathrm{d}X(t)}{\mathrm{d}t} = AX(t), \\ X(0) = X_0, \end{cases}$$

where $A = \begin{pmatrix} -0.96 & 0.91 & 0.86 \\ 0.44 & -2.55 & -5.7 \\ -0.52 & 1.64 & 1.12 \end{pmatrix}$, $X_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

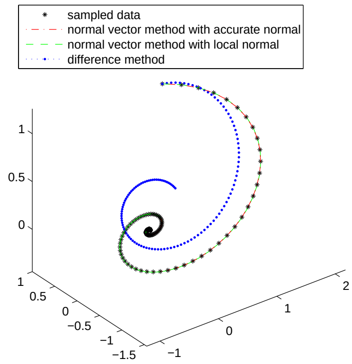

Figure 3 Reconstructions of the curve in Example 4.3 by our algorithm and difference method with $h = 0.05$

h	exact normal vectors		approximate normal vectors		difference method	
	error(A)	error$(X(t))$	error(A)	error$(X(t))$	error(A)	error$(X(t))$
0.1	$4.71E(-14)$	$4.34E(-14)$	$3.21E(-2)$	$6.76E(-2)$	$4.332E(0)$	$1.96E(0)$
0.05	$7.29E(-14)$	$5.90E(-14)$	$8.01E(-3)$	$1.67E(-2)$	$4.326E(0)$	$1.38E(0)$
0.01	$4.07E(-14)$	$3.65E(-14)$	$3.20E(-4)$	$6.86E(-4)$	$4.334E(0)$	$1.42E(0)$

Table 5 The errors of our algorithm and difference method in Example 4.3 from noisefree data

δ_X	normal vectors method		difference method	
	error(A)	error$(X(t))$	error(A)	error$(X(t))$
$1.0E(-3)$	$7.99E(-1)$	$1.01E(-1)$	$4.34E(0)$	$1.53E(0)$
$1.0E(-4)$	$7.70E(-3)$	$4.30E(-3)$	$4.34E(0)$	$1.53E(0)$
$1.0E(-5)$	$7.583E(-5)$	$9.91E(-5)$	$4.34E(0)$	$1.53E(0)$

Table 6 The errors of our algorithm and difference method in Example 4.3 from noisy data

From the figures and tables, we come to the conclusion that our method can get better results than the difference method from the data with or without noise.

5. Conclusion

In this paper, we propose an algorithm to reconstruct the differential system based on the normal vector from the given discrete points in order to avoid the problem of parameterization in curve fitting and approximation. We also carry out some theoretical analysis about our algorithm. We point out that when the data points are taken from the solution curve and the set composed of these data points is not degenerate, the coefficient matrix A reconstructed by our algorithm is unique from the given discrete noisefree data. And we discuss the error bounds for the approximate coefficient matrix and the solution which are reconstructed by our algorithm. Finally, the numerical experiments show the effectiveness of our algorithm.

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