Journal of Mathematical Research with Applications
Sept., 2017, Vol. 37, No. 5, pp. 543–549
DOI:10.3770/j.issn:2095-2651.2017.05.005
Http://jmre.dlut.edu.cn

On Weakly r-Clean Rings

Hanpeng GAO, Xiaobin YIN*
Department of Mathematics, Anhui Normal University, Anhui 241000, P. R. China

Abstract As generalization of r-clean rings and weakly clean rings, we define a ring R is weakly r-clean if for any $a \in R$ there exist an idempotent e and a regular element r such that $a = r + e$ or $a = r - e$. Some properties and examples of weakly r-clean rings are given. Furthermore, we prove the weakly clean rings and weakly r-clean rings are equivalent for abelian rings.

Keywords r-clean rings; weakly clean rings; weakly r-clean rings; regular rings

MR(2010) Subject Classification 16U99; 16E50

1. Introduction

All rings in this paper are assumed to be associative with identity. For a ring R, we denote the group of units, the set of idempotents, and the set of (von Neumann) regular element by $U(R)$, $\mathrm{Id}(R)$ and $\mathrm{Reg}(R)$, respectively.

A ring R is said to be exchange [1] if for every $a \in R$, there exists an idempotent $e \in Ra$ such that $1 - e \in R(1 - a)$. An element $a \in R$ is called clean if it is the sum of an idempotent and a unit. Nicholson said that R is clean if every element of R is clean in [1]. He also proved clean rings and exchange rings are equivalent for abelian rings (i.e., all idempotents are central). Recently, this work is motivated by the concept of nil clean rings, r-clean rings, weakly clean rings and so on [2–5].

It is well known that R is clean if and only if for any $a \in R$, there are an idempotent e and a unit u such that $a = u - e$. Question is asked whether R must be clean if for each $a \in R$, either $a = u + e$ or $a = u - e$ where $u \in U(R)$ and $e \in Id(R)$. Anderson and Camillo gave the negative answer [6, Example 17]. Ahn and Anderson called an element $a \in R$ is weakly clean if there exist $e \in \mathrm{Id}(R)$ and $u \in U(R)$ such that $a = u + e$ or $a = u - e$ and R is weakly clean if every element of R is weakly clean. Clearly, clean rings are weakly clean. An element $r \in R$ is called (von Neumann) regular if there exists $y \in R$ such that $r = ryr$. A ring R is (von Neumann) regular if every element of R is regular. Following Ashrafi and Nasibi [7], an element $a \in R$ is r-clean if $a = e + r$ where $e \in \mathrm{Id}(R)$ and $r \in \mathrm{Reg}(R)$ and R is r-clean if every element of R is r-clean.

In this paper, we call an element $a \in R$ is weakly r-clean if $a = r + e$ or $a = r - e$ where $e \in \mathrm{Id}(R)$ and $r \in \mathrm{Reg}(R)$ and R is weakly r-clean if every element of R is weakly r-clean. Some

Received October 13, 2016; Accepted May 17, 2017
Supported by the National Natural Science Foundation of China (Grant Nos. 11401009; 11326062).
* Corresponding author
E-mail address: hpgao07@163.com (Hanpeng GAO); xbyinzh@mail.ahnu.edu.cn (Xiaobin YIN)

basic properties of weakly r-clean rings are obtained. Some examples of weakly r-clean rings, of course, are regular rings, r-clean rings and weakly clean rings. We show weakly r-clean rings need not be weakly clean or r-clean (see Example 2.2). Furthermore, we prove the following results:

Theorem 1.1 *Let R be an abelian ring. Then R is weakly r-clean if and only if R is weakly clean.*

Theorem 1.2 *Let R be a weakly r-clean ring. Then every element of R is the sum of r-clean element and the square root of* 1.

Theorem 1.3 *Let I be a regular ideal of the ring R. If idempotents can be lifted modulo I, then R is weakly r-clean if and only if R/I is also.*

Theorem 1.4 *Let $R = \prod_{i\in I} R_i$ be a direct product of rings with $|I| \geq 2$. Then R is weakly r-clean if and only if there exists $j \in I$ such that R_j is weakly r-clean and R_k is r-clean for all $k \neq j$.*

Let A, B be rings and M be an A-B-bimodule. Finally, we consider when the formal triangular matrix $\left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$ is weakly r-clean.

2. Some properties and examples

In this section first we define weakly r-clean element and weakly r-clean rings and we show that the class of r-clean rings and weakly clean rings are a proper subset of the class of weakly r-clean rings.

Definition 2.1 *Let R be a ring. An element $a \in R$ is called weakly r-clean if there exist $e \in \mathrm{Id}(R)$ and $r \in \mathrm{Reg}(R)$ such that $a = r + e$ or $a = r - e$. A ring R is weakly r-clean if every element of R is weakly r-clean.*

Clearly, an element a is weakly r-clean if and only if $-a$ is also. Weakly clean rings and r-clean rings are weakly r-clean. The following example shows that weakly r-clean rings need not be weakly clean or r-clean.

Example 2.2 Let $A = Z_{(3)} \cap Z_{(5)}$ and B be a regular ring which is not clean ring as Bergman's example [8, Example 1]. By [6, Example 17], we have A is a weakly clean ring but not clean ring. It follows from [7, Theorem 2.2] that R is weakly r-clean but not r-clean. Take $R = A \times B$. Next, we show R is weakly r-clean ring but not weakly clean. For any $(a, b) \in R$, if $a = u + e$ for some $e \in \mathrm{Id}(A)$ and $u \in U(R)$, we write $(a, b) = (u, b) + (e, 0)$; if $a = u - e$ for some $e \in \mathrm{Id}(A)$ and $u \in U(R)$, we write $(a, b) = (u, b) - (e, 0)$. Clearly, $(u, b) \in \mathrm{Reg}(R)$ and $(e, 0) \in \mathrm{Id}(R)$, so R is weakly r-clean. Note that A is not clean, there exists $a \in A$ such that $a - e \notin U(A)$ for every $e \in Id(A)$. Similarly, there exists $b \in B$ such that $b + f \notin U(B)$ for every $f \in \mathrm{Id}(B)$. Assume that R is weakly clean, $(a, b) = (e_1, e_2) + (u_1, u_2)$ or $(a, b) = -(e_1, e_2) + (u_1, u_2)$ where $e_1 \in \mathrm{Id}(A)$, $e_2 \in \mathrm{Id}(B)$, $u_1 \in U(A)$ and $u_2 \in U(B)$, contradictorily.

An element e is very idempotent if $e^2 = e$ or $e^2 = -e$. Clearly, a ring R is weakly clean if and only if every element of R is the sum of a unit and a very idempotent. A ring R is weakly r-clean if and only if every element of R is the sum of a regular element and a very idempotent.

Lemma 2.3 *Let R be an abelian ring. Let e be a very idempotent of R and $a \in R$ be a clean element. Then*

(1) ae is weakly clean in R.

(2) If $-a$ is clean, then $a + e$ is weakly clean.

Proof (1) Since a is clean in R, we write $a = f + u$ where $f \in \mathrm{Id}(R)$ and $u \in U(R)$. Hence $ae = fe + ue$. Clearly, fe is a very idempotent and $ue \in U(eRe)$. We denote the inverse of u in eRe by v. If $(fe)^2 = fe$, $ae = (fe + 1 - e) + [ue - (1 - e)]$ where $(fe + 1 - e) \in \mathrm{Id}(R)$ and $[ue-(1-e)]$ is a unit in R with inverse $[v-(1-e)]$. If $(fe)^2 = -fe$, $ae = [fe-(1-e)]+(ue+1-e)$ where $[fe - (1 - e)]^2 = -[fe - (1 - e)]$ and $(ue + 1 - e)$ is a unit in R with inverse $(v + 1 - e)$. Hence ae is weakly clean in R.

(2) Assume that both a and $-a$ are clean. Clearly, $1 - a$ and $1 + a$ are also.

Case 1 If $e^2 = e$, $a = f + u$ and $1 + a = g + v$ where $f, g \in \mathrm{Id}(R)$, $u, v \in U(R)$. We have $a+e = ae+a(1-e)+e = (1+a)e+a(1-e) = (g+v)e+(f+u)(1-e) = ge+f(1-e)+ve+u(1-e)$. Note that R is abelian, $ge + f(1 - e)$ is an idempotent and $ve + u(1 - e)$ is an unit in R with inverse $v^{-1}e + u^{-1}(1 - e)$.

Case 2 If $e^2 = -e$, write $-a = f + u$ and $1 - a = g + v$ where $f, g \in \mathrm{Id}(R)$, $u, v \in U(R)$. We have $a + e = -ae + a(1 + e) + e = (1 - a)e - [-a(1 + e)] = (g + v)e - (f + u)(1 + e) = -[g(-e)+f(1+e)]+[ve-u(1+e)]$. Note that R is abelian, $[g(-e)+f(1+e)]^2 = g(-e)+f(1+e)$ and $ve - u(1 + e)$ is a unit in R with inverse $v^{-1}e - u^{-1}(1 + e)$. Hence $a + e$ is weakly clean. □

An element r is strongly regular if there exists $y \in R$ such that $r = ryr$ and $ry = yr$. Nicholson proved that strongly regular elements are clean [9, Theorem 1].

Theorem 2.4 *Let R be an abelian ring. Then R is weakly r-clean if and only if R is weakly clean.*

Proof Let R be a weakly r-clean ring. For every $a \in R$, there exist a very idempotent e and a regular element r such that $a = e + r$. r is strongly regular because R is abelian, so r is clean. It follows from Lemma 2.3 that a is weakly clean. The other direction is trivial. □

A ring R is weakly exchange [10] if for any $x \in R$, there exists an idempotent $e \in Rx$ such that $1 - e \in R(1 - x)$ or $1 - e \in R(1 + x)$. Let R be an abelian ring. Then R is weakly clean if and only if it is weakly exchange [11, Corollary 2.3].

Corollary 2.5 *Let R be an abelian ring. Then R is weakly r-clean if and only if R is weakly exchange.*

A ring R is reduce if for any $a \in R$, $a^2 = 0$ implies $a = 0$. It is clear that reduce rings are abelian. Hence we have the following result.

Corollary 2.6 *Let R be a reduce ring. Then R is weakly r-clean if and only if R is weakly exchange.*

Theorem 2.7 *Let R be a weakly r-clean ring. Then every element of R is the sum of an r-clean element and a square root of 1.*

Proof Suppose R is weakly r-clean, for any $a \in R$, $a-1 = r+e$ or $a-1 = r-e$ where $e \in \mathrm{Id}(R)$, $r \in \mathrm{Reg}(R)$. Hence $a = e+r+1$ or $a = e+r+(1-2e)$. Therefore we complete the proof. □

In [6], Anderson and Camillo proved if R is a ring in which 2 is invertible, then R is clean if and only if every element of R is the sum of a unit and a square root of 1. It is easy to prove that R is r-clean if and only if every element of R is the sum of a regular element and a square root of 1 if R is a ring in which 2 is invertible.

Proposition 2.8 *Let R be a weakly r-clean ring in which 2 is invertible. Then every element of R is the sum of a regular element and two square root of 1.*

Proof Suppose R is weakly r-clean and 2 is invertible, for any $a \in R$, $\frac{a}{2} = r+e$ or $\frac{a}{2} = r-e$ where $e \in \mathrm{Id}(R), r \in \mathrm{Reg}(R)$. Hence $a = 2e-1+2r+1$ or $a = (1-2e)+2r+(-1)$. Note that $2r \in \mathrm{Reg}(R)$ and $(1-2e)^2 = (2e-1)^2 = 1$, so the result is true. □

Whether the reverse of Proposition 2.8 holds or not? We give a negative answer by Example 2.14. Before answering this question, more properties of weakly r-clean rings should be given.

Proposition 2.9 *Every factor ring of a weakly r-clean ring is weakly r-clean.*

Proof Assume R is a weakly r-clean ring and I is an ideal of R, for each $\overline{a} = a+I \in R/I$, there exist $e \in \mathrm{Id}(R)$ and $r \in \mathrm{Reg}(R)$ such that $a = r+e$ or $a = r-e$. Hence $\overline{a} = \overline{r}+\overline{e}$ or $\overline{a} = \overline{r}-\overline{e}$. Clearly, $\overline{e} \in \mathrm{Id}(R/I)$ and $\overline{r} \in \mathrm{Reg}(R/I)$, it follows that R/I is weakly r-clean. □

Corollary 2.10 *The homomorphic image of weakly r-clean rings is weakly r-clean.*

Remark 2.11 The inverse of above Proposition 2.9 need not be true. For example, let p be a prime number, then $\mathbb{Z}/p\mathbb{Z} = \mathbb{Z}_p$ is weakly r-clean, but $\mathbb{Z}$ is not weakly r-clean.

Let R be a ring and I be an ideal of R. I is said to be regular if every element of I is regular.

Theorem 2.12 *Let R be a ring and I be an regular ideal of R. If idempotents can be lifted module I, then R is weakly r-clean if and only if R/I is weakly r-clean.*

Proof Assume that R is weakly r-clean, R/I is weakly r-clean by Proposition 2.9. Conversely, if R/I is weakly r-clean, then for any $a \in R$. We may write $\overline{a} = a+I = \overline{r}+\overline{e}$ or $\overline{a} = a+I = \overline{r}-\overline{e}$, where $\overline{e} \in \mathrm{Id}(R/I)$ and $\overline{r} \in \mathrm{Reg}(R/I)$. Thus $((a-e)+I)(x+I)((a-e)+I) = ((a-e)+I)$ or $((a+e)+I)(x+I)((a+e)+I) = ((a+e)+I)$ for some $x \in R$. Hence $(a-e)-(a-e)x(a-e) \in I$ or $(a+e)-(a+e)x(a+e) \in I$. Note that I is regular, we have $a-e \in \mathrm{Reg}(R)$ or $a+e \in \mathrm{Reg}(R)$ by [12, Lemma 1]. As idempotents can be lifted module I, we may suppose e is idempotent.

Hence R is weakly r-clean. □

Ashrafi and Nasibi proved that a ring R is r-clean if and only if for any $a \in R$, $a = r - e$ where $r \in \mathrm{Reg}(R)$ and $e \in \mathrm{Id}(R)$ (see [7, Theorem 2.11]).

Theorem 2.13 *Let $R = \prod_{i \in I} R_i$ be a direct product of rings with $|I| \geq 2$. Then R is weakly r-clean if and only if there exists $j \in I$ such that R_j is weakly r-clean and R_k is r-clean for all $k \neq j$.*

Proof Clearly, R_i is weakly r-clean for all $i \in I$ by Corollary 2.10. Assume that there exist $j, k \in I$ such that neither R_j nor R_k are r-clean. Then there exists $r_1 \in R_j$ such that $r_1 - e \notin \mathrm{Reg}(R)$ for all $e \in \mathrm{Id}(R_j)$. Similarly, there are $r_2 \in R_k$ Such that $r_2 + f \notin \mathrm{Reg}(R)$ for all $f \in \mathrm{Id}(R_k)$. Hence (r_1, r_2) is not weakly r-clean in $R_j \times R_k$, a contradiction. Conversely, assume R_j is weakly r-clean and R_k is r-clean for all $k \neq j$. For every $(a_i) \in R$, there exist $e_j \in \mathrm{Id}(R_j)$ and $r_j \in \mathrm{Reg}(R_j)$ such that $a_j = r_j + e_j$ or $a_j = r_j - e_j$. If $a_j = r_j + e_j$, for every $k \in I \setminus \{j\}$, we write $a_k = r_k + e_k$ for some $r_k \in \mathrm{Reg}(R_k)$ and $e_k \in \mathrm{Id}(R_k)$. If $a_j = r_j - e_j$, for every $i \in I \setminus \{j\}$, we write $a_k = r_k - e_k$ for some $r_k \in \mathrm{Reg}(R_k)$ and $e_k \in \mathrm{Id}(R_k)$. Hence $(a_i) = (r_i) + (e_i)$ or $(a_i) = (r_i) - (e_i)$. Since $(e_i) \in \mathrm{Id}(R)$ and $(r_i) \in \mathrm{Reg}(R)$, R is weakly r-clean, as needed. □

Example 2.14 Let $R = Z_{(3)} \cap Z_{(5)}$. Then $R \times R$ is not weakly r-clean by Theorem 2.13. Next, we show that $(2,2) \in U(R \times R)$ and every element of $R \times R$ is the sum of a regular element and two square root of 1.

Firstly, $(2,2) \in U(R \times R)$ because of $2 \in U(R)$. Secondly, note that R is weakly clean, for every $(a,b) \in R \times R$, write $\frac{a}{2} = u + e$ or $\frac{a}{2} = u - e$ for some $u \in U(R)$, $e \in \mathrm{Id}(R)$. Similarly, we can write $\frac{b}{2} = v + f$ or $\frac{b}{2} = v - f$ for some $u \in U(R)$, $e \in \mathrm{Id}(R)$.

Case 1 If $\frac{a}{2} = u + e$ and $\frac{b}{2} = v + f$, then $(a,b) = (2e-1, 2f-1) + (1,1) + (2u, 2v)$.

Case 2 If $\frac{a}{2} = u + e$ and $\frac{b}{2} = v - f$, then $(a,b) = (2e-1, -2f+1) + (1,-1) + (2u, 2v)$.

Case 3 If $\frac{a}{2} = u - e$ and $\frac{b}{2} = v + f$, then $(a,b) = (-2e+1, 2f-1) + (-1,1) + (2u, 2v)$.

Case 4 If $\frac{a}{2} = u - e$ and $\frac{b}{2} = v - f$, then $(a,b) = (-2e+1, -2f+1) + (-1,-1) + (2u, 2v)$.

3. Extensions of weakly r-clean rings

In this section, we consider the trivial extension and ideal extension.

Let R be a ring and M be an R-R-bimodule. The trivial extension of R by M is a ring, denoted by $T(R,M) = R \bigoplus M$. The addition and the multiplication can be defined by

$$(r_1, m_1) + (r_2, m_2) = (r_1 + r_2, m_1 + m_2), \quad (r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2).$$

We denote the ring $\{\left(\begin{smallmatrix} r & m \\ 0 & r \end{smallmatrix}\right) | r \in R, m \in M\}$ with the addition and the multiplication of matrix rings by T. Since $T(R,M) \cong T$, R is weakly r-clean if $T(R,M)$ is weakly r-clean by Corollary 2.10. It is well known that R is weakly clean if $T(R,M)$ is weakly clean [2, Theorem 1.10]. Hence, for weakly r-clean rings, we have the following result.

Proposition 3.1 *Let R be an abelian ring and M be an R-R-bimodule. Then $T(R,M)$ is weakly r-clean if and only if R is weakly r-clean.*

Proof One direction is trivial. Conversely, assume R is weakly r-clean, R is weakly clean as R is abelian. Hence $T(R,M)$ is weakly clean, it follows that $T(R,M)$ is weakly r-clean. □

Let A, B be rings and M be an A-B-bimodule. Then the formal triangular matrix $T(A,B,M) = \left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$ is a ring with the usual matrix addition and multiplication. Clearly, if $T(A,B,M)$ is weakly, then A and B both are weakly r-clean. Next, We consider when $T(A,B,M)$ is weakly r-clean.

Theorem 3.2 *Let A, B be rings and M be an A-B-bimodule. Suppose that one of the following conditions holds:*

(1) One of the rings A and B is weakly clean and the other one is r-clean.

(2) One of the rings A and B is weakly r-clean and the other one is clean.

Then $T(A,B,M)$ is weakly r-clean.

Proof (1) Suppose A is weakly clean and B is r-clean, for every $\left(\begin{smallmatrix} a & m \\ 0 & b \end{smallmatrix}\right) \in T(A,B,M)$, there exist $u \in U(A)$ and $e\,\mathrm{Id}(A)$ such that $a = u + e$ or $a = u - e$.

If $a = u + e$, write $b = r + f$ where $r \in \mathrm{Reg}(B)$ and $f\,\mathrm{Id}(B)$. Then

$$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} u & m \\ 0 & r \end{pmatrix} + \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}.$$

If $a = u - e$, write $b = r - f$ where $r \in \mathrm{Reg}(B)$ and $f\,\mathrm{Id}(B)$. Then

$$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} u & m \\ 0 & r \end{pmatrix} - \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}.$$

Note that $r \in \mathrm{Reg}(B)$, we write $r = ryr$ for some $r \in B$. Therefore,

$$\begin{pmatrix} u & m \\ 0 & r \end{pmatrix} \begin{pmatrix} u^{-1} & u^{-1}my \\ 0 & y \end{pmatrix} \begin{pmatrix} u & m \\ 0 & r \end{pmatrix} = \begin{pmatrix} u & m \\ 0 & r \end{pmatrix}.$$

Hence $\left(\begin{smallmatrix} u & m \\ 0 & r \end{smallmatrix}\right) \in \mathrm{Reg}(T(A,B,M))$, and $\left(\begin{smallmatrix} e & 0 \\ 0 & f \end{smallmatrix}\right) \in \mathrm{Id}(T(A,B,M))$. While A is clean and B is weakly clean we can prove similarly. So $T(A,B,M)$ is weakly r-clean.

(2) The proof is similar to (1). □

Let R be a ring and M be an R-R-bimodule which is a general ring (possibly with no unity) satisfying $(mn)r = m(nr)$, $(mr)n = m(rn)$ and $(rm)n = r(mn)$ for every $m, n \in M$ and $r \in R$. The ideal extension of R by M is a ring, denoted by $I(R;M) = R \bigoplus M$ whit the following addition and multiplication

$$(r_1, m_1) + (r_2, m_2) = (r_1 + r_2, m_1 + m_2), \quad (r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2 + m_1 m_2).$$

Proposition 3.3 *Let R and M be as above. Then the following results hold:*

(1) If $I(R;M)$ is weakly r-clean, then R is also.

(2) If R is weakly clean and for any $m \in M$ there is $n \in M$ such that $m + n + mn = 0$, then $I(R;M)$ is weakly clean.

(3) If R is weakly clean and for any $m \in M$ there is $n \in M$ such that $m + n + mn = 0$, then $I(R;M)$ is weakly r-clean.

(4) If R is weakly r-clean and for every $m \in M$ and $r \in \mathrm{Reg}(R)$ there exists $y \in R$ such that $rym + myr + mym = m$ and $r = ryr$, then $I(R;M)$ is weakly r-clean.

Proof (1) It is trivial because R is homomorphic image of $I(R;M)$.

(2) Since R is weakly clean, for every $(a,m) \in I(R;M)$, we write $a = u + e$ or $a = u - e$ where $u \in U(R)$ and $e \in \mathrm{Id}(R)$. Thus $(a,m) = (u,m) + (e,0)$ or $(a,m) = (u,m) - (e,0)$. Note that there is $n \in M$ such that $m + n + mn = 0$ for any $m \in M$, we claim $(u,m) \in U(I(R;M))$ (see [13, Proposition 7] and $(e,0) \in \mathrm{Id}(I(R;M))$. Therefore, $I(R;M)$ is weakly clean.

(3) It follows from (2).

(4) Given (4), for every $(a,m) \in I(R;M)$, we write $a = r+e$ or $a = r-e$ where $r \in \mathrm{Reg}(R)$ and $e \in \mathrm{Id}(R)$. Hence $(a,m) = (r,m) + (e,0)$ or $(a,m) = (r,m) - (e,0)$. By hypothesis, there exists $y \in R$ such that $rym + myr + mym = m$ and $r = ryr$. So $(r,m)(y,0)(r,m) = (ryr, rym + myr + mym) = (r,m)$. Clearly, $(e,0)$ is an idempotent. Hence $I(R;M)$ is weakly r-clean. $\square$

Acknowledgements The authors would like to thank the anonymous referees for their careful reading of the manuscript and helpful comments.

References

[1] W. K. NICHOLSON. *Lifting idempotents and exchange rings.* Trans. Amer. Math. Soc., 1977, **229**: 269–278.

[2] M. S. AHN, D. D. ANDERSON. *Weakly clean rings and almost clean rings.* Rocky Mountain J. Math., 2006, **36**(3): 783–798.

[3] V. P. CAMILLO, Huaping YU. *Exchange rings, unit and idempotents.* Comm. Algebra, 1994, **22**(12): 4737–4749.

[4] A. J. DIESL. *Nil clean rings.* J. Algebra, 2013, **383**: 197–211.

[5] W. K. NICHOLSON, Yiqiang ZHOU. *Clean general rings.* J. Algebra, 2005, **291**(1): 297–311.

[6] D. D. ANDERSON, V. P. CAMILLO. *Commutative rings whose elements are a sum of a unit and idempotent.* Comm. Algebra, 2002, **30**(7): 3327–3336.

[7] N. ASHRAF, E. NASIBI. *Rings in which elements are the sum of an idempotent and a regular element.* Bull. Iranian Math. Soc., 2013, **39**(3): 579–588.

[8] D. HANDELMAN. *Perspectivity and cancellation in regular rings.* J. Algebra, 1997, **48**(1): 1–16.

[9] W. K. NICHOLSON. *Strongly clean rings and Fitting's lemma.* Comm. Algebra, 1999, **27**(8): 3583–592.

[10] Junchao WEI. *Weakly-abel rings and weakly exchange rings.* Acta Math. Hungar, 2012, **137**(4): 254–262.

[11] A. Y. M. CHIN, K. T. QUA. *A note on weakly clean rings.* Acta Math. Hungar, 2011, **132**(1-2): 113-116.

[12] B. BROWN, N. MCCOY. *The maximal regular ideal of a ring.* Proc. Amer. Math. Soc., 1950, **1**: 165–171.

[13] W. K. NICHOLSON, Yiqiang ZHOU. *Rings in which elements are uniquely the sum of an idempotent and a unit.* Glasg. Math. J., 2004, **46**(2): 227–236.