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Coefficient Estimates for the Subclasses of Analytic Functions and Bi-Univalent Functions Associated with the Strip Domain

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Abstract The Sălăgean operator is used here to introduce a new subclass of analytic functions associated with the strip domain. We obtain the bounds of coefficients and Fekete-szegö inequality for functions in this class and coefficient estimates of bi-univalent functions for certain subclasses of this class. The results presented here extend some of the earlier results.

Keywords analytic functions; strip domain; Sălăgean operator; subordination

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1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ normalized by

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$. Also, let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all functions which are univalent in $\mathbb{U}$ (see [1]).

It is well known that every function $f\in\mathcal{S}$ of the form (1.1) has an inverse f^{-1}, defined by $f^{-1}(f(z))=z(z\in\mathbb{U})$ and $f^{-1}(f(\omega))=\omega$ $(|\omega|<r;r\geq\frac{1}{4})$, where

$$f^{-1}(\omega)=\omega-a_2\omega^2+(2a_2-a_3)\omega^3-(5a_2^2-5a_2a_3+a_4)\omega^4+\cdots \tag{1.2}$$

A function $f\in\mathcal{A}$ is bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions defined in the open unit disk $\mathbb{U}$. Recently, the bounds of coefficients of analytic and bi-univalent functions have been studied by many authors [2–7].

Let $u(z)$ and $v(z)$ be analytic in $\mathcal{A}$. We say that the function $u(z)$ is subordinate to $v(z)$ in $\mathbb{U}$, and write $u(z)\prec v(z)$, if there exists a Schwarz function $\omega(z)$, which is analytic in $\mathbb{U}$ with $\omega(0)=0$ and $|\omega(z)|<1$ such that $u(z)=v(\omega(z))$ $(z\in\mathbb{U})$.

Furthermore, if the function v is univalent in $\mathbb{U}$, then we have the following equivalence:

$$u(z)\prec v(z)\quad(z\in\mathbb{U})\Longleftrightarrow u(0)=v(0)\quad\text{and}\quad u(\mathbb{U})\subset v(\mathbb{U}).$$

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Let $\mathcal{P}$ denote the class of functions $p(z)$ of the form:

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{1.3}$$

which are analytic in $\mathbb{U}$. If $\Re(p(z)) > 0$ $(z \in \mathbb{U})$, we say that $p(z)$ is the Caratheodory function [1].

Let $S^*(\alpha)$ and $K(\alpha)$ $(0 \leq \alpha < 1)$ denote the subclass consisting of all functions, which are defined, respectively, by

$$\Re\{\frac{zf'(z)}{f(z)}\} > \alpha$$

and

$$\Re\{1 + \frac{zf''(z)}{f'(z)}\} > \alpha, \quad f(z) \in \mathcal{A}.$$

The classes $S^*(\alpha)$ and $K(\alpha)$ were introduced by Robertson [8]. Obviously, for $\alpha = 0$, we have the well-known classes S^* and K, respectively.

Also, let $M(\beta)$ and $N(\beta)$ $(\beta > 1)$ denote the subclasses consisting of all functions, which are defined, respectively, by

$$\Re\{\frac{zf'(z)}{f(z)}\} < \beta$$

and

$$\Re\{1 + \frac{zf''(z)}{f'(z)}\} < \beta, \quad f(z) \in \mathcal{A}.$$

The classes $M(\beta)$ and $N(\beta)$ were investigated by Uralegaddi, Ganigi and Sarangi [9] (see also [10]).

In [11], Kuroki and Owa defined an analytic function $S_{\alpha,\beta}(z) : \mathbb{U} \to \mathbb{C}$ as follows.

Definition 1.1 ([11]) *Let α and β be real numbers with $\alpha < 1$ and $\beta > 1$. Then the function $S_{\alpha,\beta}(z)$ defined by*

$$S_{\alpha,\beta}(z) = 1 + \frac{\beta - \alpha}{\pi} i \log\left(\frac{1 - e^{\frac{2\pi i(1-\alpha)}{\beta-\alpha}} z}{1 - z}\right), \quad z \in \mathbb{U} \tag{1.4}$$

is analytic and univalent in $\mathbb{U}$ with $S_{\alpha,\beta}(0) = 1$. In addition, $S_{\alpha,\beta}(z)$ maps $\mathbb{U}$ onto the strip domain ω with $\alpha < \Re\{\omega\} < \beta$.

We note that the function $S_{\alpha,\beta}(z)$ defined by (1.4) has the form [11]

$$S_{\alpha,\beta}(z) = 1 + \sum_{n=1}^{\infty} B_n z^n, \tag{1.5}$$

where

$$B_n = \frac{\beta - \alpha}{n\pi} i(1 - e^{\frac{2n\pi i(1-\alpha)}{\beta-\alpha}}), \quad n \in \mathbb{N}. \tag{1.6}$$

Definition 1.2 ([12]) *Let $-1 \leq B < A \leq 1$, $C \neq D$ and $-1 \leq D \leq 1$. Then the analytic function $p(z) \in P(A, B; C, D)$ if and only if $p(z)$ satisfies each of the following two subordination relationships:*

$$p(z) \prec h_1(z) = \frac{1 + Az}{1 + Bz} \tag{1.7}$$

and

$$p(z) \prec h_2(z) = \frac{1+Cz}{1+Dz}. \tag{1.8}$$

For $A = 1 - 2\alpha$ $(0 \le \alpha < 1)$, $B = -1$, $C = 1 - 2\beta$ $(\beta > 1)$ and $D = -1$ in $P(A, B; C, D)$, we obtain the following relationship:

$$p(z) \in P(\alpha, \beta) = P(1 - 2\alpha, -1; 1 - 2\beta, -1) \Longleftrightarrow \alpha < \Re\{p(z)\} < \beta. \tag{1.9}$$

From (1.4) and (1.9), we have

$$p(z) \in P(\alpha, \beta) \Longleftrightarrow p(z) \prec S_{\alpha,\beta}(z). \tag{1.10}$$

Also, from Definition 1.2, we introduce the following subclass of $p(z) \in P(A, B; C, D)$.

Definition 1.3 *Let*

$$\tilde{P}(\rho_1) = \{p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n : \Re(p(z)) > \rho_1\},$$

$$\tilde{P}(\rho_2) = \{p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n : \Re(p(z)) < \rho_2\},$$

$$\tilde{P}(\rho_1, \rho_2) = \{p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n : \rho_1 < \Re(p(z)) < \rho_2\}$$

and

$$\tilde{P}(\rho_3, \rho_4) = \{p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n : \rho_3 < \Re\{p(z)\}, \Re\{2 - p(z)\} < 1 + \rho_4\},$$

where

$$\begin{cases} \rho_1 = \max\{\frac{1-A}{1-B}, \frac{1+C}{1+D}\}, & -1 < B < A \le 1,\ -1 < C < D < 1, \\ \rho_2 = \min\{\frac{1+A}{1+B}, \frac{1-C}{1-D}\}, & -1 < B < A \le 1,\ -1 < C < D < 1, \\ \rho_3 = \{\frac{1-A}{2}\}, & B = -1, \\ \rho_4 = \{\frac{1-C}{2}\}, & D = 1. \end{cases} \tag{1.11}$$

In [13], Sălăgean defined the operator $D^m f(z) : \mathcal{A} \to \mathcal{A}$ as follows:

$$D^0 f(z) = f(z),\ D' f(z) = Df(z) = zf'(z),$$

in general,

$$D^m f(z) = D(D^{m-1} f(z)) = z + \sum_{n=2}^{\infty} n^m a_n z^n, \quad m \in \mathbb{N}_0 = \mathbb{N} \bigcup \{0\}. \tag{1.12}$$

By using the operator D^m, we introduce the following two new subclasses of $\mathcal{A}$.

Definition 1.4 *Let $m \in \mathbb{N}_0$, $0 \le \lambda$, $-1 \le B < A \le 1$, $-1 < C < D \le 1$, and $f(z) \in \mathcal{A}$. Then the function $f(z) \in S_{m,\lambda}(A, B; C, D)$ if and only if $f(z)$ satisfies the following condition:*

$$\psi(f; m, \lambda) = \frac{D^{m+1} f(z)}{D^m f(z)} + \lambda \frac{z^2 (D^m f(z))''}{D^m f(z)} \in P(A, B; C, D). \tag{1.13}$$

From the class $S_{m,\lambda}(A, B; C, D)$, we obtain the following subclasses which were studied in many earlier works:

(i) $S_{0,0}(1-2\alpha,-1;1-2\beta,-1)=S(\alpha,\beta)$ $(0\leq\alpha<1,\beta>1)$ (see [11,14]).

(ii) $S_{1,0}(1-2\alpha,-1;1-2\beta,-1)=K(\alpha,\beta)$ $(0\leq\alpha<1,\beta>1)$ (see [15]).

(iii) $S_{0,\lambda}(1-2\alpha,-1;1-2\beta,-1)=K(\lambda;\alpha,\beta)$ (see [16]).

(iv) $S_{m,0}(A,B;C,D)=S_m(A,B;C,D)$ (see [12]).

Definition 1.5 *Let $m\in\mathbb{N}_0$, $0\leq\lambda$, $-1\leq B<A\leq 1$, $-1<C<D\leq 1$, and $f(z)\in\mathcal{A}$. We denote by $S\Sigma_{m,\lambda}(A,B;C,D)$ the class of bi-univalent functions consisting of the functions in $\mathcal{A}$ such that $f\in S\Sigma_{m,\lambda}(A,B;C,D)$ and $f^{-1}\in S\Sigma_{m,\lambda}(A,B;C,D)$, where f^{-1} is the inverse function of f.*

This paper is organized as follows. We start with the function $p(z)\in P(A,B;C,D)$ if and only if $p(z)$ satisfies each of the two conditions. We obtain the bounds of coefficients and Fekete-szegö inequality for functions in this class and coefficient estimates of bi-univalent functions for certain subclasses of this class. The results presented here extend some of the earlier results.

2. Preliminary results

To prove the main results in the paper, we need the following lemmas.

Lemma 2.1 ([12]) *The function $p(z)\in P(A,B;C,D)$ if and only if $p(z)$ satisfies each of the following two conditions:*

$$\begin{cases} \mid p(z)-\sigma_i\mid< r_i, \quad i=1,2;\ -1<B<A\leq 1;\ -1<C<D<1, \\ \rho_3<\Re\{p(z)\},\ B=-1,\quad \Re\{2-p(z)\}<1+\rho_4,\ D=1, \end{cases} \tag{2.1}$$

where

$$\begin{cases} \sigma_1=\frac{1-AB}{1-B^2} \ \text{ and } \ r_1=\frac{A-B}{1-B^2}, \\ \sigma_2=\frac{1-CD}{1-D^2} \ \text{ and } \ r_2=\frac{D-C}{1-D^2}, \end{cases} \tag{2.2}$$

and ρ_3,ρ_4 are given by (1.11).

Lemma 2.2 ([12]) *Let $j=1,2,3,4$; $-1<B<A\leq 1$ and $-1<C<D<1$; $S_{\alpha,\beta}(z)$ is defined by (1.4). If $p(z)\in P(A,B;C,D)$, then*

$$p(z)\prec p_j(z)=\begin{cases} p_1(z)=S_{\frac{1-A}{1-B},\frac{1-C}{1-D}}(z), & BC-AD\geq|A-B+C-D|, \quad j=1, \\ p_2(z)=S_{\frac{1+C}{1+D},\frac{1+A}{1+B}}(z), & AD-BC\geq|A-B+C-D|, \quad j=2, \\ p_3(z)=S_{\frac{1-A}{1-B},\frac{1+A}{1+B}}(z), & |AD-BC|\leq B-A+D-C, \quad j=3, \\ p_4(z)=S_{\frac{1+C}{1+D},\frac{1-C}{1-D}}(z), & |AD-BC|\leq A-B+C-D, \quad j=4, \end{cases} \tag{2.3}$$

where $p_j(0)=1$ and

$$p_j(z)=\begin{cases} p_1(z)=1+\sum\limits_{n=1}^{\infty}B_{n,1}z^n, \quad j=1, \\ p_2(z)=1+\sum\limits_{n=1}^{\infty}B_{n,2}z^n, \quad j=2, \\ p_3(z)=1+\sum\limits_{n=1}^{\infty}B_{n,3}z^n, \quad j=3, \\ p_4(z)=1+\sum\limits_{n=1}^{\infty}B_{n,4}z^n, \quad j=4, \end{cases} \tag{2.4}$$

for

$$B_{n,j}=\begin{cases} B_{n,1}=\frac{\frac{1-C}{1-D}-\frac{1-A}{1-B}}{n\pi}i(1-e^{2n\pi i(1-\frac{1-A}{1-B})/(\frac{1-C}{1-D}-\frac{1-A}{1-B})}),\ j=1,\\ B_{n,2}=\frac{\frac{1+A}{1+B}-\frac{1+C}{1+D}}{n\pi}i(1-e^{2n\pi i(1-\frac{1+C}{1+D})/(\frac{1+A}{1+B}-\frac{1+C}{1+D})}),\ j=2,\\ B_{n,3}=\frac{\frac{1+A}{1+B}-\frac{1-A}{1-B}}{n\pi}i(1-e^{2n\pi i(1-\frac{1-A}{1-B})/(\frac{1+A}{1+B}-\frac{1-A}{1-B})}),\ j=3,\\ B_{n,4}=\frac{\frac{1-C}{1-D}-\frac{1+C}{1+D}}{n\pi}i(1-e^{2n\pi i(1-\frac{1+C}{1+D})/(\frac{1-C}{1-D}-\frac{1+C}{1+D})}),\ j=4.\end{cases}\tag{2.5}$$

Proof (i) Let $p(z)\in P(A,B;C,D)$ with $BC-AD\ge|A-B+C-D|$. Let $p(z)=1+c_1z+c_2z^2+\cdots\in P(A,B;C,D)$. Then, from Definition 1.2 and the definition of subordination, we get

$$\begin{cases} p(0)=h_1(0),\quad p(\mathbb{U})\subset h_1(\mathbb{U}),\\ p(0)=h_2(0),\quad p(\mathbb{U})\subset h_2(\mathbb{U}),\end{cases}\tag{2.6}$$

where $h_1(z)$ and $h_2(z)$ are given by (1.7) and (1.8), respectively. Therefore, we have

$$\begin{cases} p(z)=h_1(\omega_1(z)),\quad \omega_1(0)=0,\ |\omega_1(z)|<1,\\ p(z)=h_2(\omega_2(z)),\quad \omega_2(0)=0,\ |\omega_2(z)|<1.\end{cases}$$

We also deduce that

$$\begin{cases} |\omega_1(z)|=|\frac{p(z)-1}{A-Bp(z)}|<1,\quad p(z)=u+iv,\\ |\omega_2(z)|=|\frac{p(z)-1}{C-Dp(z)}|<1,\quad p(z)=u+iv.\end{cases}\tag{2.7}$$

From (2.7), we find that

$$\begin{cases} 2u(1-AB)>1-A^2+(1-B^2)(u^2+v^2),\\ 2u(1-CD)>1-C^2+(1-D^2)(u^2+v^2).\end{cases}\tag{2.8}$$

Since

$$|p(z)|^2\ge[\Re(p(z))]^2,\tag{2.9}$$

from (2.8) and (2.9) we have

$$\begin{cases} \frac{1-A}{1-B}<u=\Re(p(z))<\frac{1+A}{1+B},\\ \frac{1+C}{1+D}<u=\Re(p(z))<\frac{1-C}{1-D}.\end{cases}\tag{2.10}$$

Then, from (2.10) we obtain

$$\frac{1-A}{1-B}<\Re\{p(z)\}<\frac{1-C}{1-D}.$$

By using (1.9), we get

$$p(z)\prec p_1(z)=S_{\frac{1-A}{1-B},\frac{1-C}{1-D}}(z),\quad BC-AD\ge|A-B+C-D|.$$

Also, similarly as the proof in (i), it is easy to prove that

(ii) $p(z)\prec p_2(z)=S_{\frac{1+C}{1+D},\frac{1+A}{1+B}}(z)$, $AD-BC\ge|A-B+C-D|$,

(iii) $p(z)\prec p_3(z)=S_{\frac{1-A}{1-B},\frac{1+A}{1+B}}(z)$, $|AD-BC|\le B-A+D-C$

and

(iv) $p(z)\prec p_4(z)=S_{\frac{1+C}{1+D},\frac{1-C}{1-D}}(z)$, $|AD-BC|\le A-B+C-D$.

Therefore, we complete the proof of Lemma 2.2. $\square$

The functions p_j $(j=1,2,3,4)$ maps $\mathbb{U}$ onto the strip domain (see Figures 1-1, 1-2, 1-3 and 1-4).

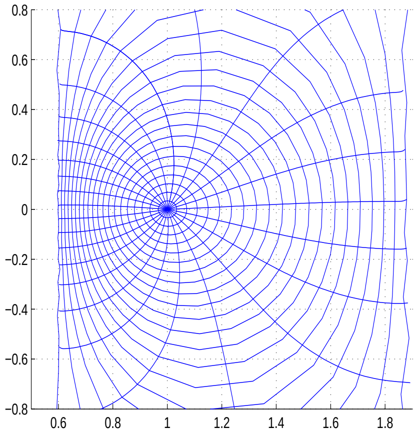

Figure 1-1 The image of $\mathbb{U}$ under $p_1(z)$ for $A = 0.1, B = -0.5, C = -0.5, D = 0.2$

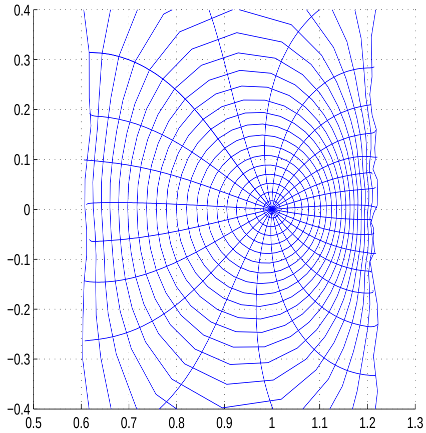

Figure 1-2 The image of $\mathbb{U}$ under $p_2(z)$ for $A = 0.7, B = 0.4, C = 0.1, D = 0.8$

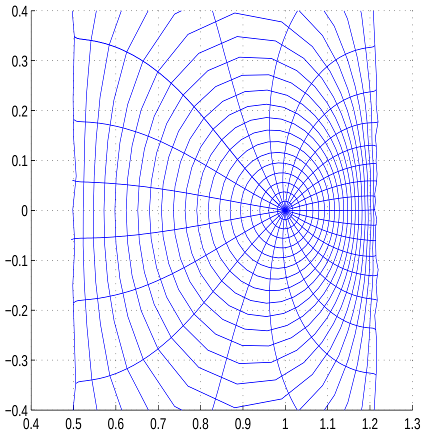

Figure 1-3 The image of $\mathbb{U}$ under $p_3(z)$ for $A = 0.7, B = 0.4, C = -0.1, D = 0.8$

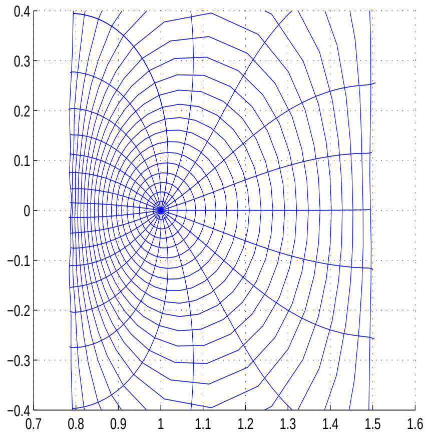

Figure 1-4 The image of $\mathbb{U}$ under $p_4(z)$ for $A = 0.9, B = 0.1, C = 0.1, D = 0.4$

Lemma 2.3 ([20]) *Let $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ be analytic and univalent in $\mathbb{U}$, and suppose that $p(z)$ maps $\mathbb{U}$ onto a convex domain. If $q(z) = 1 + q_1 z + q_2 z^2 + \cdots$ is analytic in $\mathbb{U}$ and satisfies the following subordination:*

$$q(z) \prec p(z), \quad z \in \mathbb{U}$$

then

$$| q_n | \leq | c_1 |, \quad n = 1, 2, \ldots.$$

Using Definition 1.1, Lemma 2.3 and the definition of subordination, we can obtain the following lemma.

Lemma 2.4 ([12]) *Let $-1 \leq B < A \leq 1$, $-1 < C < D \leq 1$, $i = 1, 2$; $j = 1, 2, 3, 4$ and*

$\tilde{P}(\rho_1), \tilde{P}(\rho_2), \tilde{P}(\rho_1,\rho_2)$ *and* $\tilde{P}(\rho_3,\rho_4)$ *are given by Definition 1.3. If* $p(z) = 1 + c_1 z + c_2 z^2 + \cdots \in P(A,B;C,D)$*, then*

$$|c_n| \le \chi(\delta_i;\rho_j) = \begin{cases} 2\delta_1, & p \in \tilde{P}(\rho_1), \\ 2\delta_2, & p \in \tilde{P}(\rho_2), \\ 2\min\{\delta_1,\delta_2\}, & p \in \tilde{P}(\rho_1,\rho_2), \\ 2\min\{\frac{1+A}{2}, \frac{1-C}{2}\}, & p \in \tilde{P}(\rho_3,\rho_4), \end{cases} \tag{2.11}$$

where

$$\begin{cases} \delta_1 = \min\{\frac{A-B}{1-B}, \frac{D-C}{1+D}\}, \\ \delta_2 = \min\{\frac{A-B}{1+B}, \frac{D-C}{1-D}\}, \end{cases} \tag{2.12}$$

and ρ_j *are given by (1.11).*

Lemma 2.5 ([21]) *Let the function* $p(z)$ *be given by (1.3). If* $p(z) \in \mathcal{P}$*, then for any complex number* γ*,*

$$|c_2 - \gamma c_1^2| \le 2\max\{1, |\ 2\gamma - 1\ |\},$$

and the result is sharp for the functions given by $p(z) = \frac{1+z^2}{1-z^2}$, $p(z) = \frac{1+z}{1-z}$.

3. Main results

Using Lemma 2.1 and Definition 1.4, we easily get

Theorem 3.1 *Let* $\psi(f;m,\lambda)$ *be defined by (1.13). The function* $f(z) \in S_{m,\lambda}(A,B;C,D)$ *if and only if* $f(z)$ *satisfies each of the following two conditions:*

$$\begin{cases} |\ \psi(f;m,\lambda) - \sigma_i\ | < r_i, \quad i = 1,2; -1 < B < A \le 1; -1 < C < D < 1, \\ \rho_3 < \Re\{\psi(f;m,\lambda)\},\ B = -1, \quad \Re\{2 - \psi(f;m,\lambda)\} < 1 + \rho_4,\ D = 1, \end{cases}$$

where σ_i *and* r_i $(i = 1,2)$ *are given by (2.2) and* ρ_k $(k = 3,4)$ *are given by (1.11).*

Theorem 3.2 *Let* $m \in \mathbb{N}_0, \lambda \ge 0, |a_1| = 1$ *and the function* $f(z)$ *be given by (1.1). If* $f(z) \in S_{m,\lambda}(A,B;C,D)$*, then*

$$|a_n| \le M_{n,j}(m,\lambda) = \begin{cases} \frac{|B_{1,j}|}{2^m(2\lambda+1)}, & n = 2, \\ \frac{|B_{1,j}|}{(n-1)(n\lambda+1)n^m} \prod_{k=2}^{n-1}(1 + \frac{|B_{1,j}|}{(k-1)(k\lambda+1)}), & n \ge 3, \end{cases} \tag{3.1}$$

where $|B_{1,j}|$ $(j = 1,2,3,4)$ *are defined by (2.5).*

Proof According to Definition 1.2 and the subordination relationship, we have

$$\frac{D^{m+1}f(z)}{D^m f(z)} + \lambda\frac{z^2(D^m f(z))''}{D^m f(z)} \in h_1(\mathbb{U}) \tag{3.2}$$

and

$$\frac{D^{m+1}f(z)}{D^m f(z)} + \lambda\frac{z^2(D^m f(z))''}{D^m f(z)} \in h_2(\mathbb{U}), \tag{3.3}$$

where the functions $h_1(z)$ and $h_2(z)$ are given by (1.7) and (1.8), respectively.

Applying (3.2) and (3.3), we get

$$\frac{D^{m+1}f(z)}{D^m f(z)}+\lambda\frac{z^2(D^m f(z))''}{D^m f(z)}=p(z),\quad \exists p(z)=1+c_1z+c_2z^2+\cdots\in P(A,B;C,D),$$

or, equivalently,

$$D^{m+1}f(z)+\lambda z^2(D^m f(z))''=p(z)D^m f(z),\quad \exists p(z)=1+c_1z+c_2z^2+\cdots\in P(A,B;C,D). \tag{3.4}$$

Then, comparing the coefficients of z^n in the both sides of (3.4), we have

$$(n-1)(n\lambda+1)n^m a_n=(c_{n-1}+c_{n-2}2^m a_2+\cdots+c_1(n-1)^m a_{n-1}). \tag{3.5}$$

Using Lemma 2.2, Lemma 2.3 and (3.5), we obtain

$$\begin{aligned}|a_n|&\le\frac{1}{(n-1)(n\lambda+1)n^m}(\mid c_{n-1}\mid+\mid c_{n-2}\mid 2^m\mid a_2\mid+\cdots+\mid c_1\mid(n-1)^m\mid a_{n-1}\mid)\\&\le\frac{|B_{1,j}|}{(n-1)(n\lambda+1)n^m}\sum_{k=1}^{n-1}k^m|a_k|.\end{aligned}$$

Hence, we have $|a_2|\le M_{2,j}(m,\lambda)$. To prove the remaining part of the theorem, we need to show that

$$\sum_{k=1}^{n-1}k^m|a_k|\le\prod_{k=2}^{n-1}(1+\frac{|B_{1,j}|}{(k-1)(k\lambda+1)}), \tag{3.6}$$

for $n=3,4,5,\ldots$. We use induction to prove (3.6). The case $n=3$ is clear. Next, assume that the inequality (3.6) holds for $n=p$. Then, a straightforward calculation gives

$$\begin{aligned}\sum_{k=1}^{p}k^m|a_k|=&\sum_{k=1}^{p-1}k^m|a_k|+p^m|a_p|\\\le&(1+\frac{|B_{1,j}|}{(p-1)(p\lambda+1)})\sum_{k=1}^{p-1}k^m|a_k|\\\le&(1+\frac{|B_{1,j}|}{(p-1)(p\lambda+1)})\prod_{k=2}^{p-1}(1+\frac{|B_{1,j}|}{(k-1)(k\lambda+1)})\\=&\prod_{k=2}^{p}(1+\frac{|B_{1,j}|}{(k-1)(k\lambda+1)})\end{aligned}$$

which implies that the inequality (3.6) holds for $n=p+1$. Hence, the desired estimate for $|a_n|$ $(n\ge 3)$ follows, as asserted in (3.1). This completes the proof of Theorem 3.2. □

Remark 3.3 Taking $m=0$, $A=1-2\alpha$ $(0\le\alpha\le 1)$, $B=-1$; $C=1-2\beta$ $(1<\alpha)$, $D=-1$, we obtain the improved result of Theorem 3.1 in the paper [16]. Also, setting $m=0$, $\lambda=0$, we obtain the improved result of Theorem 3.2 in the paper [12].

Also, using Lemma 2.4 and Definition 1.4, we get

Theorem 3.4 *Let $m\in\mathbb{N}_0$, $\lambda\ge 0$, $|a_1|=1$ and the function $f(z)$ be given by (1.1). If*

$f(z)\in S_{m,\lambda}(A,B;C,D)$, then

$$|a_n|\le \Psi_{n,j}(m,\lambda)=\begin{cases}\frac{\chi(\delta_i;\rho_j)}{2^m(2\lambda+1)}, & n=2,\\ \frac{\chi(\delta_i;\rho_j)}{(n-1)(n\lambda+1)n^m}\prod_{k=2}^{n-1}(1+\frac{\chi(\delta_i;\rho_j)}{(k-1)(k\lambda+1)}), & n\ge 3,\end{cases}\tag{3.7}$$

where $\chi(\delta_i;\rho_j)$ $(i=1,2;j=1,2,3,4)$ are defined by (2.11).

Remark 3.5 Setting $m=0,\lambda=0$, we obtain the improved result of Theorem 3.1 in [12].

Theorem 3.6 *Let $m\in\mathbb{N}_0$, $\lambda\ge 0$, $-1<B<A\le 1$, $-1<C<D<1, 0\le\mu\le 1$ and $p_j(z)=1+\sum_{n=1}^{\infty}B_{n,j}z^n$ $(j=1,2,3,4)$. If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in S_m(A,B;C,D)$, then*

$$|\ a_3-\mu a_2^2\ |\le \frac{|B_{1,j}|}{2\cdot 3^m(3\lambda+1)}\max\{1, |\ \frac{B_{2,j}}{B_{1,j}}-\frac{2(3\lambda+1)(\frac{3}{4})^m\mu-(2\lambda+1)}{(2\lambda+1)^2}B_{1,j}\ |\},\tag{3.8}$$

where $|B_{i,j}|$ $(i=1,2;j=1,2,3,4)$ are defined by (2.5).

Proof If $f(z)\in S_m(A,B;C,D)$, then there exists a Schwarz function $\omega(z)$ in $\mathbb{U}$ such that

$$\frac{D^{m+1}f(z)}{D^mf(z)}+\lambda\frac{z^2(D^mf(z))''}{D^mf(z)}=p_j(\omega(z)),\quad z\in\mathbb{U},\tag{3.9}$$

where $p_j(z)$ $(j=1,2,3,4)$ are defined by (2.3).

Let the function $p(z)$ be given by

$$p(z)=\frac{D^{m+1}f(z)}{D^mf(z)}+\lambda\frac{z^2(D^mf(z))''}{D^mf(z)}.\tag{3.10}$$

Then, from (3.9) and (3.10) we have $p(z)\prec p_j(z)$. Let

$$q(z)=\frac{1+\omega(z)}{1-\omega(z)}=1+q_1z+q_2z^2+\cdots.\tag{3.11}$$

Then $q(z)$ is analytic and has positive real part in $\mathbb{U}$. From (3.11), we get

$$\omega(z)=\frac{q(z)-1}{q(z)+1}=\frac{1}{2}[q_1z+(q_2-\frac{q_1^2}{2})z^2+\cdots].\tag{3.12}$$

We see from (3.12) that

$$p(z)=p_j(\frac{q(z)-1}{q(z)+1})=1+\frac{1}{2}B_{1,j}q_1z+[\frac{1}{2}B_{1,j}(q_2-\frac{q_1^2}{2})+\frac{B_{2,j}q_1^2}{4}]z^2+\cdots.\tag{3.13}$$

Using (3.10) and (3.13), we obtain

$$(2\lambda+1)2^ma_2=\frac{B_{1,j}q_1}{2},$$

$$2(3\lambda+1)3^ma_3-(2\lambda+1)4^ma_2^2=\frac{B_{1,j}q_2}{2}-\frac{q_1^2}{4}(B_{1,j}-B_{2,j}),$$

which imply that

$$a_3-\mu a_2^2=\frac{B_{1,j}}{4\cdot 3^m(3\lambda+1)}[q_2-\gamma_jq_1^2],\tag{3.14}$$

where, for convenience,

$$\gamma_j=\frac{1}{2}[1-\frac{B_{2,j}}{B_{1,j}}+\frac{2(3\lambda+1)(\frac{3}{4})^m\mu-(2\lambda+1)}{(2\lambda+1)^2}B_{1,j}].$$

Then, applying Lemma 2.5, we have

$$\begin{aligned} |\, a_3 - \mu a_2^2 \,| &\le \frac{|B_{1,j}|}{4\cdot 3^m(3\lambda+1)}|q_2 - \gamma_j q_1^2| \le \frac{|B_{1,j}|}{2\cdot 3^m(3\lambda+1)}\max\{1, |\,1-2\gamma_j\,|\} \\ &\le \frac{|B_{1,j}|}{2\cdot 3^m(3\lambda+1)}\max\Big\{1, \Big|\,\frac{B_{2,j}}{B_{1,j}} - \frac{2(3\lambda+1)(\frac{3}{4})^m\mu - (2\lambda+1)}{(2\lambda+1)^2}B_{1,j}\,\Big|\Big\}. \end{aligned}$$

The estimate is sharp for the function $f_j(z)$ $(j = 1, 2, 3, 4)$ defined by

$$f_j(z) = D^{-m}\Big[\int_0^z \Big(\exp\Big(\int_0^\eta \frac{p_j(\xi)-1}{\xi}\mathrm{d}\xi\Big)\Big)\mathrm{d}\eta\Big], \tag{3.15}$$

where the function $p_j(z)$ $(j = 1, 2, 3, 4)$ are given by (2.3) (see Figures 2-1, 2-2, 2-3 and 2-4). Hence we complete the proof of Theorem 3.6. $\square$

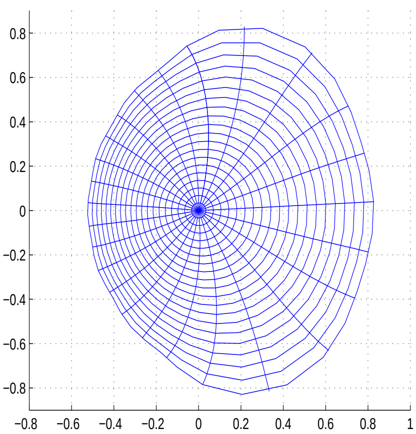

Figure 2-1 The image of $\mathbb{U}$ under $f_1(z)$ for $A = 0.1, B = -0.5, C = -0.5, D = 0.2, m = 0$

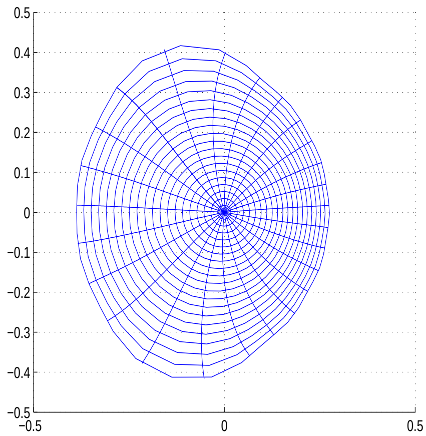

Figure 2-2 The image of $\mathbb{U}$ under $f_2(z)$ for $A = 0.7, B = 0.4, C = 0.1, D = 0.8, m = 0$

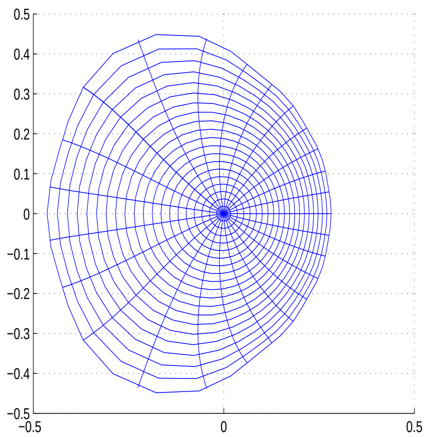

Figure 2-3 The image of $\mathbb{U}$ under $f_3(z)$ for $A = 0.7, B = 0.4, C = -0.1, D = 0.8, m = 0$

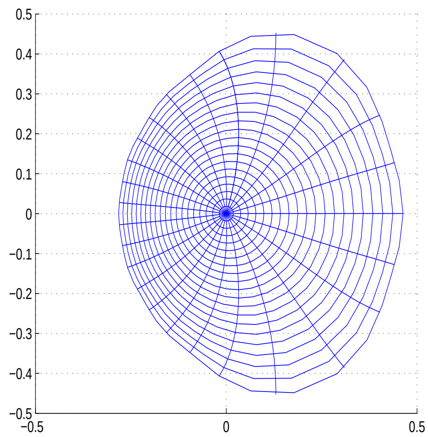

Figure 2-4 The image of $\mathbb{U}$ under $f_4(z)$ for $A = 0.9, B = 0.1, C = 0.1, D = 0.4, m = 0$

Remark 3.7 Setting $m = 0, A = 1-2\alpha$ $(0 \le \alpha \le 1)$, $B = -1$; $C = 1-2\beta$ $(1 < \alpha)$, $D = -1$, we obtain the improved result of Theorem 2 in the paper [16]. Also, taking $m = 0, \lambda = 0$, we have the improved result of Theorem 3.3 in the paper [12].

Using Theorem 3.6, we can easily get the following result.

Corollary 3.8 *Let $m \in \mathbb{N}_0$, $\lambda \geq 0$, $-1 < B < A \leq 1$, $-1 < C < D < 1$, and f^{-1} be the inverse function of f. If $f(z) \in S_m(A, B; C, D)$, and*

$$f^{-1}(\omega) = \omega + \sum_{n=2}^{\infty} b_n \omega^n, \quad |\omega| < r; r \geq \frac{1}{4},$$

then

$$|b_2| \leq \frac{|B_{1,j}|}{2^m(2\lambda+1)} \text{ and } |b_3| \leq \frac{|B_{1,j}|}{2 \cdot 3^m(3\lambda+1)} \max\{1, |\frac{B_{2,j}}{B_{1,j}} - \frac{4(3\lambda+1)(\frac{3}{4})^m - (2\lambda+1)}{(2\lambda+1)^2} B_{1,j}|\},$$

where $|B_{i,j}|$ $(i = 1, 2; j = 1, 2, 3, 4)$ are defined by (2.5).

Proof The relations (1.2) and $f^{-1}(\omega) = \omega + b_2\omega^2 + \cdots$ yield $b_2 = -a_2$ and $b_3 = 2a_2^2 - a_3$. Thus, in view of (3.1) and the identity $|b_2| = |a_2|$, the estimate for $|b_2|$ follows immediately. Furthermore, applying Theorem 3.6 with $\mu = 2$ gives the estimate for $|b_3|$. $\Box$

Finally, we will estimate some initial coefficients for the bi-univalent functions f.

Theorem 3.9 *Let $m \in \mathbb{N}_0$, $\lambda \geq 0$, $-1 < B < A \leq 1$, $-1 < C < D < 1$. If $f \in S\Sigma_{m,\lambda}(A, B; C, D)$, then*

$$|a_2| \leq \frac{|B_{1,j}|\sqrt{|B_{1,j}|}}{\sqrt{|B_{1,j}^2[2(3\lambda+1)3^m - (2\lambda+1)4^m] + 4^m(2\lambda+1)^2(B_{1,j} - B_{2,j})|}}$$

and

$$|a_3| \leq \frac{|B_{1,j}|\{2|4(3\lambda+1)3^m - (2\lambda+1)4^m| + 2(2\lambda+1)4^m\} + 8(3\lambda+1)3^m|B_{1,j} - B_{2,j}|}{4(3\lambda+1)3^m|4(3\lambda+1)3^m - 2(2\lambda+1)4^m|}, \quad (3.16)$$

where $|B_{i,j}|$ $(i = 1, 2; j = 1, 2, 3, 4)$ are defined by (2.5).

Proof If $f(z) \in S\Sigma_m(A, B; C, D)$, then $f(z) \in S_{m,\lambda}(A, B; C, D)$ and $g = f^{-1} \in S_{m,\lambda}(A, B; C, D)$. Hence

$$G(z) = \frac{D^{m+1}f(z)}{D^m f(z)} + \lambda \frac{z^2(D^m f(z))''}{D^m f(z)} \prec p_j(z), \quad z \in \mathbb{U};\ j = 1, 2, 3, 4$$

and

$$H(z) = \frac{D^{m+1}g(z)}{D^m g(z)} + \lambda \frac{z^2(D^m g(z))''}{D^m g(z)} \prec p_j(z), \quad z \in \mathbb{U};\ j = 1, 2, 3, 4,$$

where the function $p_j(z)$ is given by (2.3). Let

$$\varsigma(z) = \frac{1 + p_j^{-1}(G(z))}{1 - p_j^{-1}(G(z))} = 1 + \varsigma_1 z + \varsigma_2 z^2 + \cdots, \quad z \in \mathbb{U};\ j = 1, 2, 3, 4$$

and

$$\tau(z) = \frac{1 + p_j^{-1}(H(z))}{1 - p_j^{-1}(H(z))} = 1 + \tau_1 z + \tau_2 z^2 + \cdots, \quad z \in \mathbb{U};\ j = 1, 2, 3, 4.$$

Then ς and τ are analytic and have positive real part in $\mathbb{U}$, and satisfy the estimates

$$|\varsigma_n| \leq 2 \text{ and } |\tau_n| \leq 2, \quad n \in \mathbb{N}. \tag{3.17}$$

Therefore, we have

$$G(z) = p_j(\frac{\varsigma(z)-1}{\varsigma(z)+1}) \text{ and } H(z) = p_j(\frac{\tau(z)-1}{\tau(z)+1}), \quad z \in \mathbb{U};\ j = 1,2,3,4.$$

By comparing the coefficients, we get

$$(2\lambda+1)2^m a_2 = \frac{B_{1,j}\varsigma_1}{2}, \tag{3.18}$$

$$2(3\lambda+1)3^m a_3 - (2\lambda+1)2^{2m}a_2^2 = \frac{B_{1,j}\varsigma_2}{2} - \frac{\varsigma_1^2}{4}(B_{1,j} - B_{2,j}), \tag{3.19}$$

$$-(2\lambda+1)2^m a_2 = \frac{B_{1,j}\tau_1}{2} \tag{3.20}$$

and

$$-2(3\lambda+1)3^m a_3 + [4(3\lambda+1)3^m - (2\lambda+1)4^m]a_2^2 = \frac{B_{1,j}\tau_2}{2} - \frac{\tau_1^2}{4}(B_{1,j} - B_{2,j}), \tag{3.21}$$

where $B_{i,j}$ $(i = 1,2; j = 1,2,3,4)$ are given by (2.5). From (3.18) and (3.20), we obtain

$$\varsigma_1 = -\tau_1. \tag{3.22}$$

Also, from (3.19)–(3.22), we see that

$$a_2^2 = \frac{B_{1,j}^3(\varsigma_2 + \tau_2)}{4B_{1,j}^2[2(3\lambda+1)3^m - (2\lambda+1)4^m] + 4^{m+1}(2\lambda+1)^2(B_{1,j} - B_{2,j})}$$

and

$$a_3 = \frac{B_{1,j}\{[4(3\lambda+1)3^m - (2\lambda+1)4^m]\varsigma_2 + (2\lambda+1)4^m\tau_2\} - 2(3\lambda+1)3^m(B_{1,j} - B_{2,j})\varsigma_1^2}{4(3\lambda+1)3^m[4(3\lambda+1)3^m - 2(2\lambda+1)4^m]}.$$

These equations, together with (3.17), give the bounds on $|a_2|$ and $|a_3|$ as asserted in (3.16). This completes the proof of Theorem 3.9. □

Remark 3.10 Letting $m = 0$, $A = 1 - 2\alpha$ $(0 \leq \alpha \leq 1)$, $B = -1$; $C = 1 - 2\beta$ $(1 < \alpha)$, $D = -1$, we get the improved result of Theorem 3.6 in the paper [16].

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