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Weak Convergence to the Two-Parameter Volterra Multifractional Process in Besov Spaces

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Abstract In this paper, we prove that two-parameter Volterra multifractional process can be approximated in law in the topology of the anisotropic Besov spaces by the family of processes $\{B_n(s,t)\}_{n\in\mathbb{N}}$ defined by

$$B_n(s,t)=\int_0^s\int_0^t K_{\alpha(s)}(s,u)K_{\beta(t)}(t,v)\theta_n(u,v)\mathrm{d}u\mathrm{d}v,$$

where $\{\theta_n(u,v)\}_{n\in\mathbb{N}}$ is a family of processes, converging in law to a Brownian sheet as $n\to\infty$, based on the well known Donsker's theorem.

Keywords multifractional Brownian sheet; Poisson process; weak convergence

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1. Introduction

The fractional Brownian sheet $B^{H_1,H_2}_{s,t}$ of Hurst parameter $(H_1,H_2)\in(0,1)^2$ is a two-parameter centered Gaussian process, starting from $(0,0)$, with the covariance function given by

$$E[B^{H_1,H_2}_{s,t}B^{H_1,H_2}_{s',t'}]=\frac{1}{4}(s^{2H_1}+t^{2H_1}-|s-t|^{2H_1})\left((s')^{2H_2}+(t')^{2H_2}-|s'-t'|^{2H_2}\right). \tag{1}$$

Note that this process can be also defined by a Wiener integral with respect to the Brownian sheet W (see [1])

$$B^{H_1,H_2}_{s,t}=\int_0^s\int_0^t K_{H_1}(s,u)K_{H_2}(t,v)\mathrm{d}W(u,v), \tag{2}$$

where for $i=1,2$

$$K_{H_i}(s,u)=d_{H_i}(s-u)^{H_i-1/2}+d_{H_i}(1/2-H_i)\int_u^s(x-u)^{H_i-3/2}\big(1-(\frac{u}{x})^{1/2-H_i}\big)\mathrm{d}x,$$

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and d_{H_i} is a positive constant depending only on $H_i, i = 1, 2$. One property of fractional Brownian sheet is that the regularity may be prescribed by its Hurst parameters (H_1, H_2). However the main limitation of fractional Brownian sheet is that the Hölder regularity is constant along the paths. In order to consider phenomena which have more intricate structures with variations in irregularities, Mendy [2] extended fractional Brownian sheet to two-parameter Volterra multifractional process by replacing the constant exponent (H_1, H_2) by $(\alpha(s), \beta(t))$, where $\alpha : \mathbb{R}_+ \to (0, 1)$ and $\beta : \mathbb{R}_+ \to (0, 1)$. This time-varying exponent $(\alpha(s), \beta(t))$ describes the local variations of the irregularity of two-parameter Volterra multifractional process.

Let $\underline{M}$ and $\overline{M}$ be two real numbers satisfying $1/2 < \underline{M} < \overline{M} < 1$. Throughout the paper, we consider two functions $\alpha(\cdot) : \mathbb{R}_+ \to [\underline{M}, \overline{M}]$ and $\beta(\cdot) : \mathbb{R}_+ \to [\underline{M}, \overline{M}]$. Moreover, we suppose that $\alpha(\cdot)$ is a α_1-Hölder function and $\beta(\cdot)$ is a α_2-Hölder function with $0 < \alpha_1, \alpha_2 < 1$ where α_1 and α_2 are real numbers.

Definition 1.1 *The two-parameter Volterra multifractional process $B^{\alpha(s),\beta(t)}_{s,t}, (t, s) \in [0, T]$ is the centered Gaussian process given by the following Volterra-type representation*

$$B^{\alpha(s),\beta(t)}_{s,t} = \int_0^s \int_0^t K_{\alpha(s)}(s, u) K_{\beta(t)}(t, v) W(\mathrm{d}u, \mathrm{d}v), \tag{3}$$

where

$$K_{\alpha(s)}(s, u) = u^{1/2-\alpha(s)} \int_u^s y^{\alpha(s)-1/2}(y - u)^{\alpha(s)-3/2} \mathrm{d}y,$$
$$K_{\beta(t)}(t, v) = v^{1/2-\beta(t)} \int_v^t y^{\beta(t)-1/2}(y - v)^{\beta(t)-3/2} \mathrm{d}y,$$

and W is a Brownian sheet.

If $\alpha(\cdot) = H_1$ and $\beta(\cdot) = H_2$ are two constants, $B^{H_1,H_2}$is a fractional Brownian sheet up to a multiplicative constant. The two-parameter Volterra multifractional process has properties analogous to those of fractional Brownian sheet, such as self-similarity and Hölder paths. Therefore, it seems interesting to study the two-parameter Volterra multifractional process.

On the other hand, weak convergence to fractional Brownian motion, fractional Brownian sheet and related processes have been studied extensively since the works of Taqqu [3] and Delgado and Jolis [4]. The classical framework for this kind of limit theorems is in $C([0, 1])$, the Banach space of continuous processes on $[0, 1]$, and the Skorohod space $D([0, 1])$, for discontinuous ones. For example, Bardina et al. [1,5–7] gave a weak approximation of the Brownian sheet in d-parameter case ($d \geq 2$). Sottinen [8], Nieminen [9], Li and Dai [10] investigated weak approximation of fractional Brownian motion. Bardina et al [11], Boufoussi and Hajji [12], Mellall and Ouknine [13] considered weak approximation of stochastic differential equation driven by fractional noise. More works for the fields can be found in Shen et al. [14], Bardina et al. [15], Dai and Li [16], Wang et al. [17] and the references therein.

In recent years, many results of this kind of limit theorems have been obtained with stronger topologies. For example, Tudor [18] showed that fractional Brownian sheet can be approximated in law by Kac-Stroock type process in a class of Besov spaces. Dai [19] shown that multifractional

Brownian motion of Riemann-Liouville type can be approximated in law in Besov spaces. More works for the fields can be found in Boufoussi and Guerbaz [20], Boufoussi and Lakhel [21] and the references therein.

From representation (3), a natural way to approximate in law two-parameter Volterra multifractional process is to define the sequence of processes $B_n(s,t)$ by

$$B_n(s,t) = \int_0^s \int_0^t K_{\alpha(s)}(s,u) K_{\beta(t)}(t,v) \theta_n(u,v) \mathrm{d}u \mathrm{d}v, \tag{4}$$

where $\{\theta_n(u,v)\}_{n\in\mathbb{N}}$ is a "weak approximation of a Brownian sheet", i.e., $\{\theta_n(u,v)\}_{n\in\mathbb{N}}$ is a family of processes, defined on some probability space, such that for

$$X_n(s,t) = \int_0^s \int_0^t \theta_n(u,v) \mathrm{d}u \mathrm{d}v,$$

$\{X_n(s,t), (s,t) \in [0,1]^2\}_{n\in\mathbb{N}}$ converges in law to the standard Brownian sheet in plane as $n \to \infty$.

In this paper we consider $\{\theta_n(u,v)\}$ which comes from the well-known Donsker's theorem. It states that the Brownian sheet can be approximated in law by a sequence of random variables. More precisely, let $\{Z_k, k \in \mathbb{N}^2\}$ be an independent family of identically distributed and centered random variables, with $E(Z_k^2) = 1$ for all $k \in \mathbb{N}^2$, and such that $E(|Z_k|^m) < +\infty$ for all $k \in \mathbb{N}^2$ and some sufficiently large $m \in \mathbb{N}$. For any $n \in \mathbb{N}$, we define the kernels

$$\theta_n(u,v) = n \sum_{k=(k^1,k^2)\in\mathbb{N}^2} Z_k \cdot 1_{[k^1-1,k^1)\times[k^2-1,k^2)}(un,vn), \quad (u,v) \in [0,1]^2, \tag{5}$$

then

$$n \int_0^s \int_0^t \theta_n(u,v) \mathrm{d}u \mathrm{d}v \Rightarrow \text{ Brownian Sheet.}$$

where the symbol " $\Rightarrow$ " denotes convergence in law and $1_A(\cdot)$ denotes the indicator function of set A.

The rest of this paper is organized as follows. In Section 2, we begin by making some notations and by recalling some basic preliminaries which will be needed later. In Section 3, we will prove weak limit theorems for two-parameter Volterra multifractional process given by Eq. (3) in Besov spaces. Most of the estimates of this paper contain unspecified constants. An unspecified positive and finite constant will be denoted by C, which may not be the same in each occurrence. Sometimes we shall emphasize the dependence of these constants upon parameters.

2. Preliminaries

In this section, we briefly recall some basic elements in two-dimensional Besov spaces. We refer to Kamont [22], for some complete descriptions of two-dimensional Besov spaces.

Suppose $T^2 = [0,1]^2$ and $D = \{1,2\}$. For a function $f : T^2 \to R, i \in D, h \in R$, $e_1 = (1,0)$ and $e_2 = (0,1)$ unit vectors, let

$$\Delta_{h,i} f(x) = \begin{cases} f(x+he_i) - f(x), & \text{if } x, x+he_i \in T^2 \\ 0, & \text{otherwise} \end{cases}$$

denote the progressive difference of f in the direction e_i.

Let

$$\Delta_{h_1,h_2}f = \Delta_{h,1} \circ \Delta_{h,2}, \text{ for } (h_1, h_2) \in \mathbb{R}^2.$$

If $A = \{i\}$ is a subset of D with one element, then we put $\Delta_{(h_1,h_2),A}f = \Delta_{h_i,i}f$ and $\Delta_{(h_1,h_2),A}f = f$ if $A = \varnothing$.

Now, for $f \in L^p(T^2)$ if $1 \le p < \infty$ or $f \in C(T^2)$ (the space of continuous functions on T^2) if $p = \infty$, we define its L^p-modulus of continuity by

$$\omega_{p,A}(f, (s_1, s_2)) = \sup_{0 \le h_1 \le s_1, 0 \le h_2 \le s_2} ||\Delta_{(h_1,h_2),A}f||_p \text{ for } (s_1, s_2) \in \mathbb{R}^2_+.$$

Let b be real and $\bar{a} = (a_1, a_2), a_1, a_2 > 0$. We consider the real valued application on T^2 given by

$$\omega_{a,b}((s_1, s_2), A) = \prod_{i \in A} (s_i)^{a_i} \Big(1 + \sum_{i \in A} \log \frac{1}{s_i}\Big)^b$$

for any $A \subset D$ with $\omega_{\bar{a},b}((s_1, s_2), \varnothing) = 1$.

Definition 2.1 *Suppose $\bar{a} = (a_1, a_2), a_1, a_2 > 0$, $b \in \mathbb{R}$ and $1 \le p \le \infty$. The anisotropic Besov space $\mathrm{Lip}_p(\bar{a}, b)$ is defined by*

$$\mathrm{Lip}_p(\bar{a}, b) = \Big\{f \in L^p(T^2); \sum_{A \subset D} \sup_{s_1, s_2 > 0} \frac{\omega_{p,A}(f, (s_1, s_2))}{\omega_{\bar{a},b}((s_1, s_2), A)} < \infty\Big\}$$

and this space is endowed with the norm

$$\|f\|_p^{\bar{a},b} = \sum_{A \subset D} \sup_{s_1, s_2 > 0} \frac{\omega_{p,A}(f, (s_1, s_2))}{\omega_{\bar{a},b}((s_1, s_2), A)}.$$

In this way $\mathrm{Lip}_p(\bar{a}, b)$ becomes a non-separable Banach space.

We also introduce the subspace $\mathrm{Lip}^*_p(\bar{a}, b)$ of $\mathrm{Lip}_p(\bar{a}, b)$ by

$$\mathrm{Lip}^*_p(\bar{a}, b) = \big\{f \in L^p(T^2); \forall \emptyset \ne A \subset E, \lim_{\delta_A(s_1,s_2)} \frac{\omega_{p,A}(f, (s_1, s_2))}{\omega_{\bar{a},b}((s_1, s_2), A)} = 0\big\}$$

where $\delta_A(s_1, s_2) = \min\{s_i, i \in A\}$. Similar to [2, Lemma 2.1], one can get

Lemma 2.2 *(i) There exists a function $\Phi \in^2 ((0, 1]^2, \mathbb{R}^2_+)$ such that for every $(u, v) \in (0, 1]^2$*

$$\sup_{(\lambda,\lambda') \in [a,b]^2, (t,s) \in (0,1]^2} \Big|\frac{\partial^2[K_\lambda(s, u)K_{\lambda'}(t, v)]}{\partial\lambda\partial\lambda'}\Big| \le \Phi(u, v)$$

(ii) For every $(s, t) \in [0, 1]^2$ and every $(\lambda_i, \lambda'_i) \in [\underline{M}, \overline{M}]^2$, $i = 1, 2$, there exists a constant $C > 0$ satisfying

$$\int_0^1 \int_0^1 \big[K_{\lambda'_1}(s, u)K_{\lambda'_2}(t, v) - K_{\lambda_1}(s, u)K_{\lambda_2}(t, v)\big]^2 \mathrm{d}u\mathrm{d}v \le C|\lambda'_1 - \lambda_1|^2|\lambda'_2 - \lambda_2|^2.$$

(iii) For every $(s, t) \in [0, 1]^2$, $(s + h, t + k) \in [0, 1]^2$ and every $(\lambda_1, \lambda_2) \in [\underline{M}, \overline{M}]^2$, there exists a constant $C > 0$ satifying

$$\int_0^1 \int_0^1 [K_{\lambda_1}(s + h, u)K_{\lambda_2}(t + k, v) - K_{\lambda_1}(s, u)K_{\lambda_2}(t, v)]^2 \mathrm{d}u\mathrm{d}v \le C|h|^{2\lambda_1}|k|^{2\lambda_2}.$$

Furthermore, one can follow the similar proof in [2, Proposition 2.5] to conclude the following

Lemma 2.3 *For any* $2 < p < \infty$ *and* $t, s \in [0,1]$, *there holds*

$$P\big(B_{s,t}^{\alpha(s),\beta(t)} \in \mathrm{Lip}_p((\alpha_1,\alpha_2),0)\big) = 1 \text{ and } P\big(B_{s,t}^{\alpha(s),\beta(t)} \in \mathrm{Lip}_p^*((\alpha_1,\alpha_2),0)\big) = 0,$$

where $(\alpha_1,\alpha_2) \in (0,\underline{M})^2$.

Remark 2.4 The similar results on fractional Brownian motion and fractional Brownian sheet have been proved in in Kamont [22] and Tudor [18], respectively.

3. Main result and its proof

The purpose of this section is to prove that the laws of $\{B_n(s,t),(s,t) \in [0,1]^2\}$ given by (4) converge weakly to the law of two-parameter Volterra multifractional process $B_{s,t}^{\alpha(s),\beta(t)}$ given by (3) in a class of Besov spaces as $n \to \infty$. The main result is stated as follows.

Theorem 3.1 *Let* $(\alpha_1,\alpha_2) \in (0,\underline{M}]^2$. *For any* $b > 0$ *and* $p > \frac{1}{\alpha_1} \vee \frac{1}{\alpha_2} \vee \frac{2}{b}$, *the law of processes* $\{B_n(s,t),(s,t) \in [0,1]^2\}$ *given by (4) with donsker kernels, converges weakly, as* $n \to \infty$, *in the space* $\mathrm{Lip}_p^*(\alpha_1,\alpha_2)$, *to the multifractional Liouville process* $B_{s,t}^{\alpha(s),\beta(t)}$ *given by (3).*

In order to prove Theorem 3.1, we need to check the following two points:

(i) We firstly need to check that the family of $\{B_n(s,t),(s,t) \in [0,1]^2\}$ is tight in the space $\mathrm{Lip}_p^*(a,b)$ with $p > \frac{1}{\alpha_1} \vee \frac{1}{\alpha_2} \vee \frac{2}{b}$ for any $b > 0$.

(ii) We secondly need to prove the convergence of the finite dimensional distribution of $\{B_n(s,t),(s,t) \in [0,1]^2\}$ to those of $B_{s,t}^{\alpha(s),\beta(t)}$ as n tends to infinity.

The first point is a consequence of the following lemma.

Lemma 3.2 *Let* $(\alpha_1,\alpha_2) \in (0,\underline{M})^2$. *Then, for any* $(s_1,t_1),(s_2,t_2) \in [0,1]^2$ *such that* $s_1 \le s_2, t_1 \le t_2$ *and every even* $m \in \mathbb{N}$

$$\sup_m E\,|\Delta_{s_1,t_1} B_n(s_2,t_2)|^m \le C_m |s_1 - s_2|^{m\alpha_1} \cdot |t_1 - t_2|^{m\alpha_2}. \tag{6}$$

Proof Here we use the similar method shown in Bardina and Florit [7] to prove this lemma. For $(s_1,t_1),(s_2,t_2) \in [0,1]^2$ such that $s_1 \le s_2, t_1 \le t_2$, we have

$$\begin{aligned}
&\Delta_{s_1,t_1} B_n(s_2,t_2)\\
&= \int_0^1 \int_0^1 \big(K_{\alpha(s_1)}(s_1,u) - K_{\alpha(s_2)}(s_2,u)\big)\big(K_{\beta(t_1)}(t_1,v) - K_{\beta(t_2)}(t_2,v)\big)\,\theta_n(u,v)\mathrm{d}u\mathrm{d}v.
\end{aligned}$$

Following the ideas in Tudor [18], we know that the Besov norms are increasing in m, it suffices to prove that the result is true for m even. Then for every even $m \in \mathbb{N}$, we obtain

$$\begin{aligned}
&E\,|\Delta_{s_1,t_1} B_n(s_2,t_2)|^m\\
&= E\Big|\int_0^1 \int_0^1 \big(K_{\alpha(s_1)}(s_1,u) - K_{\alpha(s_2)}(s_2,u)\big)\big(K_{\beta(t_1)}(t_1,v) - K_{\beta(t_2)}(t_2,v)\big)\,\theta_n(u,v)\mathrm{d}u\mathrm{d}v\Big|^m\\
&= \int_{[0,1]^{2m}} \prod_{i=1}^m \big(K_{\alpha(s_1)}(s_1,u_i) - K_{\alpha(s_2)}(s_2,u_i)\big)\big(K_{\beta(t_1)}(t_1,v_i) - K_{\beta(t_2)}(t_2,v_i)\big)\cdot
\end{aligned}$$

$$E\Big[\prod_{i=1}^{m}\theta_n(u_i,v_i)\Big]\mathrm{d}u_1\mathrm{d}v_1\cdots\mathrm{d}u_m\mathrm{d}v_m. \tag{7}$$

Note that

$$\begin{aligned}E\Big[\prod_{i=1}^{m}\theta_n(u_i,v_i)\Big]&=n^mE\Big[\prod_{i=1}^{m}\Big(\sum_{k=(k^1,k^2)\in\mathbb{N}^2}Z_k1_{[k^1-1,k^1)}(u_in)1_{[k^2-1,k^2)}(v_in)\Big)\Big]\\&=n^m\sum_{k_1,\ldots,k_m\in\mathbb{N}^2}E(Z_{k_1}\cdots Z_{k_m})\prod_{i=1}^{m}\big(1_{[k^1-1,k^1)}(u_in)1_{[k^2-1,k^2)}(v_in)\big).\end{aligned} \tag{8}$$

Since the sequences $Z_{k_1}\cdots Z_{k_m}$ are I.I.D., one can get

$$E(Z_{k_1}\cdots Z_{k_m})=0,$$

if for some $i\in\{1,\ldots,m\}$, we have that $k_i\neq k_j$ for all $j\in\{1,\ldots,m\}\ /\{i\}$; that is, for some variable Z_{k_i} appears only once in the product $Z_{k_1}\cdots Z_{k_m}$.

On the other hand, since $E(|Z_k|^m)<\infty$ for all $k\in\mathbb{N}^2$, then $E(Z_{k_1}\cdots Z_{k_m})$ is bounded for all $k_1,\ldots,k_m\in\mathbb{N}^2$. Hence, we have

$$E\Big[\prod_{i=1}^{m}\theta_n(u_i,v_i)\Big]\le C_mn^m\sum_{(k_1,\ldots,k_m)\in\mathbb{B}^m}\big(1_{[k^1-1,k^1)}(u_in)1_{[k^2-1,k^2)}(v_in)\big),$$

where

$$\begin{aligned}B_m=\{&(k_1,\ldots,k_m)\in\mathbb{N}^{2m};\ \text{for all } j\in\{1,\ldots,m\},\\&\text{there exists } i\in\{1,\ldots,m\}/\{j\} \text{ such that } k_i=k_j\}.\end{aligned}$$

In addition,

$$\sum_{(k_1,\ldots,k_m)\in\mathbb{B}^m}\prod_{i=1}^{m}\big(1_{[k^1-1,k^1)}(u_in)1_{[k^2-1,k^2)}(v_in)\big)\le 1_{E^m}(u_1,\ldots,u_m;v_1,\ldots,v_m),$$

where E^m denotes the set of $(u_1,\ldots,u_m;v_1,\ldots,v_m)\in[0,1]^{2m}$ satisfying the following property: for all $j\in\{1,\ldots,m\}$, there exists $i\in\{1,\ldots,m\}/\{j\}$ such that $|u_i-u_j|<\frac{1}{n}$ and $|v_i-v_j|<\frac{1}{n}$ and, moreover, if there is some $l\neq i,j$ verifying $|u_j-u_l|<\frac{1}{n}$ and $|v_j-v_l|<\frac{1}{n}$, then $|u_i-u_l|<\frac{1}{n}$ and $|v_i-v_l|<\frac{1}{n}$.

Next, we should note that $1_{E^m}(u_1,\ldots,u_m;v_1,\ldots,v_m)$ can be bounded by a finite sum of products of indicators, where in each product of indicators appear all the m variables $u_1,\ldots,u_m$ and all the m variables $v_1,\ldots,v_m$, but each indicator concerns only two or three of them, and each variable appears only in one of the indicators of each product. Then Eq. (7) can be bounded by the following two kinds of terms:

(i) For some $j,i\in\{1,\ldots,m\}$ such that $j\neq i$, one obtains

$$\begin{aligned}C_mn^2\int_{[0,1]^4}&\big(K_{\alpha(s_1)}(s_1,u_i)-K_{\alpha(s_2)}(s_2,u_i)\big)\big(K_{\beta(t_1)}(t_1,v_i)-K_{\beta(t_2)}(t_2,v_i)\big)\cdot\\&\big(K_{\alpha(s_1)}(s_1,u_j)-K_{\alpha(s_2)}(s_2,u_j)\big)\big(K_{\beta(t_1)}(t_1,v_j)-K_{\beta(t_2)}(t_2,v_j)\big)\cdot\\&1_{[0,\frac{1}{n})}(|u_i-u_j|)1_{[0,\frac{1}{n})}(|v_i-v_j|)\mathrm{d}u_i\mathrm{d}u_j\mathrm{d}v_i\mathrm{d}v_j.\end{aligned} \tag{9}$$

(ii) For some $i \neq j \neq l$, $i,j,l \in \{1,\ldots,m\}$, one gets

$$
\begin{aligned}
&C_m n^3 \int_{[0,1]^6} \left(K_{\alpha(s_1)}(s_1,u_i) - K_{\alpha(s_2)}(s_2,u_i)\right)\left(K_{\beta(t_1)}(t_1,v_i) - K_{\beta(t_2)}(t_2,v_i)\right)\cdot \\
&\quad \left(K_{\alpha(s_1)}(s_1,u_j) - K_{\alpha(s_2)}(s_2,u_j)\right)\left(K_{\beta(t_1)}(t_1,v_j) - K_{\beta(t_2)}(t_2,v_j)\right)\cdot \\
&\quad \left(K_{\alpha(s_1)}(s_1,u_l) - K_{\alpha(s_2)}(s_2,u_l)\right)\left(K_{\beta(t_1)}(t_1,v_l) - K_{\beta(t_2)}(t_2,v_l)\right)\cdot \\
&\quad 1_{[0,\frac{1}{n})}(|u_i-u_j|)1_{[0,\frac{1}{n})}(|v_i-v_j|)1_{[0,\frac{1}{n})}(|u_i-u_l|)1_{[0,\frac{1}{n})}(|v_i-v_l|)\cdot \\
&\quad 1_{[0,\frac{1}{n})}(|u_j-u_l|)1_{[0,\frac{1}{n})}(|v_j-v_l|)\mathrm{d}u_i\mathrm{d}u_j\mathrm{d}v_i\mathrm{d}v_j\mathrm{d}u_l\mathrm{d}v_l.
\end{aligned}
\tag{10}
$$

Then, in order to conclude the proof, it suffices to prove that

$$
\begin{aligned}
A =&C_m n^2 \int_{[0,1]^4} \left(K_{\alpha(s_1)}(s_1,u_i) - K_{\alpha(s_2)}(s_2,u_i)\right)\left(K_{\beta(t_1)}(t_1,v_i) - K_{\beta(t_2)}(t_2,v_i)\right)\cdot \\
&\left(K_{\alpha(s_1)}(s_1,u_j) - K_{\alpha(s_2)}(s_2,u_j)\right)\left(K_{\beta(t_1)}(t_1,v_j) - K_{\beta(t_2)}(t_2,v_j)\right)\cdot \\
&1_{[0,\frac{1}{n})}(|u_i-u_j|)1_{[0,\frac{1}{n})}(|v_i-v_j|)\mathrm{d}u_i\mathrm{d}u_j\mathrm{d}v_i\mathrm{d}v_j \\
\le& C_m(s_1-s_2)^{2\alpha_1}(t_1-t_2)^{2\alpha_2},
\end{aligned}
\tag{11}
$$

and

$$
\begin{aligned}
B =&C_m n^3 \int_{[0,1]^6} \left(K_{\alpha(s_1)}(s_1,u_i) - K_{\alpha(s_2)}(s_2,u_i)\right)\left(K_{\beta(t_1)}(t_1,v_i) - K_{\beta(t_2)}(t_2,v_i)\right)\cdot \\
&\left(K_{\alpha(s_1)}(s_1,u_j) - K_{\alpha(s_2)}(s_2,u_j)\right)\left(K_{\beta(t_1)}(t_1,v_j) - K_{\beta(t_2)}(t_2,v_j)\right)\cdot \\
&\left(K_{\alpha(s_1)}(s_1,u_l) - K_{\alpha(s_2)}(s_2,u_l)\right)\left(K_{\beta(t_1)}(t_1,v_l) - K_{\beta(t_2)}(t_2,v_l)\right)\cdot \\
&1_{[0,\frac{1}{n})}(|u_i-u_j|)1_{[0,\frac{1}{n})}(|v_i-v_j|)1_{[0,\frac{1}{n})}(|u_i-u_l|)1_{[0,\frac{1}{n})}(|v_i-v_l|)\cdot \\
&1_{[0,\frac{1}{n})}(|u_j-u_l|)1_{[0,\frac{1}{n})}(|v_j-v_l|)\mathrm{d}u_i\mathrm{d}u_j\mathrm{d}v_i\mathrm{d}v_j\mathrm{d}u_l\mathrm{d}v_l \\
\le& C(s_1-s_2)^{3\alpha_1}(t_1-t_2)^{3\alpha_2}.
\end{aligned}
\tag{12}
$$

Now we show Eq. (11) holds. By the fact that $2ab \le (a^2+b^2)$ for any $a,b \in R$,

$$
\begin{aligned}
A \le& n^2 \int_{[0,1]^4} \Big(\left(K_{\alpha(s_1)}(s_1,u_i) - K_{\alpha(s_2)}(s_2,u_i)\right)\left(K_{\beta(t_1)}(t_1,v_i) - K_{\beta(t_2)}(t_2,v_i)\right) \Big)^2\cdot \\
&1_{[0,\frac{1}{n})}(|u_i-u_j|)1_{[0,\frac{1}{n})}(|v_i-v_j|)\mathrm{d}u_i\mathrm{d}u_j\mathrm{d}v_i\mathrm{d}v_j + \\
&n^2 \int_{[0,1]^4} \Big(\left(K_{\alpha(s_1)}(s_1,u_j) - K_{\alpha(s_2)}(s_2,u_k)\right)\left(K_{\beta(t_1)}(t_1,v_j) - K_{\beta(t_2)}(t_2,v_j)\right) \Big)^2\cdot \\
&1_{[0,\frac{1}{n})}(|u_i-u_j|)1_{[0,\frac{1}{n})}(|v_i-v_j|)\mathrm{d}u_i\mathrm{d}u_j\mathrm{d}v_i\mathrm{d}v_j \\
&:\equiv A_1 + A_2.
\end{aligned}
\tag{13}
$$

The two terms A_1 and A_2 can be done in the similar way. Then we only need deal with A_1. In fact, following the similar lines in the proof of [2, Proposition 2.5], we obtain

$$
\begin{aligned}
A_1 \le& \int_{[0,1]^2} \left(\left(K_{\alpha(s_1)}(s_1,u_i) - K_{\alpha(s_2)}(s_2,u_i)\right)\left(K_{\beta(t_1)}(t_1,v_i) - K_{\beta(t_2)}(t_2,v_i)\right)\right)^2 \mathrm{d}u_i\mathrm{d}v_i \\
=& \int_0^1 \left(K_{\alpha(s_1)}(s_1,u_i) - K_{\alpha(s_2)}(s_2,u_i)\right)^2 \mathrm{d}u_i \int_0^1 \left(K_{\beta(t_1)}(t_1,v_i) - K_{\beta(t_2)}(t_2,v_i)\right)^2 \mathrm{d}v_i \\
\le& C(s_1-s_2)^{2\alpha_1}(t_1-t_2)^{2\alpha_2}.
\end{aligned}
$$

We now proceed to prove the inequality (12). Actually, we have

$$\begin{aligned}B =&n^3\int_{[0,1]^4}\Big[\big(\big(K_{\alpha(s_1)}(s_1,u_i)-K_{\alpha(s_2)}(s_2,u_i)\big)\big(K_{\beta(t_1)}(t_1,v_i)-K_{\beta(t_2)}(t_2,v_i)\big)\big)\cdot\\&1_{[0,\frac{1}{n})}(|u_i-u_j|)1_{[0,\frac{1}{n})}(|v_i-v_j|)\Big]\cdot\\&\Big[\int_{[0,1]^2}\big(\big(K_{\alpha(s_1)}(s_1,u_j)-K_{\alpha(s_2)}(s_2,u_j)\big)\big(K_{\beta(t_1)}(t_1,v_j)-K_{\beta(t_2)}(t_2,v_j)\big)\big)\cdot\\&\big(\big(K_{\alpha(s_1)}(s_1,u_l)-K_{\alpha(s_2)}(s_2,u_l)\big)\big(K_{\beta(t_1)}(t_1,v_l)-K_{\beta(t_2)}(t_2,v_l)\big)\big)\cdot\\&1_{[0,\frac{1}{n})}(|u_i-u_l|)1_{[0,\frac{1}{n})}(|v_i-v_l|)1_{[0,\frac{1}{n})}(|u_j-u_l|)1_{[0,\frac{1}{n})}(|v_j-v_l|)\mathrm{d}u_l\mathrm{d}v_l\Big]\mathrm{d}u_i\mathrm{d}u_j\mathrm{d}v_i\mathrm{d}v_j.\end{aligned}\tag{14}$$

Then using the Hölder inequality, Eq. (14) can be bounded by $n^3B_1^{1/2}\cdot B_2^{1/2}$, where

$$\begin{aligned}B_1=&\int_{[0,1]^4}\big(\big(K_{\alpha(s_1)}(s_1,u_i)-K_{\alpha(s_2)}(s_2,u_i)\big)\big(K_{\beta(t_1)}(t_1,v_i)-K_{\beta(t_2)}(t_2,v_i)\big)\big)^2\cdot\\&1_{[0,\frac{1}{n})}(|u_i-u_j|)1_{[0,\frac{1}{n})}(|v_i-v_j|)\mathrm{d}u_i\mathrm{d}u_j\mathrm{d}v_i\mathrm{d}v_j.\end{aligned}$$

and

$$\begin{aligned}B_2=&\int_{[0,1]^4}\Big[\int_{[0,1]^2}\big(\big(K_{\alpha(s_1)}(s_1,u_j)-K_{\alpha(s_2)}(s_2,u_j)\big)\big(K_{\beta(t_1)}(t_1,v_j)-K_{\beta(t_2)}(t_2,v_j)\big)\big)\cdot\\&\big(\big(K_{\alpha(s_1)}(s_1,u_l)-K_{\alpha(s_2)}(s_2,u_l)\big)\big(K_{\beta(t_1)}(t_1,v_l)-K_{\beta(t_2)}(t_2,v_l)\big)\big)\cdot\\&1_{[0,\frac{1}{n})}(|u_i-u_l|)1_{[0,\frac{1}{n})}(|v_i-v_l|)1_{[0,\frac{1}{n})}(|u_j-u_l|)1_{[0,\frac{1}{n})}(|v_j-v_l|)\mathrm{d}u_l\mathrm{d}v_l\Big]^2\mathrm{d}u_i\mathrm{d}u_j\mathrm{d}v_i\mathrm{d}v_j.\end{aligned}$$

Using the similar method as the proof of inequality (11), we can get

$$B_1\le C\frac{1}{n^2}(s_1-s_2)^{2\alpha_1}(t_1-t_2)^{2\alpha_2}.$$

Now let us estimate the second term B_2. In fact, we have

$$\begin{aligned}B_2\le&C\int_{[0,1]^6}\big(\big(K_{\alpha(s_1)}(s_1,u_j)-K_{\alpha(s_2)}(s_2,u_j)\big)\big(K_{\beta(t_1)}(t_1,v_j)-K_{\beta(t_2)}(t_2,v_j)\big)\big)^2\cdot\\&\big(\big(K_{\alpha(s_1)}(s_1,u_l)-K_{\alpha(s_2)}(s_2,u_l)\big)\big(K_{\beta(t_1)}(t_1,v_l)-K_{\beta(t_2)}(t_2,v_l)\big)\big)^2\cdot\\&1_{[0,\frac{1}{n})}(|u_i-u_l|)1_{[0,\frac{1}{n})}(|v_i-v_l|)\cdot\\&1_{[0,\frac{1}{n})}(|u_j-u_l|)1_{[0,\frac{1}{n})}(|v_j-v_l|)\mathrm{d}u_i\mathrm{d}v_i\mathrm{d}u_j\mathrm{d}v_j\mathrm{d}u_l\mathrm{d}v_l\\\le&C(B_{2,1}+B_{2,2}),\end{aligned}$$

where

$$\begin{aligned}B_{2,1}=&\int_{[0,1]^6}\big(\big(K_{\alpha(s_1)}(s_1,u_j)-K_{\alpha(s_2)}(s_2,u_j)\big)\big(K_{\beta(t_1)}(t_1,v_j)-K_{\beta(t_2)}(t_2,v_j)\big)\big)^2\cdot\\&\big(\big(K_{\alpha(s_1)}(s_1,u_l)-K_{\alpha(s_2)}(s_2,u_l)\big)\big(K_{\beta(t_1)}(t_1,v_l)-K_{\beta(t_2)}(t_2,v_l)\big)\big)^2\cdot\\&1_{[0,\frac{1}{n})}(|u_i-u_l|)1_{[0,\frac{1}{n})}(|v_i-v_l|)\mathrm{d}u_i\mathrm{d}v_i\mathrm{d}u_j\mathrm{d}v_j\mathrm{d}u_l\mathrm{d}v_l\\\le&\frac{1}{n^2}\int_{[0,1]^2}\big(K_{\alpha(s_1)}(s_1,u_j)K_{\beta(t_1)}(t_1,v_j)-K_{\alpha(s_2)}(s_2,u_j)K_{\beta(t_2)}(t_2,v_j)\big)^2\mathrm{d}u_j\mathrm{d}v_j\cdot\\&\int_{[0,1]^2}\big(K_{\alpha(s_1)}(s_1,u_l)K_{\beta(t_1)}(t_1,v_l)-K_{\alpha(s_2)}(s_2,u_l)K_{\beta(t_2)}(t_2,v_l)\big)^2\mathrm{d}u_{l_1}\mathrm{d}v_{l_1}\end{aligned}$$

$$\leq \frac{1}{n^2}(s_1-s_2)^{4\alpha_1}(t_1-t_2)^{4\alpha_2}.$$

and

$$B_{2,2}=\int_{[0,1]^6}\left(\left(K_{\alpha(s_1)}(s_1,u_j)-K_{\alpha(s_2)}(s_2,u_j)\right)\left(K_{\beta(t_1)}(t_1,v_j)-K_{\beta(t_2)}(t_2,v_j)\right)\right)^2\cdot \\ \left(\left(K_{\alpha(s_1)}(s_1,u_l)-K_{\alpha(s_2)}(s_2,u_l)\right)\left(K_{\beta(t_1)}(t_1,v_l)-K_{\beta(t_2)}(t_2,v_l)\right)\right)^2\cdot \\ 1_{[0,\frac{1}{n})}(|u_j-u_l|)1_{[0,\frac{1}{n})}(|v_j-v_l|)\mathrm{d}u_i\mathrm{d}v_i\mathrm{d}u_j\mathrm{d}v_j\mathrm{d}u_l\mathrm{d}v_l.$$

We also can get

$$B_{2,2}\leq \frac{1}{n^2}(s_1-s_2)^{4\alpha_1}(t_1-t_2)^{4\alpha_2},$$

with the similar arguments. This completes the proof of this lemma. □

Then, with the tightness criterion in Bosev spaces given by Boufoussi and Dozzi [23] and Lemma 3.2, we obtain the tightness of the sequences $\{B_n(s_1,t_1)\}_{n\in\mathbb{N}}$ in a class of Besov spaces.

Lemma 3.3 *Let $(\alpha_1,\alpha_2)\in(0,\underline{M})$. For any $b>0$ and $p>\frac{1}{\alpha_1}\vee\frac{1}{\alpha_2}\vee\frac{2}{b}$, then the sequence $\{B_n(s,t),(s,t)\in[0,1]^2\}$ given by Eq. (4) with Donsker kernels is tight in the separable Banach space $\mathrm{Lip}_p^*((\alpha_1,\alpha_2),b)$.*

Secondly, let us prove the second point in order to complete the proof of Theorem 3.1. In fact, we claim that, for any $a_1,\dots,a_m\in R$ and $(s_1,t_1),\dots,(s_m,t_m)\in[0,1]^2$, the law of linear combination

$$\sum_{j=1}^m a_jB_n(s_j,t_j),$$

converges weakly to the law of a random variable defined by

$$\sum_{j=1}^m a_jB_{s_j,t_j}^{\alpha(s_j),\beta(t_j)},$$

as n tends to infinity. This will be done by proving the convergence of the corresponding characteristic functions, i.e.,

$$E\Big[\exp\Big\{i\xi\sum_{j=1}^m a_jB_n(s_j,t_j)\Big\}\Big]\to E\Big[\exp\Big\{i\xi\sum_{j=1}^m a_jB_{s_j,t_j}^{\alpha(s_j),\beta(t_j)}\Big\}\Big], \tag{15}$$

as n tends to infinity. Since for any fixed $(s,t)\in[0,1]^2$, $K_{\alpha(s)}(s,u)K_{\beta(t)}(t,v)\in L^2([0,1]^2)$, then for any $(s_j,t_j)\in[0,1]^2$, $j\in\{1,2,\dots,m\}$, there exits a sequence $\left(K_{\alpha(s_j)}(s_j,u)K_{\beta(t_j)}(t_j,v)\right)_k$ of simple functions such that $\left(K_{\alpha(s_j)}(s_j,u)K_{\beta(t_j)}(t_j,v)\right)_k$ converges to $\left(K_{\alpha(s_j)}(s_j,u)K_{\beta(t_j)}(t_j,v)\right)$ in $L^2([0,1]^2)$ as $k\to\infty$.

To simplify notation, we define

$$B_n^{j,k}=\int_0^1\int_0^1\left(K_{\alpha(s_j)}(s_j,u)K_{\beta(t_j)}(t_j,v)\right)_k\theta_n(u,v)\mathrm{d}u\mathrm{d}v,$$

and

$$B^{j,k}=\int_0^1\int_0^1\left(K_{\alpha(s_j)}(s_j,u)K_{\beta(t_j)}(t_j,v)\right)_k\mathrm{d}W(u,v).$$

Then

$$
\begin{aligned}
&\Big|E\Big[\exp\Big\{i\xi\sum_{j=1}^{m}a_jB_n(s_j,t_j)\Big\}\Big]-E\Big[\exp\Big\{i\xi\sum_{j=1}^{m}a_jB^{\alpha(s_j),\beta(t_j)}_{s_j,t_j}\Big\}\Big]\Big|\\
&\le\Big|E\Big[\exp\Big\{i\xi\sum_{j=1}^{m}a_jB_n(s_j,t_j)\Big\}\Big]-E\Big[\exp\Big\{i\xi\sum_{j=1}^{m}a_jB^{j,k}_{n}\Big\}\Big]\Big|+\\
&\quad\Big|E\Big[\exp\Big\{i\xi\sum_{j=1}^{m}a_jB^{j,k}_{n}\Big\}\Big]-E\Big[\exp\Big\{i\xi\sum_{j=1}^{m}a_jB^{j,k}\Big\}\Big]\Big|+\\
&\quad\Big|E\Big[\exp\Big\{i\xi\sum_{j=1}^{m}a_jB^{j,k}\Big\}\Big]-E\Big[\exp\Big\{i\xi\sum_{j=1}^{m}a_jB^{\alpha(s_j),\beta(t_j)}_{s_j,t_j}\Big\}\Big]\Big|\\
&:=I+J+K.
\end{aligned}
$$

We will proceed to prove (15) in three steps.

Step 1. By the mean value theorem, there exists a constant $C>0$ such that

$$
\begin{aligned}
I\le&C\max_{1\le j\le m}\big\{E|B_n(s_j,t_j)-B^{j,k}_n|\big\}\\
=&C\max_{1\le j\le m}\Big\{E\Big|\int_0^1\int_0^1\big[\big(K_{\alpha(s_j)}(s_j,u)K_{\beta(t_j)}(t_j,v)\big)-\\
&\big(K_{\alpha(s_j)}(s_j,u)K_{\beta(t_j)}(t_j,v)\big)_k\big]\theta_n(u,v)\mathrm{d}u\mathrm{d}v\Big|\Big\}.
\end{aligned}
$$

Using the same method as the proof of Lemma 3.2, by Hölder inequality we can get

$$
\begin{aligned}
&E\Big|\int_0^1\int_0^1\big[\big(K_{\alpha(s_j)}(s_j,u)K_{\beta(t_j)}(t_j,v)\big)-\big(K_{\alpha(s_j)}(s_j,u)K_{\beta(t_j)}(t_j,v)\big)_k\big]\theta_n(u,v)\mathrm{d}u\mathrm{d}v\Big|\\
&\le C\Big(\int_0^1\int_0^1\big[\big(K_{\alpha(s_j)}(s_j,u)K_{\beta(t_j)}(t_j,v)\big)-\big(K_{\alpha(s_j)}(s_j,u)K_{\beta(t_j)}(t_j,v)\big)_k\big]^2\mathrm{d}u\mathrm{d}v\Big)^{\frac{1}{2}}.
\end{aligned}
$$

So I uniformly converges to 0 with respect to n.

Step 2. We proceed to deal with J. Using the mean value theorem again,

$$
\begin{aligned}
J\le&C\max_{1\le j\le m}\big\{E|B^{j,k}_n-B^{j,k}|\big\}\\
=&C\max_{1\le j\le m}\Big\{E\Big|\int_0^1\int_0^1\big(K_{\alpha(s_j)}(s_j,u)K_{\beta(t_j)}(t_j,v)\big)_k\theta_n(u,v)\mathrm{d}u\mathrm{d}v-\\
&\int_0^1\int_0^1\big(K_{\alpha(s_j)}(s_j,u)K_{\beta(t_j)}(t_j,v)\big)_k\,\mathrm{d}W(u,v)\Big|\Big\}.
\end{aligned}
$$

Thanks to Donsker's theorem, as $n\to\infty$, the laws of processes

$$\int_0^1\int_0^1\big(K_{\alpha(s_j)}(s_j,u)K_{\beta(t_j)}(t_j,v)\big)_k\theta_n(u,v)\mathrm{d}u\mathrm{d}v$$

converge weakly to the law of

$$\int_0^1\int_0^1\big(K_{\alpha(s_j)}(s_j,u)K_{\beta(t_j)}(t_j,v)\big)_k\mathrm{d}W(u,v),$$

since $(K_{\alpha(s_j)}(s_j,u)K_{\beta(t_j)}(t_j,v))_k$ is a simple function. So we can get $J\to 0$, as $n\to\infty$.

Step 3. Finally we deal with K. Using the mean value theorem again, one can get

$$K \leq C \max_{1\leq j\leq m} \Big\{ E\Big| \int_0^1 \int_0^1 \big(K_{\alpha(s_j)}(s_j,u)K_{\beta(t_j)}(t_j,v)\big)_k \mathrm{d}W(u,v) - \int_0^1 \int_0^1 \big(K_{\alpha(s_j)}(s_j,u)K_{\beta(t_j)}(t_j,v)\big)\mathrm{d}W(u,v)\Big| \Big\}.$$

Then by the Hölder inequality and the variance for a stochastic integral, we get

$$\begin{aligned} &E\Big| \int_0^1 \int_0^1 \big(K_{\alpha(s_j)}(s_j,u)K_{\beta(t_j)}(t_j,v)\big)_k \mathrm{d}W(u,v) - \\ &\qquad \int_0^1 \int_0^1 \big(K_{\alpha(s_j)}(s_j,u)K_{\beta(t_j)}(t_j,v)\big)\mathrm{d}W(u,v)\Big| \\ &\leq \Big(\int_0^1 \int_0^1 [\big(K_{\alpha(s_j)}(s_j,u)K_{\beta(t_j)}(t_j,v)\big)_k - K_{\alpha(s_j)}(s_j,u)K_{\beta(t_j)}(t_j,v)]^2 \mathrm{d}u\mathrm{d}v\Big)^{1/2}. \end{aligned}$$

So, $K \to 0$, for all $1 \leq j \leq m$, as $n \to \infty$. Then we can conclude the proof of the second point.

Finally, let us make several comments.

Remark 3.4 As stated in Theorem 3.1, we prove that two-parameter Volterra multifractional process introduced in Mendy [2] can be approximated in law in the topology of the anisotropic Besov spaces which is different from the results proved in Dai [19], because they studied the weak convergence of multifractional Brownian motion of Riemann-Liouville type in Besov spaces. Moreover Theorem 3.1 is also different from the results in Dai and Li [16] where the authors proved approximations of multifractional Brownian motions with moving-average representations in the space of continuous functions on $[0,1]$ based on Poisson processes.

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