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# Quasi-Zero-Divisor Graphs of Non-Commutative Rings

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**Abstract** In this paper, a new class of rings, called FIC rings, is introduced for studying quasi-zero-divisor graphs of rings. Let $R$ be a ring. The quasi-zero-divisor graph of $R$, denoted by $\Gamma_*(R)$, is a directed graph defined on its nonzero quasi-zero-divisors, where there is an arc from a vertex $x$ to another vertex $y$ if and only if $xRy = 0$. We show that the following three conditions on an FIC ring $R$ are equivalent: (1) $\chi(R)$ is finite; (2) $\omega(R)$ is finite; (3) $\mathrm{Nil}_*R$ is finite where $\mathrm{Nil}_*R$ equals the finite intersection of prime ideals. Furthermore, we also completely determine the connectedness, the diameter and the girth of $\Gamma_*(R)$.

**Keywords** quasi-zero-divisor; zero-divisor graph; chromatic number; clique number; FIC ring

**MR(2010) Subject Classification** 05C25; 05C15; 16U99

## 1. Introduction

Given a ring $R$, there are many ways to associate a directed or undirected graph to $R$ in order to study the properties of $R$ in terms of some invariants of the resulting graphs. In 1988, Beck [1] introduced the notion of the zero-divisor graph for a commutative ring, and mainly studied the coloring problem of rings. In 1999, Anderson and Livingston [2] associated a graph $\Gamma(R)$ to a commutative ring $R$, called the zero-divisor graph of $R$, with vertices set $Z(R)^* = Z(R) \setminus \{0\}$ of all nonzero zero-divisors, in which two distinct vertices $x, y \in Z(R)^*$ are adjacent if and only if $xy = 0$, and then they investigated the interplay between the ring-theoretic properties of $R$ and the graph-theoretic properties of $\Gamma(R)$. In 2002, Redmond [3] extended the definition to non-commutative rings. He defined a directed graph $\Gamma(R)$ for a ring $R$ with the vertices set $Z(R)^*$, where $x \longrightarrow y$ is an edge between distinct vertices $x$ and $y$ if and only if $xy = 0$. In 2006, Akbari and Mohammadian [4] defined an undirected graph $\overline{\Gamma}(R)$ for an arbitrary ring $R$

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with identity, where the vertices set of the graph $\overline{\Gamma}(R)$ is $Z(R)^*$, and two distinct vertices $x$ and $y$ in the graph are adjacent if and only if either $xy = 0$ or $yx = 0$ holds. In 2008, Behboodi and Beyranvand [5] introduced the strong zero-divisor graph $\widetilde{\Gamma}(R)$ for a ring $R$. In 2015, Alibemani and Bakhtyiari, etc. [6], introduced the annihilator ideal graph for rings. Motivated by previous studies, we introduce a new graph for a ring (not necessarily commutative and not necessarily contains the identity) and study its properties.

Let $R$ be a ring. An element $a \in R$ is said to be a left quasi-zero-divisor (resp., right quasi-zero-divisor) if there exists $0 \neq b \in R$ such that $aRb = 0$ (resp., $bRa = 0$). An element in $R$ is called a quasi-zero-divisor of $R$ if it is a left or a right quasi-zero-divisor. The sets of all left quasi-zero-divisors, right quasi-zero-divisors and quasi-zero-divisors of $R$ are denoted by $Q_l(R)$, $Q_r(R)$ and $Q(R)$, respectively. The quasi-zero-divisor graph of the ring $R$, denoted by $\Gamma_*(R)$, is a directed graph with the vertices set $Q(R)^* = Q(R) \setminus \{0\}$ of all nonzero quasi-zero-divisors and with an arc from $x$ to $y$, denoted by $x \longrightarrow y$, if and only if $xRy = 0$ for distinct $x, y \in Q(R)^*$. It is clear that $\Gamma_*(R)$ is an empty graph if and only if $R$ is a prime ring. The basis graph of $\Gamma_*(R)$, denoted by $\overline{\Gamma}_*(R)$, is an undirected graph with the vertices set $Q(R)^*$, where two distinct vertices $x$ and $y$ are adjacent, denoted by $x — y$, if and only if either $xRy = 0$ or $yRx = 0$ holds.

Now, let us recall some notions that will be used in our paper. A graph (resp., directed graph) $G$ is an ordered pair $(V(G), E(G))$ consisting of a set $V(G)$ of vertices and a set $E(G)$ of edges, together with an incidence function $\psi_G$ that associates with each edge of $G$ an unordered (resp., ordered) pair of vertices of $G$. The chromatic number of $G$, denoted by $\chi(G)$, is defined to be the minimal index $k$ if one assigns $k$ colors to each vertex of $G$ in such a way that every two adjacent vertices have different colors. A subset $\{x_1, \ldots, x_m\}$ of $V(G)$ is called an *m-clique* of $G$ if $x_i$ and $x_j$ are adjacent for $1 \leq i \neq j \leq m$. The clique number of $G$, denoted by $\omega(G)$, is defined to be the maximal number $m$ if $G$ has an $m$-clique. Undefined notions and notations in graph theory, please refer to [7]. For simplification, the chromatic number and the clique number of $\Gamma_*(R)$ are denoted by $\chi(R)$ and $\omega(R)$, respectively.

Now we state the key definition that will be used in this paper. A ring $R$ is called FIC, if it satisfies the following condition: for any finite ideal $I$ of $R$, $R/I$ having an infinite clique implies that $R$ contains an infinite clique. It is easy to say that all finite rings are FIC. If $R$ is an FIC ring, then the quotient ring $R/I$ is also FIC for the finite ideal $I$ of $R$. In fact, for any finite ideal $K$ $(I \subseteq K)$ of $R$, if $(R/I)/(K/I)\,(\cong R/K)$ has an infinite clique, then $R$ has an infinite clique $C$ and $\overline{C} = \{\overline{c} \in R/I : c \in C\}$ is an infinite clique of $R/I$. Thus $R/I$ is an FIC ring.

## 2. Some properties of FIC rings

As usual, let $\mathbb{Z}_n$ and $\mathbb{N}$ denote the ring of integers mod $n$ and the set of all positive integers, respectively. The set of all $n \times n$ matrices over $R$ is a matrix ring denoted by $\mathbb{M}_n(R)$. The cardinal of a set $A$ is denoted by $|A|$. An element $x$ in $R$ is called finite if both the left ideal $Rx$ and the right ideal $xR$ of $R$ are finite; otherwise $x$ is called infinite.

**Proposition 2.1** *Let $R$ be a ring. If $R$ contains infinitely many finite elements, then $R$ contains* *an infinite clique.*

**Proof** Let $X = \{x_i \in R : x_i\,(\neq 0)$ is a finite element for $i \in \mathbb{N}\}$ be an infinite set. Now we construct an infinite clique of $R$. Since $x_1$ is a finite element, we can write $x_1R = \{a_1, a_2, \ldots, a_m\}$ and $Rx_1 = \{b_1, b_2, \ldots, b_k\}$. Then we have $x_1Rx_i \subseteq x_1R$ and $a_tx_i \in x_1R$ for all $x_i \in X$ where $1 \leq t \leq m$ and so there exists an infinite set $X_1 = \{x_i^1 \in X : a_1x_i^1 = a_1x_1^1$ for all $i \in \mathbb{N}\}$. Repeating this process, we can also obtain an infinite set $X_m = \{x_i^m \in X_{m-1} \subseteq X : a_tx_i^m = a_tx_1^m$ where $i \in \mathbb{N}$ and $1 \leq t \leq m\}$. Let

$$Y = \{y_i \in R : y_i = x_1^m - x_i^m \text{ and } y_i \neq 0, x_1 \text{ for all } x_i^m \in X_m\}.$$

Then $Y$ is an infinite set of finite elements with $x_1Ry_i = 0$ for all $y_i \in Y$. Similarly, we can get an infinite set $Y_k = \{y_i^k \in Y : y_i^kb_t = y_1^kb_t$ and $x_1Ry_i^k = 0$ where $i \in \mathbb{N}$ and $1 \leq t \leq k\}$. Let

$$Z = \{z_i \in R : z_i = y_1^k - y_i^k \text{ and } z_i \neq 0, x_1 \text{ for all } y_i^k \in Y_k\}.$$

Then $Z$ is an infinite set of finite elements with $z_iRx_1 = 0$ and $x_1Rz_i = 0$ for all $z_i \in Z$.

Repeating the process of constructing the set $Y$ and the set $Z$, then we can obtain an infinite clique of $R$. $\square$

**Theorem 2.2** *The commutative ring $R$ is FIC.*

**Proof** Let $I$ be a finite ideal of $R$. If $R/I$ contains no infinite cliques, we are done. Now we suppose that $R/I$ contains an infinite clique $\overline{H} = \{\overline{x_i} \in R/I : x_i \in R$ for all $i \in \mathbb{N}\}$. In order to prove that $R$ is an FIC ring, we need show that $R$ contains an infinite clique. Write $H = \{x_i \in R$: all $\overline{x_i} \in \overline{H}\}$ and

$$H_1^2 = \{x_ix_j : x_ix_j \neq 0 \text{ for all } x_i \neq x_j \in H\}.$$

Let $x_1$ be an element of $H$. If $|H_1^2| = \infty$, then $\overline{H_1^2} = \{\overline{y_i} \in R/I :$ all $y_i \in H_1^2\}$ is also an infinite clique in $R/I$. So we get an infinite set $Z = \{z_i \in H_1^2 : x_1z_i = x_1z_1$ where all $i \in \mathbb{N}\}$. Let

$$\overline{H_1} = \{\overline{c_i} \in R/I : c_i = z_1 - z_i \text{ and } c_i \neq 0, x_1 \text{ for all } z_i \in Z\}.$$

Then $\overline{H_1}$ is also an infinite clique of $R/I$ such that $x_1Rc_i = c_iRx_1 = 0$ for all $c_i \in \overline{H_1}$. Repeating this process, we can obtain an infinite clique of $R$. If $|H_1^2| < \infty$, then we have $(x_1 \cdot H)\backslash\{x_1^2\} \subseteq H_1^2$. So we get an infinite set $\{t_i \in H : x_1t_i = x_1t_1$ where $i \in \mathbb{N}\}$. Similar to the proof of the case of $|H_1^2| = \infty$, we can also obtain an infinite clique of $R$ as the case of $|H_1^2| \neq \infty$. $\square$

**Lemma 2.3** ([7, Theorem 3.1]) *Let $R$ be a ring and $\mathbb{M}_n(R)$ be the matrix ring over $R$. Then the ideal of $\mathbb{M}_n(R)$ is of the form $\mathbb{M}_n(I)$ for a uniquely determined ideal $I$ of $R$.*

**Theorem 2.4** *Let $R$ be a commutative ring with identity. Then $\mathbb{M}_n(R)$ is an FIC ring for $n \in \mathbb{N}$.*

**Proof** Let $M$ be a finite ideal of $\mathbb{M}_n(R)$. If $\mathbb{M}_n(R)/M$ contains no infinite cliques, we are done. Now we suppose that $\mathbb{M}_n(R)/M$ contains an infinite clique $\overline{C} = \{\overline{A_i} \in \mathbb{M}_n(R)/M :$ all $i \in \mathbb{N}\}$. By Lemma 2.3, there exists a uniquely finite ideal $I$ of $R$ such that $M = \mathbb{M}_n(I)$. In order to prove that $\mathbb{M}_n(R)$ is an FIC ring, we need show that $\mathbb{M}_n(R)$ contains an infinite clique. Now write $C = \{A_i \in \mathbb{M}_n(R) : \text{all } \overline{A_i} \in \overline{C}\}$. So we have $A_i\mathbb{M}_n(R)A_j \subseteq M = \mathbb{M}_n(I)$ for all $A_i \neq A_j \in C$. Let $A_i \in C$ and

$$X_i = \{0 \neq a_{kl} \in R : A_i = (a_{kl})_{n\times n} \text{ for } 1 \leq k, l \leq n\},$$

where $i \in \mathbb{N}$. Then it is easy to see that there exists $t_1 \in \mathbb{N}$ such that $Y_1 = (X_2 \cup \cdots \cup X_{t_1}) \setminus Y_0 \neq \emptyset$ for $Y_0 = X_1$. Pick up $y_1 \in Y_1$, without loss of generality, we can assume that $y_1 \in X_{t_1}$. So there exists $t_2 \in \mathbb{N}$ such that $Y_2 = (X_{t_1+1} \cup \cdots \cup X_{t_2}) \setminus (Y_0 \cup Y_1) \neq \emptyset$. Now pick up $y_2 \in Y_2$, without loss of generality, we can assume that $y_2 \in X_{t_2}$. Repeating this process, we can obtain an infinite set $Y = \{y_i \in R : \text{all } i \in \mathbb{N}\}$. Let

$$\overline{Y} = \{\overline{y_i} \in R/I : \text{all } y_i \in Y \subseteq R\}.$$

Then we need to prove that $\overline{Y}$ is an infinite clique of $R/I$. Let $A_{t_i} = (a_{kl})_{n\times n}$ and $A_{t_j} = (b_{st})_{n\times n} \in C$ for $i \neq j \in \mathbb{N}$. For every $r \in R$, there exist $P, Q \in \mathbb{M}_n(R)$ such that the $k$-th row of $A_{t_i}P$ is $(a_{kl}r, 0, \ldots, 0)$ and the $t$-th column of $QA_{t_j}$ is $(b_{st}, 0, \ldots, 0)^T$. So we have $a_{kl}rb_{st} \in I$ and $y_iRy_j \subseteq I$ for $y_i, y_j \in Y$ where $1 \leq k, l, s, t \leq n$. Thus $\overline{Y}$ is an infinite clique of $R/I$. By Theorem 2.2, we know that $R$ contains an infinite clique $C' = \{c_i \in R : \text{all } i \in \mathbb{N}\}$. Now let $C''$ denote the set

$$\{C_k = (c_{ij})_{n\times n} : (1,1)\text{-entry of } C_k \text{ is } c_k \text{ and others entries are } 0 \text{ for all } c_k \in C'\}.$$

Then $C''$ is an infinite clique of $\mathbb{M}_n(R)$. Therefore $\mathbb{M}_n(R)$ is an FIC ring. $\square$

A nonempty subset $S$ of a ring $R$ is called an $m$-system if for any $a, b \in S$, there exists $r \in R$ such that $arb \in S$. For an ideal $I$ of $R$, the radical ideal of $I$, denoted by $\sqrt{I}$, is defined to be the set of $\{s \in R : \text{every } m\text{-system containing } s \text{ meets } I\}$ in [8]. By [8, Theorem 10.7], we know that the radical ideal of $I$ equals the intersection of all the prime ideals containing $I$. An ideal $I$ of $R$ is called semiprime if for any ideal $K$ of $R$, $K^2 \subseteq I$ implies that $K \subseteq I$. The lower nilradical of $R$, denoted by $\mathrm{Nil}_*R$, is defined to be the set of $\sqrt{\langle 0\rangle}$. It is that $\mathrm{Nil}_*R$ is the smallest semiprime ideal where it equals the intersection of all the prime ideals of $R$. We also know that $\mathrm{Nil}_*R$ is the intersection of all the minimal prime ideals of $R$ (see [8, Exercise 10.14]). The following two lemmas are also from [8].

**Lemma 2.5** ([8, Proposition 10.16]) *The following conditions are equivalent:*

*(1) $R$ is a semiprime ring;*

*(2) $Nil_*R = 0$;*

*(3) $R$ has no nonzero nilpotent ideal;*

*(4) $R$ has no nonzero nilpotent left ideal.*

Note that if $R$ is a semiprime ring, then for any $x, y \in R$, we have $xRy = 0$ if and only if $yRx = 0$. In fact, if $xRy = 0$ and $yRx \neq 0$, then there exists $r \in R$ such that $yrx \neq 0$. But we have $yr \cdot x(yrx)y \cdot rx \in yr \cdot xRy \cdot rx = 0$ and $yrx \in \mathrm{Nil}_*(R) = 0$. This is a contradiction and so the case of $xRy = 0$ and $yRx \neq 0$ is not true. Similarly, if $R$ is a semiprime ring, then the case of $xRy \neq 0$ and $yRx = 0$ is also not true. Thus for a semiprime ring $R$, we know that $xRy = 0$

if and only if $yRx = 0$.

**Lemma 2.6** ([8, Proposition 10.2]) *For a proper ideal $P$ of a ring $R$, the following statements are equivalent:*

*(1) $P$ is prime;*

*(2) For any $a, b \in R$, $\langle a\rangle\langle b\rangle \subseteq P$ implies that $a \in P$ or $b \in P$;*

*(3) For any $a, b \in R$, $aRb \subseteq P$ implies that $a \in P$ or $b \in P$;*

*(4) For any left ideals $I, K$ of $R$, $IK \subseteq P$ implies that $I \subseteq P$ or $K \subseteq P$;*

*(4′) For any right ideals $I, K$ of $R$, $IK \subseteq P$ implies that $I \subseteq P$ or $K \subseteq P$.*

To investigate the relationships between prime ideals and $\Gamma_*(R)$ of a ring $R$, we introduce the following definition. Let $I$ be an ideal of $R$. The quasi-annihilator ideal of $I$, denoted by $Q$-ann$(I)$, is defined to be the set of $\{a \in R : aRI = IRa = 0\}$. We will denote by $\mathcal{Q}(R)$ the set of all quasi-annihilator ideals of every ideal of a ring $R$.

**Proposition 2.7** *Let $R$ be a semiprime FIC ring. If $R$ contains no infinite cliques, then $\mathcal{Q}(R)$ satisfies the condition of ACC and every maximal element of $\mathcal{Q}(R)$ is a prime ideal.*

**Proof** If the proposition is not true, then we can assume that there exists an infinite strictly ascending chain

$$\text{Q-ann}(\langle a_1\rangle) \subset \text{Q-ann}(\langle a_2\rangle) \subset \cdots \subset \text{Q-ann}(\langle a_n\rangle) \subset \cdots .$$

Let $x_{n+1} \in \text{Q-ann}(\langle a_{n+1}\rangle) \setminus \text{Q-ann}(\langle a_n\rangle)$ and $y_n \in (x_{n+1}Ra_n \cup a_nRx_{n+1}) \setminus \{0\}$ for $n \in \mathbb{N}$. Then there exists $r_i \in R$ such that $y_i = x_{i+1}r_ia_i \neq 0$ or $y_i = a_ir_ix_{i+1} \neq 0$. Without loss of generality, we can set $y_i = x_{i+1}r_ia_i$ and $y_j = x_{j+1}r_ja_j$ where $i < j \in \mathbb{N}$, then we have $y_iRy_j = x_{i+1}(r_ia_iRx_{j+1}r_j)a_j \subseteq x_{i+1}Ra_j = 0$. Thus we have $y_iRy_j = y_jRy_i = 0$ for all $i \neq j \in \mathbb{N}$. Since $R$ contains no infinite cliques, there exist some $i \neq j \in \mathbb{N}$ such that $y_i = y_j$. Then we get $y_iRy_j = y_iRy_i = 0$ and $y_i^3 = 0$. By Lemma 2.5, we know that $y_i \in \text{Nil}_*R = 0$, leading to a contradiction. Thus $\mathcal{Q}(R)$ satisfies the condition of ACC.

Let $\text{Q-ann}(\langle x\rangle)$ be a maximal element of $\mathcal{Q}(R)$ and $a, b \in R \setminus \text{Q-ann}(\langle x\rangle)$. In order to prove that $\text{Q-ann}(\langle x\rangle)$ is a prime ideal, by Lemma 2.6, we only need to show that $aRb \nsubseteq \text{Q-ann}(\langle x\rangle)$. If it is not true, then it is to say $aRb \subseteq \text{Q-ann}(\langle x\rangle)$. Since $a \notin \text{Q-ann}(\langle x\rangle)$, we have $aRx \neq 0 \neq xRa$ and also have $bRx \neq 0 \neq xRb$. So there exists $r_1 \in R$ such that $br_1x \neq 0$. Since $aRb \subseteq \text{Q-ann}(\langle x\rangle)$, it follows that $aRbRx = 0$. So $aRbr_1x = 0$ and $br_1xRa = 0$. Now we need show that $\text{Q-ann}(\langle x\rangle)$ is not a maximal element of $\mathcal{Q}(R)$. It is easy to see that $\text{Q-ann}(\langle x\rangle) \subseteq \text{Q-ann}(\langle br_1x\rangle)$ and $a \in \text{Q-ann}(\langle br_1x\rangle) \setminus \text{Q-ann}(\langle x\rangle)$. Since $\text{Nil}_*R = 0$, we have $br_1x \notin \text{Q-ann}(\langle br_1x\rangle)$. So $\text{Q-ann}(\langle br_1x\rangle) \neq R$ and $\text{Q-ann}(\langle x\rangle) \nsubseteq \text{Q-ann}(\langle br_1x\rangle) \in \mathcal{Q}(R)$. That is to say $\text{Q-ann}(\langle x\rangle)$ is not a maximal element of $\mathcal{Q}(R)$. Thus $aRb \nsubseteq \text{Q-ann}(\langle x\rangle)$. $\square$

**Proposition 2.8** *Let $R$ be a semiprime ring and $x, y \in R$. If $Q$-ann$(\langle x\rangle)$ and $Q$-ann$(\langle y\rangle)$ are different prime ideals, then $xRy = 0$ and $yRx = 0$.*

**Proof** By the statements after Lemma 2.5, we only need to show that $xRy = 0$. If it is not true, then it is to say $xRy \neq 0$. Since $R$ is semiprime, it follows that $yRx \neq 0$. Then we have $\langle x\rangle \nsubseteq$ Q-ann$(\langle y\rangle)$ and $\langle x\rangle \cdot$ Q-ann$(\langle x\rangle) = 0 \subseteq$ Q-ann$(\langle y\rangle)$. By Lemma 2.6, we know that Q-ann$(\langle x\rangle) \subseteq$ Q-ann$(\langle y\rangle)$, and also have Q-ann$(\langle y\rangle) \subseteq$ Q-ann$(\langle x\rangle)$. Then Q-ann$(\langle x\rangle) =$ Q-ann$(\langle y\rangle)$ and so $xRy = 0$. $\square$

**Proposition 2.9** *Let $R$ be a semiprime FIC ring. Then the following conditions are equivalent:*

(1) *$\chi(R)$ is finite;*

(2) *$\omega(R)$ is finite;*

(3) *$\langle 0\rangle$ is the finite intersection of prime ideals of $R$;*

(4) *$R$ contains no infinite cliques.*

**Proof** The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (4) are trivial.

(4) $\Rightarrow$ (3). Since $R$ contains no infinite cliques, $\mathcal{Q}(R)$ satisfies the condition of ACC by Proposition 2.7. Let

$$\mathcal{Q}_1(R) = \{\text{Q-ann}(\langle x\rangle) \in \mathcal{Q}(R) : \text{Q-ann}(\langle x\rangle) \text{ is a maximal element of } \mathcal{Q}(R)\}.$$

Then $\mathcal{Q}_1(R)$ is a set of prime ideals by Proposition 2.7. Since $R$ contains no any infinite cliques, $|\mathcal{Q}_1(R)|$ is finite by Proposition 2.8. Let $a \in Q(R)^*$. Now we will prove that there exists Q-ann$(\langle x\rangle) \in \mathcal{Q}_1(R)$ such that $a \notin$ Q-ann$(\langle x\rangle)$. Since Q-ann$(\langle a\rangle)$ is in $\mathcal{Q}(R)$ and $\mathcal{Q}(R)$ satisfies the condition of ACC, there exists Q-ann$(\langle x\rangle) \in \mathcal{Q}_1(R)$ such that Q-ann$(\langle a\rangle) \subseteq$ Q-ann$(\langle x\rangle)$. In there we know that $aRx \neq 0$ and $xRa \neq 0$. If it is not true, then it is to say $aRx = xRa = 0$. Then we have $x \in$ Q-ann$(\langle a\rangle) \subseteq$ Q-ann$(\langle x\rangle)$. So we have $xRx = 0$ with $x^3 = 0\,(x \in \text{Nil}_* R = 0)$ and this is a contradiction. Thus $a \notin$ Q-ann$(\langle x\rangle)$ and

$$\bigcap_{\text{Q-ann}(\langle x\rangle) \in \mathcal{Q}_1(R)} \text{Q-ann}(\langle x\rangle) \subseteq R \setminus Q(R)^*,$$

where $\mathcal{Q}_1(R) \subseteq \operatorname{Spec} R$ and $|\mathcal{Q}_1(R)| < \infty$. Finally, we will prove that

$$\bigcap_{\text{Q-ann}(\langle x\rangle) \in \mathcal{Q}_1(R)} \text{Q-ann}(\langle x\rangle) = 0.$$

Assume to the contrary, then we have $C = \cap_{\text{Q-ann}(\langle x\rangle) \in \mathcal{Q}_1(R)} \text{Q-ann}(\langle x\rangle) \neq 0$ and $\cap_{P_i \in \mathcal{S}(R)} P_i = \text{Nil}_* R = 0$ with $\mathcal{Q}_1(R) \subseteq \mathcal{S}(R) \subseteq \operatorname{Spec} R$, where $\mathcal{S}(R)$ is a minimum set satisfying $\cap_{P_i \in \mathcal{S}(R)} P_i = \text{Nil}_* R$. Pick up $0 \neq c_0 \in C$ and $p_k \in P_k \setminus C$ for some $P_k \in \mathcal{S}(R) \setminus \mathcal{Q}_1(R)$. Then we have $c_0 R p_k \neq 0$ and there exists $r_k \in R$ such that $0 \neq c_1 = c_0 r_k p_k \in C \cap P_k$. Pick up $p_l \in P_l \setminus (C \cap P_k)$ for some $P_l \in \mathcal{S}(R) \setminus (\mathcal{Q}_1(R) \cup \{P_k\})$. Then we have $c_1 R p_l \neq 0$ and there exists $r_l \in R$ such that $0 \neq c_2 = c_1 r_l p_l \in C \cap P_k \cap P_l$. Repeating this process, we can get $0 \neq c \in C \cap (\cap_{P_i \in \mathcal{S}(R)} P_i) = 0$. Thus $\cap_{\text{Q-ann}(\langle x\rangle) \in \mathcal{Q}_1(R)} \text{Q-ann}(\langle x_i\rangle) = 0$.

(3) $\Rightarrow$ (1). Let $\langle 0\rangle = P_1 \cap \cdots \cap P_k$, where $P_1, \ldots, P_k$ are prime ideals of $R$. Define a function $f$ on $\Gamma_*(R)$ by

$$f(x) = \begin{cases} \min\{i : x \notin P_i\}, & \text{if } x \in Q(R)^* \bigcap (P_1 \bigcup \cdots \bigcup P_k), \\ k+1, & \text{if } x \in Q(R)^* \setminus (P_1 \bigcup \cdots \bigcup P_k). \end{cases}$$

Let $x \neq y \in Q(R)^*$. In order to prove that $f(x)$ is a coloring function on $\Gamma_*(R)$, we need show that if $f(x) = f(y)$, then we have $xRy \neq 0 \neq yRx$. If $f(x) = f(y) = l$ with $1 \leq l \leq k$, then we have $x, y \notin P_l$. If $f(x) = f(y) = k+1$, then we have $x, y \notin P_i$ for $1 \leq i \leq k$. By Lemma 2.6, we get $xRy \neq 0 \neq yRx$. Thus we need at most $k+1$ colors to color all of $Q(R)^*$. □

**Theorem 2.10** *Let $R$ be a semiprime FIC ring. If $\chi(R)$ is finite, then $R$ has only finitely many minimal prime ideals. Moreover, if the number of minimal prime ideals of $R$ is equal to $n$, then we have $n \leq \omega(R) \leq \chi(R) = n+1$.*

**Proof** By Proposition 2.9, we know that $\langle 0 \rangle$ is the finite intersection of prime ideals of $R$. So there are only finitely many minimal prime ideals in $R$, say $P_1, P_2, \ldots, P_n$. Since $\mathrm{Nil}_* R = P_1 \cap P_2 \cap \cdots \cap P_n = 0$, we have $\chi(R) \leq n+1$ by the proof of Proposition 2.9. Let $x_i \notin P_i$ and

$$x_i \in P_1 \bigcap \cdots \bigcap P_{i-1} \bigcap P_{i+1} \bigcap \cdots \bigcap P_n,$$

where $1 \leq i \leq n$. Then $x_i R x_j \subseteq P_1 \cap \cdots \cap P_{i-1} \cap P_{i+1} \cap \cdots \cap P_n$ and $x_i R x_j \subseteq P_1 \cap \cdots \cap P_{j-1} \cap P_{j+1} \cap \cdots \cap P_n$ where $i \neq j$. So $x_i R x_j \subseteq P_1 \cap P_2 \cap \cdots \cap P_n = 0$ for $1 \leq i \neq j \leq n$, and thus $\{x_1, x_2, \ldots, x_n\}$ is a clique of $R$. So we have $n \leq \omega(R)$ and $n \leq \omega(R) \leq \chi(R) \leq n+1$. □

**Proposition 2.11** *Let $R$ be an FIC ring. If $R$ contains an infinite element $x$ such that $\langle x \rangle$ is a nilpotent ideal, then $R$ contains an infinite clique.*

**Proof** By assumption, there exists $k \in \mathbb{N}$ such that $|\langle x \rangle^k| = \infty$ and $|\langle x \rangle^{k+1}| < \infty$. If every element in $\langle x \rangle^k$ is finite, then $R$ has an infinite clique by Proposition 2.1. If there exists $a \in \langle x \rangle^k$ such that $a$ is infinite, then either $aR/\langle x \rangle^{k+1}$ or $Ra/\langle x \rangle^{k+1}$ is an infinite clique of $R/\langle x \rangle^{k+1}$, where $aR/\langle x \rangle^{k+1} = \{\overline{r} \in R/\langle x \rangle^{k+1} : \text{all } r \in aR\}$ and $Ra/\langle x \rangle^{k+1} = \{\overline{r} \in R/\langle x \rangle^{k+1} : \text{all } r \in Ra\}$. Since $R$ is an FIC ring and $\langle x \rangle^{k+1}$ is finite, it follows that $R$ also contains an infinite clique. □

**Theorem 2.12** *Let $R$ be an FIC ring. Then the following conditions are equivalent:*

*(1)* $\chi(R)$ *is finite;*

*(2)* $\omega(R)$ *is finite;*

*(3)* $\mathrm{Nil}_* R$ *is finite and it equals the finite intersection of prime ideals of $R$;*

*(4)* *$R$ contains no infinite cliques.*

**Proof** The implications (1)⇒(2) and (2)⇒(4) are trivial.

(4)⇒(3). It is clear that if $x$ is in $\mathrm{Nil}_* R$, then $\langle x \rangle$ is a nilpotent ideal of $R$. Since there are not infinite cliques in $R$, we have $\mathrm{Nil}_* R$ is finite and $R/\mathrm{Nil}_* R$ contains no infinite cliques by Propositions 2.1 and 2.11. By Proposition 2.9, it follows that the zero ideal $\langle 0 \rangle$ of $R/\mathrm{Nil}_* R$ is the finite intersection of prime ideals of $R/\mathrm{Nil}_* R$.

(3)⇒(1). Let $\mathrm{Nil}_* R = P_1 \cap P_2 \cap \cdots \cap P_k$, where $P_1, P_2, \ldots, P_k$ are prime ideals of $R$. Define a function on $\Gamma_*(R)$ by

$$f(x) = \begin{cases} \min\{i : x \notin P_i\}, & \text{if } x \in (Q(R)^* \bigcap (P_1 \bigcup \cdots \bigcup P_k)) \setminus \mathrm{Nil}_*(R), \\ k+1, & \text{if } x \in (Q(R)^* \setminus (P_1 \bigcup \cdots \bigcup P_k)) \setminus \mathrm{Nil}_*(R), \\ j+k+1, & \text{if } y_j \in \mathrm{Nil}_*(R) \setminus \{0\}. \end{cases}$$

Let $x \neq y \in Q(R)^* \setminus \mathrm{Nil}_*(R)$. If $f(x) = f(y) = l$ with $1 \leq l \leq k$, then we have $x, y \notin P_l$. If $f(x) = f(y) = k+1$, then we have $x, y \notin P_i$ for $1 \leq i \leq k$. By Lemma 2.6, we get $xRy \neq 0 \neq yRx$. Since $\mathrm{Nil}_* R$ is finite, we need only a finite number of colors to color all of $Q(R)^*$. $\Box$

**Theorem 2.13** *Let $R$ be an FIC ring with $\chi(R) < \infty$. Then the radical ideal of any finite ideal of $R$ is finite and it equals the finite intersection of prime ideals. Moreover, there exist finitely many finite ideals in $R$.*

**Proof** Let $K$ be a finite ideal of $R$. Since $R$ is an FIC ring and $\chi(R) < \infty$, we know that $R/K$ has no infinite cliques. By Theorem 2.12, we have $\chi(R/K) < \infty$ and $\mathrm{Nil}_*(R/K) = \sqrt{K/K}$ is the finite intersection of prime ideals of $R/K$. Now we want to show that $\sqrt{K/K}$ is finite. If it is not true, then it is to say $\mathrm{Nil}_*(R/K) = \infty$. Then we have two cases to consider:

**Case 1** If $x$ is finite for every $x \in \mathrm{Nil}_*(R/K)$, then $R/K$ contains an infinite clique by Proposition 2.1. So $R$ contains an infinite clique and this is a contradiction.

**Case 2** If there exists $x \in \mathrm{Nil}_*(R/K)$ such that $x$ is infinite, then $\langle x \rangle$ is nilpotent and $R/K$ contains an infinite clique by Proposition 2.11. So $R$ contains an infinite clique and this is also a contradiction.

Thus $\sqrt{K/K}$ is finite. Since $K$ is finite, it follows that $\sqrt{K}$ is finite.

Let $A = \{x \in R : x \text{ is finite}\}$. Since $\chi(R) < \infty$, we have $|A| < \infty$ and $\omega(R) < \infty$ by Proposition 2.1 and Theorem 2.12. Since $A$ contains every finite ideal, we have the number of the finite ideals of $R$ is finite. $\Box$

## 3. Some properties of $\Gamma_*(R)$ and $\overline{\Gamma}_*(R)$

In this section, we will study the properties of $\Gamma_*(R)$ and $\overline{\Gamma}_*(R)$. We first present three examples to illustrate the relationships among the graphs $\Gamma(R), \Gamma_*(R), \overline{\Gamma}(R), \overline{\Gamma}_*(R)$ and $\widetilde{\Gamma}(R)$ for a ring $R$. Note that if $R$ has the identity, $\Gamma_*(R)$ may be a subgraph of $\Gamma(R)$ and $\overline{\Gamma}_*(R)$ may also be a subgraph of $\overline{\Gamma}(R)$. The following example shows that the subgraphs could be proper.

**Example 3.1** Let $\mathbb{M}_2(\mathbb{Z})$ be a matrix ring over the integer numbers ring $\mathbb{Z}$. Since $\mathbb{M}_2(\mathbb{Z})$ has the identity, it follows that $a\mathbb{M}_2(\mathbb{Z})b = 0$ implies $ab = 0$ for $a, b \in \mathbb{M}_2(\mathbb{Z})$. Note that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbb{M}_2(\mathbb{Z}) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \neq 0 \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbb{M}_2(\mathbb{Z}) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq 0,$$

but $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0$. Thus $\Gamma_*(\mathbb{M}_2(\mathbb{Z}))$ and $\overline{\Gamma}_*(\mathbb{M}_2(\mathbb{Z}))$ are proper subgraphs of $\Gamma(\mathbb{M}_2(\mathbb{Z}))$ and $\overline{\Gamma}(\mathbb{M}_2(\mathbb{Z}))$, respectively.

**Example 3.2** Let $R = \{0, 2, 4, 6\}$ be a subring of $\mathbb{Z}_8$. Then we have $2\,R\,6 = 6\,R\,2 = 0$, but $2 \cdot 6 = 6 \cdot 2 = 4 \neq 0$. Note that $\Gamma_*(R)$ is a directed complete graph and $\overline{\Gamma}_*(R)$ is an undirected complete graph. Figures 3.1–3.3 show diagrams of $\Gamma(R), \Gamma_*(R), \overline{\Gamma}_*(R)$ and $\widetilde{\Gamma}(R)$. Thus $\Gamma(R)$ and

$\widetilde{\Gamma}(R)$ are proper subgraphs of $\overline{\Gamma}_*(R)$.

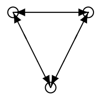

Figure 3.1 $\Gamma_*(R)$

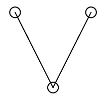

Figure 3.2 $\Gamma(R)$ and $\widetilde{\Gamma}(R)$

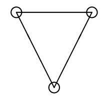

Figure 3.3 $\overline{\Gamma}_*(R)$

**Example 3.3** Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in \mathbb{Z}_2 \right\}$. Then $\Gamma_*(R)$ has three vertices and two edges with

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Figures 3.4 and 3.5 show diagrams of $\Gamma(R), \Gamma_*(R), \overline{\Gamma}(R), \overline{\Gamma}_*(R)$ and $\widetilde{\Gamma}(R)$. We can see that $\Gamma_*(R)$ and $\Gamma(R)$ are not connected.

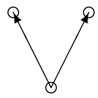

Figure 3.4 $\Gamma(R)$ and $\Gamma_*(R)$

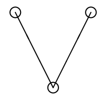

Figure 3.5 $\overline{\Gamma}(R), \overline{\Gamma}_*(R)$ and $\widetilde{\Gamma}(R)$

Recall that a (directed) graph $G$ is connected if there is a (directed) path between any two vertices of $G$; otherwise the (directed) graph is disconnected. The distance between vertices $a$ and $b$ in a (directed) graph, denoted by $d(a, b)$, is the number of edges in a shortest (directed) path. The diameter of a (directed) graph $G$, denoted by $\mathrm{diam}(G)$, is the greatest distance between any distinct two vertices of $G$. In [3], Redmond proved that $\Gamma(R)$ is connected if and only if $Z_l(R) = Z_r(R)$ and also proved that if $\Gamma(R)$ is connected, then $\mathrm{diam}(\Gamma(R)) \leq 3$. We have a similar conclusion on $\Gamma_*(R)$.

**Theorem 3.4** *Let $R$ be a ring. Then $\Gamma_*(R)$ is connected if and only if $Q_l(R) = Q_r(R)$. Moreover, if $\Gamma_*(R)$ is connected, then* $\mathrm{diam}(\Gamma_*(R)) \leq 3$.

**Proof** Suppose that $\Gamma_*(R)$ is connected. Then for every vertex $x \in Q(R)^*$, there exist two directed edges beginning at $x$ and ending at $x$. So $x$ is both a left quasi-zero-divisor and a right quasi-zero-divisor. Thus $Q_l(R) = Q_r(R)$. Conversely, suppose that $Q_l(R) = Q_r(R)$. Then we get $Q(R) = Q_l(R) = Q_r(R)$. In order to prove that $\Gamma_*(R)$ is connected, we need show that for every $x \neq y \in Q(R)^*$, there is a directed path from $x$ to $y$. Then we have five cases to consider:

**Case 1** $xRy = 0$. Then $x \longrightarrow y$ is a path of length 1.

**Case 2** $xRy \neq 0$ and $xRx = yRy = 0$. Then there exists $r \in R$ such that $xry \neq 0$. In this case, $xR(xry) = 0 = (xry)Ry$. So $x \longrightarrow xry \longrightarrow y$ is a path of length 2 (Note that $xry \neq x, y$, otherwise $x \longrightarrow y$ is a path and $xRy = 0$, and this a contradiction).

**Case 3** $xRy \neq 0, xRx = 0$ and $yRy \neq 0$. Since $y \in Q(R)^* = Q_r(R) \setminus \{0\}$, there exists $a \in Q(R)^* \setminus \{x, y\} = Q_l(R) \setminus \{0, x, y\}$ such that $aRy = 0$. If $xRa = 0$, then $x \longrightarrow a \longrightarrow y$ is a path of length 2. If $xRa \neq 0$, then there exists $r \in R$ such that $xra \neq 0$. So $x \longrightarrow xra \longrightarrow y$ is a path of length 2.

**Case 4** $xRy \neq 0, xRx \neq 0$ and $yRy = 0$. As the case 3, there is also a path of length 2 between $x$ and $y$.

**Case 5** $xRy \neq 0, xRx \neq 0$ and $yRy \neq 0$. Since $x \in Q(R)^* = Q_l(R) \setminus \{0\}$ and $y \in Q(R)^* = Q_r(R) \setminus \{0\}$, there exist $a \in Q(R)^* = Q_r(R) \setminus \{0, x, y\}$ and $b \in Q(R)^* = Q_l(R) \setminus \{0, x, y\}$ such that $xRa = 0$ and $bRy = 0$. If $a = b$, then $x \longrightarrow a \longrightarrow y$ is a path of length 2. If $a \neq b$ and $aRb = 0$, then $x \longrightarrow a \longrightarrow b \longrightarrow y$ is a path of length 3. If $a \neq b$ and $aRb \neq 0$, then there exists $r \in R$ such that $arb \neq 0$. So $x \longrightarrow arb \longrightarrow y$ is a path of length 2.

By the above proof, we always have a path from $x$ to $y$ and thus $\Gamma_*(R)$ is connected. The above proof also shows that if $\Gamma_*(R)$ is connected, then $\operatorname{diam}(\Gamma_*(R)) \leq 3$. $\Box$

**Theorem 3.5** *Let $R$ be a ring. Then $\overline{\Gamma}_*(R)$ is connected and* $\operatorname{diam}(\overline{\Gamma}_*(R)) \leq 3$.

**Proof** For each pair $x \neq y \in Q(R)^*$, in order to prove the theorem, we need show that there exists a path of length at most three between $x$ and $y$. If $xRy = 0$ or $yRx = 0$, then $x$ — $y$ is a path of length 1. Now we suppose that $xRy \neq 0$ and $yRx \neq 0$ and then there exists $r \in R$ such that $xry \neq 0$. Then we have four cases to consider:

**Case 1** $xRx = yRy = 0$. Then $x$ — $xry$ — $y$ is a path of length 2.

**Case 2** $xRx = 0$ and $yRy \neq 0$. Then there exists $a \in R \setminus \{0, x, y\}$ such that $a$ and $y$ are adjacent. If $x$ and $a$ are adjacent, then $x$ — $a$ — $y$ is a path of length 2. If $x$ and $a$ are not adjacent, then there exists $r \in R$ with $xra \neq 0$. So $x$ — $xra$ — $y$ is a path of length 2.

**Case 3** $xRx \neq 0$ and $yRy = 0$. As the case 2, there is also a path of length 2 between $x$ and $y$.

**Case 4** $xRx \neq 0$ and $yRy \neq 0$. Then there exist $a, b \in R \setminus \{0, x, y\}$ such that $x$ — $a$ and $b$ — $y$ are two edges in $\overline{\Gamma}_*(R)$. If $a = b$, then $x$ — $a$ — $y$ is a path of length 2. If $a \neq b$, and $a$ — $b$ is in $\overline{\Gamma}_*(R)$, then $x$ — $a$ — $b$ — $y$ is a path of length 3. If $a \neq b$, and $a$ — $b$ is not in $\overline{\Gamma}_*(R)$, then there exists $r \in R$ such that $arb \neq 0$. In this case, $x$ — $arb$ — $y$ is a path of length 2.

Therefore, $\overline{\Gamma}_*(R)$ is connected and $\operatorname{diam}(\overline{\Gamma}_*(R)) \leq 3$. $\Box$

Recall the definition of the girth of a graph $G$, denoted by $gr(G)$, is the length of a shortest cycle contained in $G$. If $G$ contains no cycle, then the girth of $G$ is defined to be $\infty$.

**Theorem 3.6** ([4]) *Let $R$ be a ring. Then $gr(\overline{\Gamma}_*(R)) \in \{3, 4, \infty\}$.*

**Proof** It is clear that $\operatorname{gr}(\overline{\Gamma}_*(R)) \geq 3$. If $\overline{\Gamma}_*(R)$ contains no cycles, then $\operatorname{gr}(\overline{\Gamma}_*(R)) = \infty$. Now we suppose that $\overline{\Gamma}_*(R)$ contains a shortest cycle, say, $x_1$ — $x_2$ — $\cdots$ — $x_n$ — $x_1$. If $n \leq 4$, we are done. So we can suppose that $n > 4$, then $x_3$ and $x_n$ are not adjacent and $x_3Rx_n \neq 0 \neq x_nRx_3$. Thus there exists $r \in R$ with $x_3rx_n \in Q(R)^*$, and then we have three cases to consider:

**Case 1** If $x_3rx_n = x_1$, then $x_1$ — $x_2$ — $x_3$ — $x_4$ — $x_1$ is a cycle of length 4 and this is a contradiction.

**Case 2** If $x_3rx_n = x_2$, then $x_1 — x_2 — x_4 — \cdots — x_n — x_1$ is a cycle of length $n-1$ and this is a contradiction.

**Case 3** If $x_3rx_n \neq x_1, x_2$, then $x_1 — x_2 — x_3rx_n — x_1$ is a cycle of length 3 and this is a contradiction.

Therefore, we have $\mathrm{gr}(\overline{\Gamma}_*(R)) \in \{3, 4, \infty\}$. $\square$

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