

## An Investigation on Left Hyperideals of Ordered Semihypergroups

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**Abstract** In this paper, the concepts of minimal and maximal left hyperideals in ordered semihypergroups are introduced, and several related properties are investigated. Furthermore, we introduce the concepts of weakly prime, quasi-prime, quasi-semiprime and weakly quasi-prime left hyperideals of an ordered semihypergroup, and establish the relationship among the four classes of left hyperideals. Moreover, we give some characterizations of weakly quasi-prime left hyperideals by the left hyperideals and weakly  $m$ -systems. We also characterize the quasi-prime left hyperideals in terms of the  $m$ -systems. In particular, we prove that an ordered semihypergroup  $S$  is strongly semisimple if and only if every left hyperideal of  $S$  is the intersection of all quasi-prime left hyperideals of  $S$  containing it.

**Keywords** ordered semihypergroup; minimal left hyperideal; maximal left hyperideal; weakly prime left hyperideal; quasi-prime left hyperideal; weakly quasi-prime left hyperideal

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### 1. Introduction

The first step of the development of hyperstructure theory, in particular hypergroup theory, can be traced back to the 8th Congress of Scandinavian Mathematicians in 1934, when Marty [1] introduced the concept of hypergroups, analyzed its properties and applied it to groups, rational fractions and algebraic functions. Later on, people have observed that hyperstructures have many applications to several branches of both pure and applied sciences (see [2,3]). In particular, semihypergroups are the simplest algebraic hyperstructures which possess the properties of closure and associativity. Nowadays semihypergroups have been found useful for dealing with problems in different areas of algebraic hyperstructures. Many authors studied different aspects of semihypergroups, for instance, Anvariye et al. [4], Davvaz [5], Hila et al. [6], Leoreanu [7] and Salvo et al. [8].

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As we know, an ordered semigroup is a semigroup together with a partial order that is compatible with the semigroup operation. Ordered semigroups have several applications in the theory of sequential machines, formal languages, computer arithmetics and error-correcting codes. There are several results which have been added to the theory of ordered semigroups by many researchers. For more details, the reader is referred to [9–15]. In particular, Kehayopulu [10,11] introduced the concepts of prime ideals and weakly prime ideals of ordered semigroups. Xie and Wu defined and studied quasi-prime and weakly quasi-prime left ideals of ordered semigroups in [14]. Since then, Cao and Xu [9] investigated the minimal and maximal left ideals of ordered semigroups.

The study of ordered hyperstructures is an interesting research topic of algebraic hyperstructure theory. We noticed that the relationship between ordered semigroups and algebraic hyperstructures have been already considered by Changphas, Corsini, Davvaz, Heidari, Tang and others, for instance, the reader can refer to [16–19]. It is worth pointing out that Heidari and Davvaz [18] applied the theory of hyperstructures to ordered semigroups and introduced the concept of ordered semihypergroups, which is a generalization of the concept of ordered semigroups. Motivated by the study of hyperideals in semihypergroups, and also motivated by Davvaz's works in ordered hyperstructures, we attempt in the present paper to study left hyperideals of ordered semihypergroups in detail. We first introduce the concepts of minimal and maximal left hyperideals in ordered semihypergroups, and study several their related properties. Furthermore, we define and investigate the weakly prime, quasi-prime, quasi-semiprime and weakly quasi-prime left hyperideals of an ordered semihypergroup. Moreover, we give some characterizations of weakly quasi-prime and quasi-prime left hyperideals in terms of weakly  $m$ -systems and  $m$ -systems, respectively. Finally, characterizations of strongly semisimple ordered semihypergroups by means of quasi-prime left hyperideals are given. Especially, we prove that an ordered semihypergroup  $S$  is strongly semisimple if and only if every left hyperideal of  $S$  can be expressed as the intersection of all quasi-prime left hyperideals of  $S$  containing it.

## 2. Preliminaries and some notations

Recall that a hypergroupoid  $(S, \circ)$  is a nonempty set  $S$  together with a hyperoperation, that is a map  $\circ : S \times S \rightarrow P^*(S)$ , where  $P^*(S)$  denotes the set of all the nonempty subsets of  $S$ . The image of the pair  $(x, y)$  is denoted by  $x \circ y$ . If  $x \in S$  and  $A, B$  are nonempty subsets of  $S$ , then  $A \circ B$  is defined by

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b.$$

Also  $A \circ x$  is used for  $A \circ \{x\}$  and  $x \circ A$  for  $\{x\} \circ A$ . Generally, the singleton  $\{x\}$  is identified by its element  $x$ .

We say that a hypergroupoid  $(S, \circ)$  is a semihypergroup if the hyperoperation “ $\circ$ ” is associative, that is,  $(x \circ y) \circ z = x \circ (y \circ z)$  for all  $x, y, z \in S$  (see [20]). A nonempty subset  $T$  of a semihypergroup  $S$  is called a subsemihypergroup if  $T \circ T \subseteq T$ .

We now recall the notion of ordered semihypergroups from [18].

**Definition 2.1** An algebraic hyperstructure  $(S, \circ, \leq)$  is called an ordered semihypergroup (also called po-semihypergroup in [18]) if  $(S, \circ)$  is a semihypergroup and  $(S, \leq)$  is a partially ordered set such that: for any  $x, y, a \in S$ ,  $x \leq y$  implies  $a \circ x \preceq a \circ y$  and  $x \circ a \preceq y \circ a$ . Here, if  $A, B \in P^*(S)$ , then we say that  $A \preceq B$  if for every  $a \in A$  there exists  $b \in B$  such that  $a \leq b$ . In particular, if  $A = \{a\}$ , then we write  $a \preceq B$  instead of  $\{a\} \preceq B$ .

Clearly, every ordered semigroup can be regarded as an ordered semihypergroup. Also see [19]. Throughout this paper, unless stated otherwise,  $S$  stands for an ordered semihypergroup.

**Definition 2.2** ([18]) A nonempty subset  $A$  of an ordered semihypergroup  $S$  is called a left (resp., right) hyperideal of  $S$  if (1)  $S \circ A \subseteq A$  (resp.,  $A \circ S \subseteq A$ ) and (2) If  $a \in A$  and  $S \ni b \leq a$ , then  $b \in A$ . If  $A$  is both a left and a right hyperideal of  $S$ , then it is called a hyperideal of  $S$ .

Let  $S$  be an ordered semihypergroup. For  $\emptyset \neq H \subseteq S$ , we define

$$(H] := \{t \in S \mid t \leq h \text{ for some } h \in H\}.$$

For  $H = \{a\}$ , we write  $(a]$  instead of  $(\{a\}]$ . For any  $a \in S$ , the intersection of all left hyperideals of  $S$  containing  $a$  is called the principal left hyperideal of  $S$  generated by  $a$ , denoted by  $L(a)$ . One can easily prove that  $L(a) = (a \cup S \circ a] = (a] \cup (S \circ a]$ .

**Lemma 2.3** Let  $S$  be an ordered semihypergroup. Then the following statements hold:

- (1)  $A \subseteq (A]$  and  $((A]) = (A], \forall A \in P^*(S)$ .
- (2) If  $A, B \in P^*(S), A \subseteq B$ , then  $(A] \subseteq (B]$ .
- (3)  $(A] \circ (B] \subseteq (A \circ B]$  and  $((A] \circ (B]) = (A \circ B], \forall A, B \in P^*(S)$ .
- (4) For every left (resp., right) hyperideal  $T$  of  $S$ , we have  $(T] = T$ .
- (5) If  $A, B$  are left hyperideals of  $S$ , then  $(A \circ B]$  is a left hyperideal of  $S$ .
- (6) For every  $a \in S, (S \circ a \circ S]$  and  $(S \circ a]$  are a hyperideal and a left hyperideal of  $S$ , respectively.
- (7) If  $L$  is a left hyperideal of  $S$  and  $A, B$  are two nonempty subsets of  $S$  such that  $A \preceq B \subseteq L$ , then  $A \subseteq L$ .
- (8) For any two nonempty subsets  $A, B$  of  $S$  such that  $A \preceq B$ , we have  $C \circ A \preceq C \circ B$  and  $A \circ C \preceq B \circ C$  for any nonempty subset  $C$  of  $S$ .

**Proof** Straightforward.  $\square$

**Lemma 2.4** Let  $S$  be an ordered semihypergroup and  $\{L_i \mid i \in I\}$  a family of left hyperideals of  $S$ . Then  $\bigcup_{i \in I} L_i$  is a left hyperideal of  $S$  and  $\bigcap_{i \in I} L_i$  is also a left hyperideal of  $S$  if  $\bigcap_{i \in I} L_i \neq \emptyset$ .

**Proof** The proof is straightforward verification, and hence we omit the details.  $\square$

Let  $S$  be an ordered semihypergroup. An element  $a$  of  $S$  is called a zero element of  $S$  if  $x \circ a = a \circ x = \{a\}$  and  $a \leq x$  for all  $x \in S$  and denote it by  $0$ . Clearly,  $\{0\}$  is a left hyperideal of  $S$ . A left hyperideal  $L$  of  $S$  is called proper if  $L \neq S$ . An ordered semihypergroup  $S$  without zero is called left simple if it has no proper left hyperideals. An ordered semihypergroup  $S$  with zero is called left 0-simple if it has no nonzero proper left hyperideals and  $S \circ S \neq \{0\}$ .

Let  $(S, \circ, \leq)$  be an ordered semihypergroup. A subsemihypergroup (or left hyperideal)  $T$  of

$S$  is called left simple (left 0-simple), if the ordered semihypergroup  $(T, \circ, \leq)$  is left simple (left 0-simple).

**Lemma 2.5** *Let  $S$  be an ordered semihypergroup without zero. Then the following statements are equivalent:*

- (1)  $S$  is left simple.
- (2)  $(S \circ a] = S$  for all  $a \in S$ .
- (3)  $L(a) = S$  for all  $a \in S$ .

**Proof** (1) $\implies$ (2). Since  $S$  is left simple and for all  $a \in S$   $(S \circ a]$  is a left hyperideal of  $S$ , we have  $(S \circ a] = S$ .

(2) $\implies$ (3). Suppose that  $(S \circ a] = S$  for all  $a \in S$ . Then we have

$$L(a) = (a \cup S \circ a] = (a] \cup (S \circ a] = (a] \cup S = S.$$

(3) $\implies$ (1). Assume that  $L(a) = S$  for all  $a \in S$ . Let  $L$  be a left hyperideal of  $S$  and  $a \in L$ . By hypothesis,  $L(a) = S$ . It thus follows that  $S = L(a) \subseteq L \subseteq S$ , and so  $L = S$ . Hence  $S$  is left simple.  $\square$

**Lemma 2.6** *Let  $S$  be an ordered semihypergroup with zero. Then the following statements hold:*

- (1) If  $S$  is left 0-simple, then  $L(a) = S$  for all  $a \in S \setminus \{0\}$ .
- (2) If  $L(a) = S$  for all  $a \in S \setminus \{0\}$ , then either  $S \circ S = \{0\}$  or  $S$  is left 0-simple.

**Proof** (1) Assume that  $S$  is left 0-simple. Then for any  $a \in S \setminus \{0\}$ ,  $L(a)$  is a nonzero left hyperideal of  $S$ , and we obtain that  $L(a) = S$ .

(2) Suppose that  $L(a) = S$  for all  $a \in S \setminus \{0\}$  and  $S \circ S \neq \{0\}$ . Now, let  $L$  be a nonzero left hyperideal of  $S$  and  $x \in L \setminus \{0\}$ . By hypothesis, we have  $S = L(x) \subseteq L \subseteq S$ , which implies that  $L = S$ . Consequently,  $S$  is left 0-simple.  $\square$

**Lemma 2.7** *Let  $S$  be an ordered semihypergroup,  $L$  a left hyperideal of  $S$  and  $K$  a subsemihypergroup of  $S$ . Then the following statements hold:*

- (1) If  $K$  is left simple such that  $K \cap L \neq \emptyset$ , then  $K \subseteq L$ .
- (2) If  $K$  is left 0-simple such that  $K \setminus \{0\} \cap L \neq \emptyset$ , then  $K \subseteq L$ .

**Proof** (1) Suppose that  $K$  is left simple such that  $K \cap L \neq \emptyset$ . Let  $a \in K \cap L$ . Then, by Lemma 2.5,  $(K \circ a] = K$ . Thus we have

$$K = (K \circ a] \subseteq (K \circ L] \subseteq (S \circ L] \subseteq (L] = L.$$

(2) Assume that  $K$  is left 0-simple such that  $K \setminus \{0\} \cap L \neq \emptyset$ . Then there exists  $a \in K \setminus \{0\} \cap L$ . By Lemma 2.6(1),  $L_K(a) = K$ . Thus we have

$$K = L_K(a) = ((a] \cup (K \circ a]) \cap K \subseteq (a] \cup (K \circ a] \subseteq (a] \cup (S \circ a] = L(a) \subseteq L,$$

from which we conclude that  $K \subseteq L$ .  $\square$

The reader is referred to [2,15] for notation and terminology not defined in this paper.

### 3. Minimal left hyperideals of ordered semihypergroups

In this section, we shall characterize the minimal left hyperideals and 0-minimal left hyperideals of ordered semihypergroups.

**Definition 3.1** Let  $S$  be an ordered semihypergroup without zero. A left hyperideal  $L$  of  $S$  is called minimal if there is no left hyperideal  $A$  of  $S$  such that  $A \subset L$ . Equivalently, if for any left hyperideal  $A$  of  $S$  such that  $A \subseteq L$ , we have  $A = L$ .

**Definition 3.2** Let  $S$  be an ordered semihypergroup with zero. A nonzero left hyperideal  $L$  of  $S$  is called 0-minimal if there is no a nonzero left hyperideal  $A$  of  $S$  such that  $A \subset L$ . Equivalently, if for any nonzero left hyperideal  $A$  of  $S$  such that  $A \subseteq L$ , we have  $A = L$ .

**Theorem 3.3** Let  $S$  be an ordered semihypergroup without zero and  $L$  a left hyperideal of  $S$ . Then  $L$  is minimal if and only if  $L$  is left simple.

**Proof** Suppose that  $L$  is a minimal left hyperideal of  $S$ . Let  $A$  be a left hyperideal of  $L$ . Then  $L \circ A \subseteq A$ . Let

$$H := \{h \in A \mid h \preceq k \circ a \text{ for some } k \in L \text{ and } a \in A\}.$$

Then  $\emptyset \neq H \subseteq A \subseteq L$ . To show that  $H$  is a left hyperideal of  $S$ , let  $r \in S$  and  $h \in H$ . Then  $h \preceq k \circ a$  for some  $k \in L$  and  $a \in A$ . Thus, by Lemma 2.3(8), we have  $r \circ h \preceq r \circ (k \circ a) = (r \circ k) \circ a$ . Then, since  $r \circ h \subseteq S \circ H \subseteq S \circ L \subseteq L$ ,  $(r \circ k) \circ a \subseteq (S \circ L) \circ A \subseteq L \circ A \subseteq A$ , and  $A$  is a left hyperideal of  $L$ , by Lemma 2.3(7) we have  $r \circ h \subseteq A$ . Also, since  $r \circ h \preceq (r \circ k) \circ a$ ,  $r \circ k \subseteq S \circ L \subseteq L$ ,  $a \in A$ , we can easily show that  $r \circ h \subseteq H$ . Thus  $S \circ H \subseteq H$ . Furthermore, let  $t \in S$ ,  $h \in H$  be such that  $t \leq h$ . Since  $h \in H$ , we have  $h \in A$  and  $h \preceq k \circ a$  for some  $k \in L$ ,  $a \in A$ . Hence  $t \preceq k \circ a \subseteq S \circ L \subseteq L$ . By Lemma 2.3(7), we have  $t \in L$ . Also, since  $k \circ a \subseteq L \circ A \subseteq A$ ,  $t \in L$ ,  $t \preceq k \circ a \subseteq A$  and  $A$  is a left hyperideal of  $L$ , we have  $t \in A$ . Then, since  $t \in A$ ,  $t \preceq k \circ a$ ,  $k \in L$  and  $a \in A$ , we have  $t \in H$ . Thus  $H$  is a left hyperideal of  $S$ . Since  $L$  is a minimal left hyperideal of  $S$ , we have  $H = L$  and so  $A = L$ . Therefore,  $L$  is left simple.

Conversely, assume that  $L$  is left simple. Now, let  $J$  be a left hyperideal of  $S$  such that  $J \subseteq L$ . Then  $L \cap J \neq \emptyset$ , it follows from Lemma 2.7(1) that  $L \subseteq J$ . Hence  $J = L$ , and  $L$  is a minimal left hyperideal of  $S$ .  $\square$

**Theorem 3.4** Let  $S$  be an ordered semihypergroup with zero and  $L$  a nonzero left hyperideal of  $S$ . Then the following statements hold:

- (1) If  $L$  is a 0-minimal left hyperideal of  $S$ , then either  $L \circ A = \{0\}$  for some nonzero left hyperideal  $A$  of  $L$  or  $L$  is left 0-simple.
- (2) If  $L$  is left 0-simple, then  $L$  is a 0-minimal left hyperideal of  $S$ .

**Proof** (1) The proof is similar to that of necessary condition of Theorem 3.3.

(2) By Lemma 2.7(2), it is similar to the proof of sufficient condition of Theorem 3.3.  $\square$

**Theorem 3.5** Let  $S$  be an ordered semihypergroup without zero and  $S$  has proper left hyperideals. Then every proper left hyperideal of  $S$  is minimal if and only if  $S$  contains exactly

one proper left hyperideal or  $S$  contains exactly two proper left hyperideals  $L_1, L_2$  such that  $S = L_1 \cup L_2$  and  $L_1 \cap L_2 = \emptyset$ .

**Proof** Let  $J$  be a proper left hyperideal of  $S$ . Then, by hypothesis,  $J$  is a minimal left hyperideal of  $S$ . Then we have the following two cases:

**Case 1** Let  $S = L(a)$  for all  $a \in S \setminus J$ , where  $S \setminus J$  is the complement of  $J$  in  $S$ . Assume that  $K$  is also a proper left hyperideal of  $S$  and  $K \neq J$ . Then, since  $J$  is minimal, we have  $K \setminus J \neq \emptyset$ , and there exists  $a \in K \setminus J \subseteq S \setminus J$ . Thus  $S = L(a) \subseteq K \subseteq S$ , and so  $S = K$ , which is impossible. Hence  $K = J$ . Therefore, in this case,  $J$  is the unique proper left hyperideal of  $S$ .

**Case 2** Let  $S \neq L(a)$  for some  $a \in S \setminus J$ . Then  $L(a) \neq J$  and  $L(a)$  is a minimal left hyperideal of  $S$ . By lemma 2.4,  $L(a) \cup J$  is a left hyperideal of  $S$ . Since  $J$  is minimal and  $J \subset L(a) \cup J$ , we have  $S = L(a) \cup J$ . Also, since  $L(a) \cap J \subset L(a)$  and  $L(a)$  is a minimal left hyperideal of  $S$ , by Lemma 2.4 we get  $L(a) \cap J = \emptyset$ . Furthermore, let  $K$  be an arbitrary proper left hyperideal of  $S$ . Then, by hypothesis,  $K$  is a minimal left hyperideal of  $S$ . We observe that  $K = K \cap S = (K \cap L(a)) \cup (K \cap J)$ . If  $K \cap J \neq \emptyset$ , then, since  $K$  and  $J$  are also minimal left hyperideals of  $S$ , we have  $K = J$ . If  $K \cap L(a) \neq \emptyset$ , then  $K = L(a)$ . Hence, in this case,  $S$  contains exactly two proper left hyperideals  $L(a)$  and  $J$  such that  $M = L(a) \cup J$  and  $L(a) \cap J = \emptyset$ .

Conversely, let  $S$  contain exactly one proper left hyperideal  $L$ . Then it is not difficult to see that  $L$  is minimal. Now, suppose that  $S$  contains exactly two proper left hyperideals  $L_1, L_2$  such that  $S = L_1 \cup L_2$  and  $L_1 \cap L_2 = \emptyset$ . Let  $A$  be a left hyperideal of  $S$  such that  $A \subseteq L_1$ . Then  $A \subseteq L_1 \subset S$ , and so  $A$  is a proper left hyperideal of  $S$ . Since  $A \subseteq L_1$  and  $L_1 \cap L_2 = \emptyset$ , we have  $A \neq L_2$ . By hypothesis, we have  $A = L_1$ . Hence  $L_1$  is minimal. In the same way we can show that  $L_2$  is also minimal.  $\square$

**Corollary 3.6** *Let  $S$  be an ordered semihypergroup without zero and  $S$  has proper left hyperideals. Then every proper left hyperideal of  $S$  is left simple if and only if  $S$  contains exactly one proper left hyperideal or  $S$  contains exactly two proper left hyperideals  $L_1, L_2$  such that  $S = L_1 \cup L_2$  and  $L_1 \cap L_2 = \emptyset$ .*

**Proof** It is obvious by Theorem 3.3.  $\square$

**Theorem 3.7** *Let  $S$  be an ordered semihypergroup with zero and  $S$  has nonzero proper left hyperideals. Then every nonzero proper left hyperideal of  $S$  is 0-minimal if and only if  $S$  contains exactly one nonzero proper left hyperideal or  $S$  contains exactly two nonzero proper left hyperideals  $L_1, L_2$  such that  $S = L_1 \cup L_2$  and  $L_1 \cap L_2 = \emptyset$ .*

**Proof** The proof is similar to that of Theorem 3.5 with a slight modification.  $\square$

#### 4. Maximal left hyperideals of ordered semihypergroups

In the current section we discuss mainly the properties of maximal left hyperideals of ordered semihypergroups, and give some characterizations of maximal left hyperideals.

**Definition 4.1** A proper left hyperideal  $L$  of an ordered semihypergroup  $S$  is called maximal if  $T$  is a left hyperideal of  $S$  such that  $L \subset T$ , we have  $T = S$ . Equivalently, if for any proper left hyperideal  $T$  of  $S$  such that  $L \subseteq T$ , we have  $T = L$ .

**Theorem 4.2** Let  $S$  be an ordered semihypergroup and  $L$  a proper left hyperideal of  $S$ . Then  $L$  is maximal if and only if one and only one of the following two conditions is satisfied:

- (1)  $S \setminus L = \{a\}$  and  $a \circ a \subseteq L$  for some  $a \in S$ .
- (2)  $S \setminus L \subseteq (S \circ a]$  for all  $a \in S \setminus L$ .

**Proof** Assume that  $L$  is a maximal left hyperideal of  $S$ . Then we consider the following two cases:

**Case 1** Let  $(S \circ a] \subseteq L$  for some  $a \in S \setminus L$ . Then  $a \circ a \subseteq S \circ a \subseteq (S \circ a] \subseteq L$ , and we have

$$L \cup (a] = (L \cup (S \circ a]) \cup (a] = L \cup ((S \circ a] \cup (a]) = L \cup L(a).$$

Then, by Lemma 2.4,  $L \cup (a]$  is a left hyperideal of  $S$ . On the other hand, since  $a \in S \setminus L$ , we have  $L \subset L \cup (a]$ . Also, since  $L$  is maximal, we have  $L \cup (a] = S$ . Thus  $S \setminus L \subseteq (a]$ . To show that  $S \setminus L = \{a\}$ , let  $x \in S \setminus L$ . Then  $x \leq a$  and so  $(S \circ x] \subseteq (S \circ a] \subseteq L$ . From  $(S \circ x] \subseteq L$  and  $x \in S \setminus L$ , a similar argument shows that  $S \setminus L \subseteq (x]$ . Thus we have  $a \leq x$ , and so  $x = a$ . Hence we have shown that  $S \setminus L = \{a\}$ . In this case, the property (1) holds.

**Case 2** Let  $(S \circ a] \not\subseteq L$  for all  $a \in S \setminus L$ . In this case, we shall show that the property (2) holds. In fact, let  $a \in S \setminus L$ . Since  $(S \circ a]$  is a left hyperideal of  $S$ , by Lemma 2.4, we obtain that  $L \cup (S \circ a]$  is also a left hyperideal of  $S$ . On the other hand, since  $(S \circ a] \not\subseteq L$ , we have  $L \subset L \cup (S \circ a]$ . Thus, since  $L$  is maximal,  $L \cup (S \circ a] = S$ . Hence  $S \setminus L \subseteq (S \circ a]$  for all  $a \in S \setminus L$ .

Conversely, let  $T$  be a left hyperideal of  $S$  such that  $L \subset T$ . Then  $T \setminus L \neq \emptyset$ . If  $S \setminus L = \{a\}$  and  $a \circ a \subseteq L$  for some  $a \in S$ , then  $T \setminus L \subseteq S \setminus L = \{a\}$ . Thus we have  $T \setminus L = \{a\}$ , and so  $T = L \cup \{a\} = S$ . Hence  $L$  is a maximal left hyperideal of  $S$ . If  $S \setminus L \subseteq (S \circ a]$  for all  $a \in S \setminus L$ , then for any  $x \in T \setminus L$ ,  $S \setminus L \subseteq (S \circ x] \subseteq (S \circ T] \subseteq (T] = T$ . Thus  $S = (S \setminus L) \cup L \subseteq T \cup T = T$ . Therefore,  $L$  is a maximal left hyperideal of  $S$ .  $\square$

For an ordered semihypergroup  $S$ , let  $U$  denote the union of all proper left hyperideals of  $S$ .

**Remark 4.3** Let  $S$  be an ordered semihypergroup. Then  $U = S$  if and only if  $S \neq L(a)$  for all  $a \in S$ .

**Theorem 4.4** Let  $S$  be an ordered semihypergroup. Then one and only one of the following four conditions is satisfied:

- (1)  $S$  is left simple.
- (2)  $L(a) \neq S$  for all  $a \in S$ .
- (3) There exists  $a \in S$  such that  $S = L(a)$ ,  $a \notin (S \circ a]$ ,  $a \circ a \subseteq U = S \setminus \{a\}$  and  $U$  is the unique maximal left hyperideal of  $S$ .
- (4)  $S \setminus U = \{x \in S \mid (S \circ x] = S\}$  and  $U$  is the unique maximal left hyperideal of  $S$ .

**Proof** Assume that  $S$  is not left simple. Then there exists a proper left hyperideal  $L$  of  $S$ , and

so  $U \neq \emptyset$ . By Lemma 2.4,  $U$  is a left hyperideal of  $S$ . We consider the following two cases:

**Case 1** Let  $U = S$ . By Remark 4.3, we have  $L(a) \neq S$  for all  $a \in S$ . In this case, the condition (2) is satisfied.

**Case 2** Let  $U \neq S$ . Then  $U$  is a maximal left hyperideal of  $S$ . Moreover,  $U$  is the unique maximal left hyperideal of  $S$ . Indeed, assume that  $T$  is a maximal left hyperideal of  $S$ . Since  $T$  is a proper left hyperideal of  $S$ , we have  $T \subseteq U \subset S$ . Also, since  $T$  is a maximal left hyperideal of  $S$ , we have  $T = U$ . Hence  $U$  is the unique maximal left hyperideal of  $S$ . By Theorem 4.2, one and only one of the following two conditions is satisfied:

- (i)  $S \setminus U = \{a\}$  and  $a \circ a \subseteq U$  for some  $a \in S$ .
- (ii)  $S \setminus U \subseteq (S \circ a]$  for all  $a \in S \setminus U$ .

Assume  $S \setminus U = \{a\}$  and  $a \circ a \subseteq U$  for some  $a \in S$ . Then, in this case, the condition (3) is satisfied. In fact, we have:

- 1)  $L(a) = S$ . Indeed, let  $L(a) \neq S$ . Then  $L(a)$  is a proper left hyperideal of  $S$ , and we have  $a \in L(a) \subseteq U$ , which is a contradiction. Thus  $L(a) = S$ .
- 2)  $a \notin (S \circ a]$ . Indeed, let  $a \in (S \circ a]$ . Then  $(a] \subseteq ((S \circ a]) = (S \circ a]$ , and by 1) we have  $S = L(a) = (a] \cup (S \circ a] = (S \circ a]$ . Then we have  $a \preceq s \circ a$  for some  $s \in S = (S \circ a]$ , and  $s \preceq t \circ a$  for some  $t \in S$ . Hence we have

$$a \preceq s \circ a \preceq (t \circ a) \circ a = t \circ (a \circ a) \subseteq S \circ U \subseteq U.$$

Thus, by Lemma 2.3(7), we have  $a \in U$ . It is impossible, so  $a \notin (S \circ a]$ .

- 3)  $a \circ a \subseteq U = S \setminus \{a\}$ . Indeed, by the condition (i),  $a \circ a \subseteq U$ . Also, since  $S \setminus U = \{a\}$ , we have  $U = S \setminus \{a\}$ . Therefore,  $a \circ a \subseteq U = S \setminus \{a\}$ .

Now, let  $S \setminus U \subseteq (S \circ a]$  for all  $a \in S \setminus U$ . Then, in this case, the condition (4) is satisfied. To show that  $S \setminus U = \{x \in S \mid (S \circ x] = S\}$ , let  $x \in S \setminus U$ . Then, by hypothesis,  $x \in S \setminus U \subseteq (S \circ x]$ , and we have  $(x] \subseteq (S \circ x]$ . Thus  $L(x) = (x] \cup (S \circ x] = (S \circ x]$ . Also, since  $x \notin U$ , we have  $L(x) = S$ . Hence  $S = L(x) = (S \circ x]$ . Conversely, let  $x \in S$  be such that  $(S \circ x] = S$ . If  $x \in U$ , then, since  $U \neq S$ ,  $L(x) \subseteq U \subset S$ . It is impossible, since  $L(x) = (x] \cup (S \circ x] = (x] \cup S = S$ . Hence  $x \in S \setminus U$ . Thus  $S \setminus U = \{x \in S \mid (S \circ x] = S\}$ .  $\square$

## 5. Some types of left hyperideals of ordered semihypergroups

In this section, we define and study the weakly prime, quasi-prime, quasi-semiprime and weakly quasi-prime left hyperideals of ordered semihypergroups.

**Definition 5.1** Let  $L$  be a left hyperideal of an ordered semihypergroup  $S$ . Then  $L$  is called weakly prime if for all hyperideals  $I_1, I_2$  of  $S$  such that  $I_1 \circ I_2 \subseteq L$ , we have  $I_1 \subseteq L$  or  $I_2 \subseteq L$ .

**Definition 5.2** Let  $L$  be a left hyperideal of an ordered semihypergroup  $S$ . Then  $L$  is called quasi-prime if for any two left hyperideals  $L_1, L_2$  of  $S$  such that  $L_1 \circ L_2 \subseteq L$ , we have  $L_1 \subseteq L$  or  $L_2 \subseteq L$ .  $L$  is called quasi-semiprime if for any left hyperideal  $P$  of  $S$  such that  $P \circ P \subseteq L$ , we have  $P \subseteq L$ .

**Definition 5.3** Let  $L$  be a left hyperideal of an ordered semihypergroup  $S$ . Then  $L$  is called weakly quasi-prime if for all left hyperideals  $L_1, L_2$  of  $S$  such that  $L_1 \circ L_2 \subseteq L$  and  $L \subseteq L_1, L_2$ , we have  $L_1 = L$  or  $L_2 = L$ .

One can easily observe that the quasi-prime left hyperideals are weakly prime and weakly quasi-prime. However, the concepts of weakly prime, quasi-prime and weakly quasi-prime left hyperideals are different. We can show it by the following two examples.

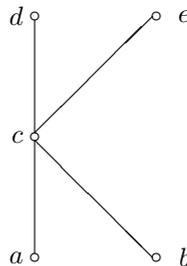
**Example 5.4** We consider a set  $S := \{a, b, c, d, e\}$  with the following hyperoperation “ $\circ$ ” and the order “ $\leq$ ”:

$\circ$	$a$	$b$	$c$	$d$	$e$
$a$	$\{a, b\}$				
$b$	$\{a, b\}$				
$c$	$\{a, b\}$	$\{a, b\}$	$\{c\}$	$\{c\}$	$\{e\}$
$d$	$\{a, b\}$	$\{a, b\}$	$\{c\}$	$\{d\}$	$\{e\}$
$e$	$\{a, b\}$	$\{a, b\}$	$\{c\}$	$\{c\}$	$\{e\}$

$$\leq := \{(a, a), (a, c), (a, d), (a, e), (b, b), (b, c), (b, d), (b, e), (c, c), (c, d), (c, e), (d, d), (e, e)\}.$$

We give the covering relation “ $\prec$ ” and the figure of  $S$  as follows:

$$\prec = \{(a, c), (b, c), (c, d), (c, e)\}.$$



Then  $(S, \circ, \leq)$  is an ordered semihypergroup. With a small amount of effort one can verify that

- (1)  $\{a, b\}, \{a, b, c\}, \{a, b, c, d\}, \{a, b, c, e\}$  and  $S$  are all left hyperideals of  $S$ .
- (2)  $\{a, b\}, \{a, b, c, e\}$  and  $S$  are all right hyperideals of  $S$ .
- (3)  $\{a, b\}, \{a, b, c, e\}$  and  $S$  are all hyperideals of  $S$ .

(4) The left hyperideal  $\{a, b, c\}$  of  $S$  is weakly prime, but it is not quasi-prime and not weakly quasi-prime. In fact,  $\{a, b, c, e\} \circ \{a, b, c, d\} = \{a, b, c\} \subseteq \{a, b, c\}$ , but

$$\{a, b, c, e\} \not\subseteq \{a, b, c\} \text{ and } \{a, b, c, d\} \not\subseteq \{a, b, c\}.$$

Moreover,  $\{a, b, c, e\} \supset \{a, b, c\}$  and  $\{a, b, c, d\} \supset \{a, b, c\}$ .

**Example 5.5** We consider a set  $S := \{a, b, c, d, e, f\}$  with the following hyperoperation “ $\circ$ ”

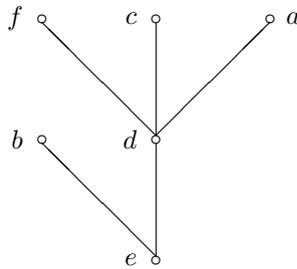
and the order “ $\leq$ ”:

$\circ$	$a$	$b$	$c$	$d$	$e$	$f$
$a$	$\{d, e\}$					
$b$	$\{d, e\}$					
$c$	$\{d, e\}$					
$d$	$\{d, e\}$	$\{e\}$	$\{d, e\}$	$\{d, e\}$	$\{d, e\}$	$\{d, e\}$
$e$	$\{d, e\}$	$\{e\}$	$\{d, e\}$	$\{d, e\}$	$\{d, e\}$	$\{d, e\}$
$f$	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{e\}$	$\{f\}$

$$\leq := \{(a, a), (b, b), (c, c), (d, d), (d, a), (d, c), (d, f), (e, e), (e, a), (e, b), (e, c), (e, d), (e, f), (f, f)\}.$$

We give the covering relation “ $\prec$ ” and the figure of  $S$  as follows:

$$\prec = \{(e, b), (e, d), (d, a), (d, c), (d, f)\}.$$



Then  $(S, \circ, \leq)$  is an ordered semihypergroup and  $\{d, e, f\}$  is a left hyperideal of  $S$ . Moreover, all left hyperideals of  $S$  containing properly  $\{d, e, f\}$  are

$$L_1 = \{a, d, e, f\}; L_2 = \{b, d, e, f\}; L_3 = \{c, d, e, f\}; L_4 = \{b, c, d, e, f\};$$

$$L_5 = \{a, c, d, e, f\}; L_6 = \{a, b, d, e, f\}; L_7 = S.$$

We claim that  $L_i \circ L_j \not\subseteq \{d, e, f\}$  ( $i, j = 1, 2, 3, 4, 5, 6, 7$ ). In fact, since for any  $i \in \{1, 2, 3, 4, 5, 6, 7\}$ , we have  $f \in L_i$ , and there exists an element  $x \in \{a, b, c\}$  such that  $x \in L_j$  ( $j \in \{1, 2, 3, 4, 5, 6, 7\}$ ). It is easy to see that  $f \circ x = \{x\} \subseteq L_i \circ L_j$  but  $x \notin \{d, e, f\}$ . This implies that  $\{d, e, f\}$  is a weakly quasi-prime left hyperideal of  $S$ . However, it is not quasi-prime. Indeed, since  $\{c, d, e\}$  is a left hyperideal of  $S$  and  $\{c, d, e\} \circ \{c, d, e\} = \{d, e\} \subseteq \{d, e, f\}$ , but  $\{c, d, e\} \not\subseteq \{d, e, f\}$ .

**Definition 5.6** An ordered semihypergroup  $S$  is called left duo if every left hyperideal of  $S$  is also a right hyperideal of  $S$ .

Clearly, every commutative ordered semihypergroup  $(S, \circ, \leq)$  (i.e.,  $a \circ b = b \circ a, \forall a, b \in S$ ) is left duo.

**Theorem 5.7** Let  $S$  be an ordered semihypergroup. Then the following statements hold:

- (1) If  $S$  is left duo (and so if  $S$  is commutative), then the concepts of weakly prime, quasi-prime and weakly quasi-prime left hyperideals of  $S$  coincide.
- (2) If the left hyperideals of  $S$  form a chain, then the concepts of quasi-prime and weakly quasi-prime left hyperideals of  $S$  coincide.

**Proof** (1) Let  $S$  be a left duo ordered semihypergroup and  $L$  a weakly prime left hyperideal of  $S$ . Then  $L$  is weakly quasi-prime. Indeed, let  $L_1, L_2$  be left hyperideals of  $S$  such that  $L_1 \circ L_2 \subseteq L$  and  $L \subseteq L_1, L_2$ . Since  $S$  is left duo,  $L_1, L_2$  are hyperideals of  $S$ . Since  $L_1 \circ L_2 \subseteq L$ , by hypothesis, we have  $L_1 \subseteq L$  or  $L_2 \subseteq L$ . Then  $L_1 = L$  or  $L_2 = L$ . Thus  $L$  is weakly quasi-prime. Furthermore, let  $S$  be left duo and  $L$  a weakly quasi-prime left hyperideal of  $S$ . Then  $L$  is quasi-prime. In fact, let  $L_1, L_2$  be left hyperideals of  $S$  such that  $L_1 \circ L_2 \subseteq L$ . By Lemma 2.4, the sets  $L_1 \cup L, L_2 \cup L$  are left hyperideals of  $S$ . Also,  $L \subseteq L_1 \cup L, L \subseteq L_2 \cup L$  and

$$\begin{aligned} (L_1 \cup L) \circ (L_2 \cup L) &= L_1 \circ L_2 \cup L \circ L_2 \cup L_1 \circ L \cup L \circ L \\ &\subseteq L_1 \circ L_2 \cup L \circ S \cup S \circ L \subseteq L_1 \circ L_2 \cup L = L. \end{aligned}$$

By hypothesis, we have  $L_1 \cup L = L$  or  $L_2 \cup L = L$ . Then  $L_1 \subseteq L$  or  $L_2 \subseteq L$ . Hence  $L$  is a quasi-prime left hyperideal of  $S$ .

(2) Assume that the left hyperideals of  $S$  form a chain. Let  $L$  be a weakly quasi-prime left hyperideal of  $S$ . Then  $L$  is quasi-prime. Indeed, let  $L_1, L_2$  be any two left hyperideals of  $S$  such that  $L_1 \circ L_2 \subseteq L$ . Suppose that  $L_1 \not\subseteq L$  and  $L_2 \not\subseteq L$ . By hypothesis,  $L \subseteq L_1$  and  $L \subseteq L_2$ . Since  $L$  is weakly quasi-prime, we have  $L_1 = L$  or  $L_2 = L$ , which is a contradiction. Therefore,  $L$  is a quasi-prime left hyperideal of  $S$ .  $\square$

**Theorem 5.8** *Let  $S$  be an ordered semihypergroup and  $\{L_i | i \in I\}$  a family of quasi-prime left hyperideals of  $S$ . Then  $\bigcap_{i \in I} L_i$  is a quasi-semiprime left hyperideal of  $S$  if  $\bigcap_{i \in I} L_i \neq \emptyset$ .*

**Proof** Let  $L_i$  be a quasi-prime left hyperideal of  $S$  for all  $i \in I$ . Assume that  $\bigcap_{i \in I} L_i \neq \emptyset$ . Then, by Lemma 2.4,  $\bigcap_{i \in I} L_i$  is a left hyperideal of  $S$ . Furthermore, we can show that  $\bigcap_{i \in I} L_i$  is quasi-semiprime. In fact, let  $P$  be a left hyperideal of  $S$  such that  $P \circ P \subseteq \bigcap_{i \in I} L_i$ . Then  $P \circ P \subseteq L_i$  for all  $i \in I$ . Hence, by hypothesis,  $P \subseteq L_i$  for all  $i \in I$ . It thus follows that  $P \subseteq \bigcap_{i \in I} L_i$ . Therefore,  $\bigcap_{i \in I} L_i$  is a quasi-semiprime left hyperideal of  $S$ .  $\square$

In the above theorem we have shown that every nonempty intersection of quasi-prime left hyperideals of an ordered semihypergroup  $S$  is quasi-semiprime. But the nonempty intersection of quasi-prime left hyperideals of  $S$  is not necessarily a quasi-prime left hyperideal of  $S$ . We can illustrate it by the following example.

**Example 5.9** Consider the ordered semihypergroup  $(S, \circ, \leq)$  given in Example 5.4. We can easily verify that  $\{a, b, c, d\}$  and  $\{a, b, c, e\}$  are quasi-prime left hyperideals of  $S$ . But  $\{a, b, c, d\} \cap \{a, b, c, e\}$  ( $= \{a, b, c\}$ ) is not quasi-prime. In fact, since  $\{a, b, c, e\} \circ \{a, b, c, d\} = \{a, b, c\} \subseteq \{a, b, c\}$ , but  $\{a, b, c, e\} \not\subseteq \{a, b, c\}$  and  $\{a, b, c, d\} \not\subseteq \{a, b, c\}$ .

In order to characterize the quasi-prime, quasi-semiprime and weakly quasi-prime left hyperideals of an ordered semihypergroup, we need the following concepts.

**Definition 5.10** *Let  $M$  be a nonempty subset of an ordered semihypergroup  $S$ .  $M$  is called  $m$ -system if for any  $a, b \in M$ , there exists  $x \in S$  such that  $(a \circ x \circ b) \cap M \neq \emptyset$ .*

The definition of  $m$ -systems in ordered semihypergroups is an extended form of the concept of  $m$ -systems of semigroups [21].

**Definition 5.11** Let  $N$  be a nonempty subset of an ordered semihypergroup  $S$ .  $N$  is called  $n$ -system if for any  $a \in N$ , there exists  $x \in S$  such that  $(a \circ x \circ a] \cap N \neq \emptyset$ .

**Definition 5.12** Let  $L$  be a left hyperideal and  $M$  a nonempty subset of an ordered semihypergroup  $S$ .  $(M, L)$  is called weakly  $m$ -system if  $L \cap M = \emptyset$  and

$$(\forall m, n \in M) ((m \cup L) \circ S \circ (n \cup L)) \cap M \neq \emptyset.$$

**Remark 5.13** If  $M$  is an  $m$ -system and  $L$  is a left hyperideal of  $S$  such that  $L \cap M = \emptyset$ . Then  $(M, L)$  is a weakly  $m$ -system. Indeed, for any two elements  $m, n \in M$ , we have  $x \in S$  such that  $(m \circ x \circ n] \cap M \neq \emptyset$ . Then we have  $((m \cup L) \circ S \circ (n \cup L)) \cap M \neq \emptyset$ .

**Theorem 5.14** Let  $L$  be a left hyperideal of an ordered semihypergroup  $S$  and  $M$  a nonempty subset of  $S$  such that  $(M, L)$  is a weakly  $m$ -system. If  $P$  is a maximal left hyperideal of  $S$  with respect to containing  $L$  and  $P \cap M = \emptyset$ , then  $P$  is a weakly quasi-prime left hyperideal of  $S$ .

**Proof** Let  $L_1, L_2$  be left hyperideals of  $S$  such that  $L_1 \circ L_2 \subseteq P$  and  $P \subseteq L_1, L_2$ . Then  $P = L_1$  or  $P = L_2$ . Indeed, if  $P \neq L_1$  and  $P \neq L_2$ , then there exist  $a \in L_1 \setminus P$  and  $b \in L_2 \setminus P$  such that

$$P \subset P \cup L(a) \subseteq L_1, \quad P \subset P \cup L(b) \subseteq L_2.$$

Thus, by hypothesis, we have  $(P \cup L(a)) \cap M \neq \emptyset$ ,  $(P \cup L(b)) \cap M \neq \emptyset$ . Let  $m_1 \in (P \cup L(a)) \cap M$ ,  $m_2 \in (P \cup L(b)) \cap M$ . Then, by Lemma 2.3, we have

$$\begin{aligned} ((m_1 \cup L) \circ S \circ (m_2 \cup L)) &\subseteq ((L_1 \cup L) \circ S \circ (L_2 \cup L)) \\ &= (L_1 \circ S \circ L_2 \cup L_1 \circ S \circ L \cup L \circ S \circ L_2 \cup L \circ S \circ L) \\ &\subseteq (L_1 \circ S \circ L_2] \quad (\text{By Lemma 2.3(2) and } L \subseteq P \subseteq L_1, L_2) \\ &\subseteq (L_1 \circ L_2] \subseteq (P] = P. \end{aligned}$$

Consequently,  $((m_1 \cup L) \circ S \circ (m_2 \cup L)) \cap M \subseteq P \cap M = \emptyset$ , which contradicts the fact that  $(M, L)$  is weakly  $m$ -system. We have thus shown that  $P$  is a weakly quasi-prime left hyperideal of  $S$ .  $\square$

**Theorem 5.15** Let  $S$  be an ordered semihypergroup and  $L$  a proper left hyperideal of  $S$ . Then the following statements are equivalent:

- (1)  $L$  is weakly quasi-prime.
- (2) For any two left hyperideals  $L_1, L_2$  of  $S$  such that  $(L \cup L_1) \circ (L \cup L_2) \subseteq L$ , we have  $L_1 \subseteq L$  or  $L_2 \subseteq L$ .
- (3) For any two left hyperideals  $L_1, L_2$  of  $S$  such that  $L \subseteq L_1$  and  $L_1 \circ L_2 \subseteq L$ , we have  $L_1 = L$  or  $L_2 \subseteq L$ .
- (4) For any two left hyperideals  $L_1, L_2$  of  $S$  such that  $(L_1 \cup L) \circ L_2 \subseteq L$ , we have  $L_1 \subseteq L$  or  $L_2 \subseteq L$ .
- (5) For all  $a, b \in S$  such that  $((a \cup L) \circ S \circ (b \cup L)) \subseteq L$ , we have  $a \in L$  or  $b \in L$ .
- (6)  $(S \setminus L, L)$  is a weakly  $m$ -system.

**Proof** (1)  $\implies$  (2). Let  $L_1, L_2$  be left hyperideals of  $S$  such that  $(L \cup L_1) \circ (L \cup L_2) \subseteq L$ . Then, since  $L \cup L_1, L \cup L_2$  are left hyperideals of  $S$  and  $L \subseteq L \cup L_1, L \cup L_2$ , by hypothesis, we have

$L \cup L_1 = L$  or  $L \cup L_2 = L$ . Then  $L_1 \subseteq L$  or  $L_2 \subseteq L$ .

(2)  $\implies$  (3). Let  $L_1, L_2$  be left hyperideals of  $S$  such that  $L \subseteq L_1$  and  $L_1 \circ L_2 \subseteq L$ . Then we have

$$\begin{aligned} (L \cup L_1) \circ (L \cup L_2) &= L \circ L \cup L_1 \circ L \cup L \circ L_2 \cup L_1 \circ L_2 \\ &\subseteq S \circ L \cup L_1 \circ L_2 \subseteq L \cup L_1 \circ L_2 = L. \end{aligned}$$

By (2), we have  $L_1 \subseteq L$  or  $L_2 \subseteq L$ . If  $L_1 \subseteq L$ , then  $L_1 = L$ .

(3)  $\implies$  (4). Let  $L_1$  and  $L_2$  be left hyperideals of  $S$  such that  $(L_1 \cup L) \circ L_2 \subseteq L$ . Since the sets  $L_1 \cup L$  and  $L_2$  are left hyperideals of  $S$ ,  $L \subseteq L_1 \cup L$  and  $(L_1 \cup L) \circ L_2 \subseteq L$ , by (3) we have  $L_1 \cup L = L$  or  $L_2 \subseteq L$ . If  $L_1 \cup L = L$ , then we have  $L_1 \subseteq L$ .

(4)  $\implies$  (5). Let  $a, b \in S$  be such that  $((a \cup L) \circ S \circ (b \cup L)) \subseteq L$ . Then  $a \circ S \circ b \subseteq L$ ,  $(L \circ S \circ b) \subseteq L$ . For the left hyperideals  $(a \cup S \circ a]$  and  $(S \circ b]$  of  $S$ , we have

$$\begin{aligned} ((a \cup S \circ a] \cup L) \circ (S \circ b] &= (a \cup S \circ a] \circ (S \circ b] \cup L \circ (S \circ b] \\ &= (a \cup S \circ a] \circ (S \circ b] \cup (L) \circ (S \circ b] \\ &\subseteq (a \circ S \circ b \cup S \circ a \circ S \circ b] \cup (L \circ S \circ b] \\ &\subseteq (L \cup S \circ L] \cup L = (L] \cup L = L. \end{aligned}$$

Then, by (4),  $(a \cup S \circ a] \subseteq L$  or  $(S \circ b] \subseteq L$ . If  $(a \cup S \circ a] \subseteq L$ , then  $a \in L$ . Let  $(S \circ b] \subseteq L$ . Since the sets  $(b \cup S \circ b]$ , and  $(b \cup S \circ b] \cup L$  are left hyperideals of  $S$ , and

$$\begin{aligned} ((b \cup S \circ b] \cup L) \circ ((b \cup S \circ b] \cup L) &= ((b \cup S \circ b] \circ (b \cup S \circ b]) \cup L \circ (b \cup S \circ b] \cup (b \cup S \circ b] \circ L \cup L \circ L \\ &\subseteq (b \circ b \cup S \circ b \circ b \cup b \circ S \circ b \cup S \circ b \circ S \circ b] \cup (L \circ b \cup L \circ S \circ b] \cup \\ &\quad (b \circ L \cup S \circ b \circ L] \cup L \circ L \\ &\subseteq (S \circ b] \cup (S \circ L] \subseteq L \cup (L] = L. \end{aligned}$$

Then, by (4), we have  $(b \cup S \circ b] \subseteq L$  or  $(b \cup S \circ b] \cup L \subseteq L$ . Then  $b \in L$ .

(5)  $\implies$  (6). Since  $L$  is a proper left hyperideal of  $S$ , we have  $\emptyset \neq S \setminus L \subseteq S$ . Let  $a, b \in S \setminus L$ . If  $((a \cup L) \circ S \circ (b \cup L)) \cap S \setminus L = \emptyset$ , then  $((a \cup L) \circ S \circ (b \cup L)) \subseteq L$ . Then, by (5),  $a \in L$  or  $b \in L$ , which contradicts the fact that  $a, b \in S \setminus L$ . Thus  $(S \setminus L, L)$  is a weakly  $m$ -system.

(6)  $\implies$  (1). Let  $L_1, L_2$  be left hyperideals of  $S$  such that  $L_1 \circ L_2 \subseteq L$  and  $L \subseteq L_1, L_2$ . Then  $L_1 = L$  or  $L_2 = L$ . Indeed, suppose that  $L \subset L_1$  and  $L \subset L_2$ . Let  $a \in L_1 \setminus L, b \in L_2 \setminus L$ . Then we have

$$\begin{aligned} ((a \cup L) \circ S \circ (b \cup L)) &\subseteq ((L_1 \cup L) \circ S \circ (L_2 \cup L)) \\ &= (L_1 \circ S \circ L_2] \subseteq (L_1 \circ L_2] \subseteq (L] = L. \end{aligned}$$

Then  $a, b \in S \setminus L$  and  $((a \cup L) \circ S \circ (b \cup L)) \cap S \setminus L = \emptyset$ . It contradicts the fact that  $(S \setminus L, L)$  is a weakly  $m$ -system. We have thus shown that  $L$  is weakly quasi-prime. This completes the proof.  $\square$

**Theorem 5.16** *Let  $S$  be an ordered semihypergroup and  $L$  a proper left hyperideal of  $S$ . Then*

the following statements are equivalent:

- (1)  $L$  is quasi-prime.
- (2) For every  $a, b \in S$  such that  $(a \circ S \circ b) \subseteq L$ , then  $a \in L$  or  $b \in L$ .
- (3)  $S \setminus L$  is an  $m$ -system.

**Proof** (1)  $\implies$  (2). Let  $a, b \in S$  such that  $(a \circ S \circ b) \subseteq L$ . Then, by Lemma 2.3, we have

$$\begin{aligned} (S \circ a] \circ (S \circ b] &\subseteq ((S \circ a) \circ (S \circ b)] = (S \circ (a \circ S \circ b)] \\ &\subseteq (S \circ (a \circ S \circ b)] \subseteq (S \circ L] \subseteq (L) = L. \end{aligned}$$

Since  $(S \circ a]$ ,  $(S \circ b]$  are left hyperideals of  $S$  and  $L$  is quasi-prime, we have  $(S \circ a] \subseteq L$  or  $(S \circ b] \subseteq L$ . Say  $(S \circ a] \subseteq L$ , then, by Lemma 2.3 we have

$$\begin{aligned} L(a) \circ L(a) &= (a \cup S \circ a] \circ (a \cup S \circ a] \\ &\subseteq (a \circ a \cup a \circ S \circ a \cup S \circ a \circ a \cup S \circ a \circ S \circ a] \\ &\subseteq (S \circ a] \subseteq L. \end{aligned}$$

Since  $L$  is quasi-prime and  $L(a)$  is a left hyperideal of  $S$ , we have  $L(a) \subseteq L$ . Then  $a \in L(a) \subseteq L$ . Similarly, say  $(S \circ b] \subseteq L$ , we have  $b \in L$ .

The proofs of (2)  $\implies$  (3)  $\implies$  (1) are similar to the proofs of (5)  $\implies$  (6)  $\implies$  (1) in Theorem 5.15 with suitable modifications.  $\square$

**Theorem 5.17** *Let  $S$  be an ordered semihypergroup and  $L$  a proper left hyperideal of  $S$ . Then the following statements are equivalent:*

- (1)  $L$  is quasi-semiprime.
- (2) For every  $a \in S$  such that  $(a \circ S \circ a] \subseteq L$ , then  $a \in L$ .
- (3)  $S \setminus L$  is an  $n$ -system.

**Proof** The proof is similar to that of Theorem 5.16 with a slight modification, and thus we omit the details.  $\square$

**Definition 5.18** *An ordered semihypergroup  $S$  is called strongly semisimple if  $(L \circ L) = L$  holds for every left hyperideal  $L$  of  $S$ .*

Now, we give some characterizations of a strongly semisimple ordered semihypergroup by means of quasi-prime left hyperideals.

**Theorem 5.19** *Let  $S$  be an ordered semihypergroup. Then the following statements are equivalent:*

- (1)  $S$  is strongly semisimple.
- (2) If  $L_1, L_2$  are left hyperideals of  $S$  such that  $L_1 \cap L_2 \neq \emptyset$ , then  $L_1 \cap L_2 \subseteq (L_1 \circ L_2]$ .
- (3)  $L(a) = (L(a) \circ L(a)]$  for every  $a \in S$ .
- (4)  $a \in (S \circ a \circ S \circ a]$  for every  $a \in S$ .
- (5) Every left hyperideal of  $S$  is quasi-semiprime.
- (6) Every left hyperideal of  $S$  is the intersection of all quasi-prime left hyperideals of  $S$  containing it.

**Proof** (1)  $\implies$  (2). Since  $L_1 \cap L_2 \neq \emptyset$ ,  $L_1 \cap L_2$  is a left hyperideal of  $S$ . By hypothesis, we have

$$L_1 \cap L_2 = ((L_1 \cap L_2) \circ (L_1 \cap L_2)) \subseteq (L_1 \circ L_2).$$

(2)  $\implies$  (3). Let  $a \in S$ . Since  $L(a)$  is a left hyperideal of  $S$ , we have  $(L(a) \circ L(a)) \subseteq (S \circ L(a)) \subseteq (L(a)) = L(a)$ . On the other hand, by (2) we have  $L(a) = L(a) \cap L(a) \subseteq (L(a) \circ L(a))$ . Thus  $L(a) = (L(a) \circ L(a))$ .

(3)  $\implies$  (4). Let  $a \in S$ . Then, by (3), we have

$$\begin{aligned} (L(a) \circ L(a)) &= ((a \cup S \circ a) \circ (a \cup S \circ a)) \\ &= ((a \cup S \circ a) \circ (a \cup S \circ a)) \text{ (By Lemma 2.3(3))} \\ &= (a \circ a \cup a \circ S \circ a \cup S \circ a \circ a \cup S \circ a \circ S \circ a) \\ &\subseteq (S \circ a). \end{aligned}$$

Thus we have

$$\begin{aligned} a \in L(a) &= (L(a) \circ L(a)) = (((L(a) \circ L(a)) \circ (L(a) \circ L(a)))) \\ &\subseteq ((S \circ a) \circ (S \circ a)) = ((S \circ a) \circ (S \circ a)) = (S \circ a \circ S \circ a). \end{aligned}$$

(4)  $\implies$  (5). Let  $L, P$  be left hyperideals of  $S$  such that  $P \circ P \subseteq L$ . Then  $P \subseteq L$ . Indeed, let  $a \in P$ . Then, by (4), we have

$$a \in (S \circ a \circ S \circ a) \subseteq (S \circ P \circ S \circ P) \subseteq ((S \circ P) \circ (S \circ P)) \subseteq (P \circ P) \subseteq (L) = L.$$

(5)  $\implies$  (6). Let  $L$  be a left hyperideal of  $S$ . Let

$$\mathcal{M} := \{K \mid K \text{ is a quasi-prime left hyperideal of } S \text{ and } K \supseteq L\}.$$

Since  $L \subseteq K$  for any  $K \in \mathcal{M}$ , we have  $L \subseteq \bigcap_{K \in \mathcal{M}} K$ . Let  $L \subset \bigcap_{K \in \mathcal{M}} K$ , and let  $a \in \bigcap_{K \in \mathcal{M}} K$  such that  $a \notin L$ . Let  $\mathcal{B} := \{P \mid P \text{ is a left hyperideal of } S, P \supseteq L \text{ and } a \notin P\}$ . Since  $L \in \mathcal{B}$ ,  $\mathcal{B} \neq \emptyset$ . Then  $(\mathcal{B}, \subseteq)$  is a partially ordered set. Let  $\mathcal{C}$  be a chain in  $\mathcal{B}$ . Then, by Lemma 2.4, the set  $\bigcup_{C \in \mathcal{C}} C$  is a left hyperideal of  $S$  and is an upper bound of  $\mathcal{C}$  in  $\mathcal{B}$ . By Zorn's Lemma,  $\mathcal{B}$  has a maximal element, say  $P_{\max}$ . Then  $a \notin P_{\max}$ . We now prove that  $P_{\max}$  is a quasi-prime left hyperideal of  $S$ . By the (1)  $\Leftrightarrow$  (2) of Theorem 5.16, let  $b, c \in S$  be such that  $(b \circ S \circ c) \subseteq P_{\max}$ . If  $b \notin P_{\max}$  and  $c \notin P_{\max}$ , then we have  $a \in L(b)$ , i.e.,  $L(a) \subseteq L(b)$ . In fact, if  $a \notin L(b)$ , then  $a \notin L(b) \cup P_{\max}$ . Also, since  $L(b) \cup P_{\max}$  is a left hyperideal of  $S$  and  $L \subseteq P_{\max} \subset L(b) \cup P_{\max}$ , we have  $L(b) \cup P_{\max} \in \mathcal{B}$ . It contradicts the fact that  $P_{\max}$  is a maximal element in  $\mathcal{B}$ . In the same way, we can show that  $a \in L(c)$ , i.e.,  $L(a) \subseteq L(c)$ . Since

$$\begin{aligned} L(a) \circ L(a) &\subseteq (L(a) \circ L(a)) \subseteq (L(b) \circ L(c)), \\ L(b) \circ L(b) &\subseteq (L(b) \circ L(b)) = ((b \cup S \circ b) \circ (b \cup S \circ b)) \\ &\subseteq (b \circ b \cup b \circ S \circ b \cup S \circ b \circ b \cup S \circ b \circ S \circ b) \subseteq (S \circ b), \\ L(c) \circ L(c) &\subseteq (L(c) \circ L(c)) = ((c \cup S \circ c) \circ (c \cup S \circ c)) \\ &\subseteq (c \circ c \cup c \circ S \circ c \cup S \circ c \circ c \cup S \circ c \circ S \circ c) \subseteq (S \circ c). \end{aligned}$$

By (5), we have  $L(a) \subseteq (L(b) \circ L(c))$ ,  $L(b) \subseteq (S \circ b]$  and  $L(c) \subseteq (S \circ c]$ . Thus we have

$$\begin{aligned} a \in L(a) &\subseteq (L(b) \circ L(c)) \subseteq ((S \circ b] \circ (S \circ c]) = (S \circ b \circ S \circ c] \\ &= (S \circ (b \circ S \circ c]) \subseteq (S \circ P_{\max}] \subseteq P_{\max}, \end{aligned}$$

which is impossible. Then  $P_{\max}$  is quasi-prime and  $P_{\max} \in \mathcal{M}$ . Thus we have  $a \in \bigcap_{K \in \mathcal{M}} K \subseteq P_{\max}$ , which contradicts the fact that  $a \notin P_{\max}$ . We have thus shown that  $L = \bigcap_{K \in \mathcal{M}} K$ .

(6)  $\implies$  (1). Suppose that  $L$  is a left hyperideal of  $S$ . Let

$$\mathcal{A} := \{K \mid K \text{ is a quasi-prime left hyperideal of } S \text{ and } K \supseteq (L \circ L)\}.$$

By (6), we have  $(L \circ L) = \bigcap_{K \in \mathcal{A}} K$ . Since  $L \circ L \subseteq (L \circ L) \subseteq K$  for any  $K \in \mathcal{A}$  and  $K$  is quasi-prime, we have  $L \subseteq K, \forall K \in \mathcal{A}$ . Consequently,  $L \subseteq \bigcap_{K \in \mathcal{A}} K = (L \circ L)$ . Also, it is obvious that  $(L \circ L) \subseteq L$ . Therefore,  $L = (L \circ L)$ , and  $S$  is strongly semisimple.  $\square$

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