Journal of Mathematical Research with Applications
Nov., 2017, Vol. 37, No. 6, pp. 649–656
DOI:10.3770/j.issn:2095-2651.2017.06.002
Http://jmre.dlut.edu.cn

The Crossing Numbers of Join of a Subdivision of $K_{2,3}$ with P_n and C_n

Zhenhua SU
Department of Mathematics, Huaihua University, Hunan 418008*, P. R. China*

Abstract Adding a new vertex to any edge of the complete bipartite graph $K_{2,3}$ gives a subdivision of $K_{2,3}$ (6-vertices graph). In the paper, we get the crossing numbers of the join graph of the specific 6-vertices graph H with n isolated vertices as well as with the path P_n on n vertices and with the cycle C_n.

Keywords crossing number; join graph; path; cycle

MR(2010) Subject Classification 05C10; 05C38

1. Introduction

Let G be a simple graph, whose vertex set and edge set are denoted by $V(G)$ and $E(G)$, respectively. A drawing of G is a representation of G in the plane such that its vertices are represented by distinct points and its edges by simple continuous arcs connecting the corresponding point pairs. For simplicity, we assume that in a drawing (a) no edge passes through any vertex other than its end-point, (b) no two edges touch each other, (c) no three edges cross at the same point.

The crossing number, $\mathrm{cr}(G)$ is the smallest number of edge crossings in any drawing of G. It is easy to see that a drawing with minimum number of crossings (an optimal drawing) is always a good drawing, meaning that (i) no edge crosses itself, (ii) no two edges cross more than once, and (iii) no two edges are incident with the same vertex cross.

Let ϕ be a drawing of graph G. We denote the number of crossings in ϕ by $\mathrm{cr}_\phi(G)$. For definitions not explained in this paper, readers are referred to [1]. By definition and notation about crossing numbers, it is easy to get the following properties:

Property 1.1 *Let D be a good drawing of G, and A, B, C be mutually edge-disjoint subgraphs of G. Then*

(i) $\mathrm{cr}_D(A \cup B, C) = \mathrm{cr}_D(A, C) + \mathrm{cr}_D(B, C)$.

(ii) $\mathrm{cr}_D(A \cup B) = \mathrm{cr}_D(A) + \mathrm{cr}_D(A, B) + \mathrm{cr}_D(B)$.

Property 1.2 *(i) Let H be a subgraph of G. Then $\mathrm{cr}(H) \le \mathrm{cr}(G)$.*

(ii) If H is isomorphic to G. Then $\mathrm{cr}(H) = \mathrm{cr}(G)$.

Received March 9, 2017; Accepted July 20, 2017
Supported by the Natural Science Foundation of Hunan Province (Grant No. 2017JJ3251).
E-mail address: szh820@163.com

In general, computing the crossing number of graphs is an NP-complete problem. At present, there are only some classes of special graphs whose crossing numbers are known. For example, these include the complete bipartite graph $K_{m,n}$ (see [2,3]) and the complete tripartite graph $K_{m,n,s}$ (see [4]) and so on. It is a very important result of $K_{m,n}$, in 1970 Kleitman [3] proved that:

$$\mathrm{cr}(K_{m,n}) = Z(m,n) = \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor,\ \ m \le 6,\ m \le n.$$

The join product of G and H, denoted by $G + H$, is obtained from vertex-disjoint copies of G and H by adding all edges between $V(G)$ and $V(H)$. Let nK_1 denote the graph on n isolated vertices, and let P_n and C_n be the path and cycle on n vertices, respectively. Recently, the crossing numbers of join product become more and more concerning. In 2007, Klešč [5] and Tang [6] obtained the crossing numbers of join of $P_n + P_n$, $P_n + C_n$ and $C_n + C_n$, respectively. And in [7] the crossing numbers of $G + P_n$ and $G + C_n$ are also known for a special graph G of order six. The up to date results of crossing numbers of G of order six with P_n and C_n are given in [7,8].

Let uv be an edge of graph G. Add a new vertex w to the edge of uv and make uw and wv replace the edge uv while the other vertices of G remain unchanged. This step is called a subdivision of an edge of graph G. Adding a new vertex w to any edge of the complete graph $K_{2,3}$ gives a subdivision of $K_{2,3}$ (see Figure 1). For convenience, we denote the subdivision graph by H, obviously H is a specific 6-vertices graph. In the paper, on the basis of result of crossing number of complete bipartite graph $\mathrm{cr}(K_{6,n}) = Z(6,n)$ by Kleitman, together with the special structure of graph H, we get the crossing numbers of the join graph of H with n isolated vertices as well as with the path P_n on n virtices and with the cycle C_n.

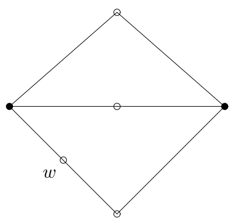


Figure 1 A subdivision of $K_{2,3}$ (the graph H)

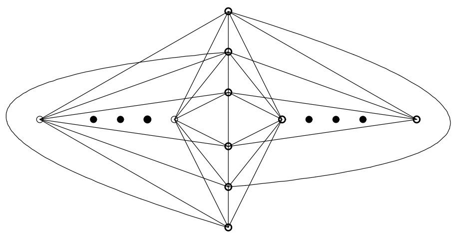


Figure 2 A good drawing of $H + nK_1$

2. The graph $H + nK_1$

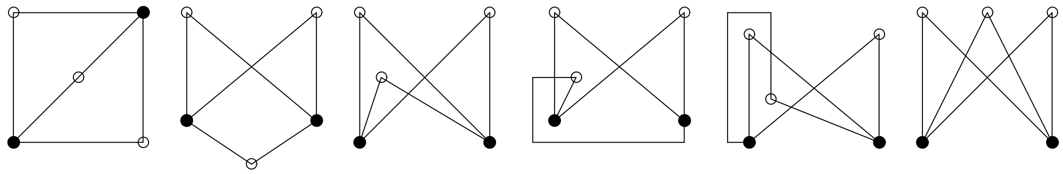
In the graph of $H + nK_1$, denote $V(H) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $V(nK_1) = \{t_1, t_2, \ldots, t_n\}$. Let for $i = 1, 2, \ldots, n$, T^i denote the subgraph of H which consists of the six edges incident with the vertex t_i. One can easily see that

$$H + nK_1 = H \cup (\bigcup_{i=1}^{n} T^i). \tag{1}$$

Lemma 2.1 *Let $H + K_1 = H \cup T^1$ and $H + 2K_1 = H \cup T^1 \cup T^2$. Then, we have* $\mathrm{cr}(H + K_1) = 1$ *and* $\mathrm{cr}(H + 2K_1) = 2$.

Proof The drawing in Figure 2 shows that $\mathrm{cr}(H + K_1) \leq 1$ and $\mathrm{cr}(H + 2K_1) \leq 2$. Since $H + K_1$ contains a subgraph which is isomorphic to the subdivision of $K_{3,3}$, and $H + 2K_1$ contains a subgraph which is isomorphic to the subdivision of $K_{3,4}$. So by Property 1.2, we have $\mathrm{cr}(H + K_1) \geq \mathrm{cr}(K_{3,3}) = 1$ and $\mathrm{cr}(H + 2K_1) \geq \mathrm{cr}(K_{3,4}) = 2$. This completes the proof. □

Lemma 2.2 ([4]) *There are 6 non-isomorphic drawings of $K_{2,3}$ (see Figure 3).*

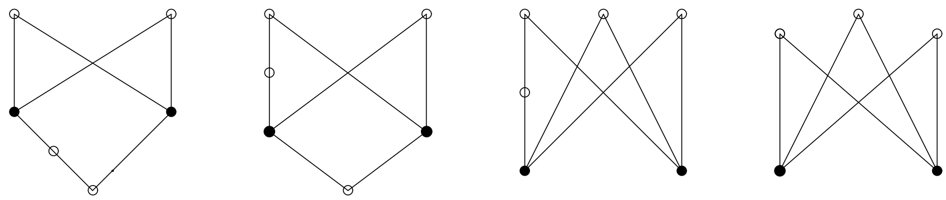


(1) (2) (3) (4) (5) (6)

Figure 3 Six good drawings of $K_{2,3}$

Lemma 2.3 *There are exactly 4 drawings of H such that a region exists with 6 vertices on its boundary (see Figure 4).*

Proof According to Lemma 2.2, we know that there are 6 non-isomorphic drawings of $K_{2,3}$ shown in Figure 3. To obtain a drawing of H such that there is a region with 6 vertices of H on its boundary, the only candidates are Figure 3(2) and 3(6). To obtain a drawing of H from Figure 3(2) and 3(6), we need to add a new vertex at any edge of $K_{2,3}$, then we get the possible good drawings of H shown in Figure 4.



(1) (2) (3) (4)

Figure 4 Four good drawings of H

Theorem 2.4 *For $n \geq 1$, we have $\mathrm{cr}(H + nK_1) = Z(6, n) + n$.*

Proof The good drawing of $H + nK_1$ in Figure 2 shows that $\mathrm{cr}(H + nK_1) \leq Z(6, n) + n$. We prove the reverse inequality by induction on n. By Lemma 2.1, the theorem is true for $n = 1$ and n=2. Suppose now for $n \geq 3$, $\mathrm{cr}(H + (n - 2)K_1) \geq Z(6, n - 2) + n - 2$, and consider such a good drawing D of $H + nK_1$ that

$$\mathrm{cr}_D(H + nK_1) < Z(6, n) + n. \tag{2}$$

Claim 2.5 There is at least one T^i, such that $\mathrm{cr}_D(H, T^i) = 0$.

Otherwise, for all t_i, $\mathrm{cr}_D(H, T^i) \geq 1$. Using (1), we have $\mathrm{cr}_D(H + nK_1) \geq \mathrm{cr}_D(\bigcup_{i=1}^n T^i) + \mathrm{cr}_D(H, \bigcup_{i=1}^n T^i) \geq Z(6, n) + n$. This contradicts (2).

Claim 2.6 For all $i, j = 1, 2, \ldots, n$, $i \neq j$, there holds $\mathrm{cr}_D(T^i, T^j) \geq 1$.

Otherwise, assume T^1 and T^2 satisfy $\mathrm{cr}_D(T^1,T^2)=0$. Since H contains two 5-cycle, $\mathrm{cr}_D(H,T^1\cup T^2)\geq 2$. As $T^i\cup T^1\cup T^2$ is isomorphic to $K_{3,6}$, by $\mathrm{cr}(K_{3,6})=6$, $\mathrm{cr}_D(T^i,T^1\cup T^2)\geq 6$ for $i=3,4,\ldots,n$. Together with (1) and Properties 2.1 and 2.2, we have

$$\begin{aligned}\mathrm{cr}_D(H_n)&=\mathrm{cr}_D(H\cup\bigcup_{i=3}^{n}T^i\cup T^1\cup T^2)\\&=\mathrm{cr}_D(H\cup\bigcup_{i=3}^{n}T^i)+\mathrm{cr}_D(T^1\cup T^2)+\mathrm{cr}_D(H,T^1\cup T^2)+\mathrm{cr}_D(\bigcup_{i=3}^{n}T^i,T^1\cup T^2)\\&\geq Z(6,n-2)+(n-2)+2+6(n-2)\geq Z(6,n)+n.\end{aligned}$$

This contradicts (2). Hence $\mathrm{cr}_D(T^i,T^j)\geq 1$.

Next we get contradiction from restricted condition of Claims 2.5 and 2.6.

By Claim 2.5, assume T^1 satisfies $\mathrm{cr}_D(H,T^1)=0$. Since $\mathrm{cr}_D(H,T^1)=0$, there is a disk such that the vertices of H are all placed on the boundary of disk. From Lemma 2.3, the good drawing of H is shown in Figure 4. Adding the edges of T^1, we have the subdrawing of $H\cup T^n$ as shown in Figure 5.

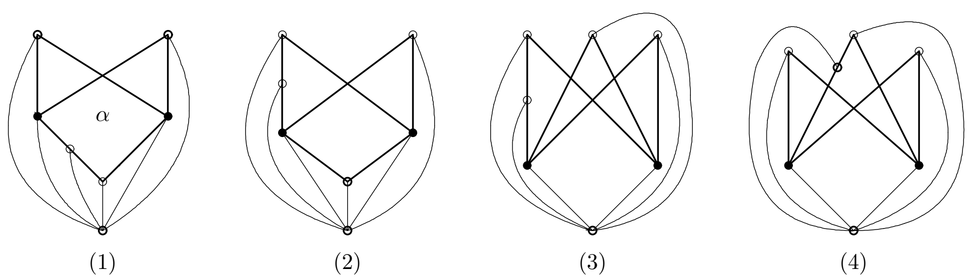


Figure 5 Four good drawings of $H\cup T^n$

(i) In Figure 5(1), when t_i $(2\leq i\leq n)$ are placed in the region α, together with Claim 2.2, $\mathrm{cr}_D(T^i,T^j)\geq 1$, we have $\mathrm{cr}_D(H\cup T^1,T^i)\geq 3$, and "$=$" holds if and only if $\mathrm{cr}_D(H,T^i)=2$ and $\mathrm{cr}_D(T^1,T^i)=1$. When t_i $(2\leq i\leq n)$ are placed in the other regions, together with $\mathrm{cr}_D(T^i,T^j)\geq 1$, we have $\mathrm{cr}_D(H\cup T^1,T^i)\geq 5$. Now let x be the number of vertices t_i which are placed in the region α. As for all this t_i, there holds $\mathrm{cr}_D(H,T^i)\geq 2$. Using (1), we have $\mathrm{cr}_D(H+nK_1)\geq Z(6,n)+2x$. This together with (2), implies that $x\leq\frac{n-1}{2}$. Hence, we have

$$\begin{aligned}\mathrm{cr}_D(H+nK_1)&=\mathrm{cr}_D(\bigcup_{i=2}^{n}T^i)+\mathrm{cr}_D(H\cup T^1,\bigcup_{i=2}^{n}T^i)+\mathrm{cr}_D(H\cup T^1)\\&\geq Z(6,n-1)+3x+5(n-1-x)+1\geq Z(6,n-1)+5n-5-2x\geq Z(6,n)+n.\end{aligned}$$

This contradicts (2).

(ii) In Figure 5(2)(3)(4), no matter which regions the vertex t_i are placed in, and by $\mathrm{cr}_D(T^i,T^j)\geq 1$, there always hold $\mathrm{cr}_D(H\cup T^1,T^i)\geq 4$. Moreover, together with $\mathrm{cr}_D(H\cup T^1)\geq 1$, we have

$$\mathrm{cr}_D(H+nK_1)=\mathrm{cr}_D(\bigcup_{i=2}^{n}T^i)+\mathrm{cr}_D(H\cup T^1,\bigcup_{i=2}^{n}T^i)+\mathrm{cr}_D(H\cup T^1)$$

$$\geq Z(6,n-1)+4(n-1)+1 \geq Z(6,n)+n.$$

This contradicts (2).

Therefore, we always have $\mathrm{cr}_D(H+nK_1) \geq Z(6,n)+n$. This completes the proof. □

3. The graph $H + P_n$

The graph $H+P_n$ contains $H+nK_1$ as a subgraph. We will use the same notion as $H+nK_1$. Let P_n^* denote the path on n vertices of $H+P_n$ not belonging to the subgraph H. One can easily see that

$$H+P_n = H \cup (\bigcup_{i=1}^{n} T^i) \cup P_n^*. \tag{3}$$

Lemma 3.1 ([7]) *Let D be a good drawing of mK_1+C_n, $m \geq 2$, $n \geq 3$, in which no edges of C_n is crossed, and C_n does not separate the other vertices of the graph. Then for all $i,j=1,2,\ldots,n$, two different subgraphs T^i and T^j cross each other in D at least $\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ times.*

Theorem 3.2 *For $n \geq 2$, we have $\mathrm{cr}(H+P_n) = Z(6,n)+n+1$.*

Proof One can easily see that in Figure 2 it is possible to add $n-1$ edges which form the path P_n^* on the vertices of nK_1 in such a way that only one edge of P_n^* is crossed by an edge of H. Hence $\mathrm{cr}(H+P_n) \leq Z(6,n)+n+1$. Next we assume there exists a good drawing D of $H+P_n$ such that

$$\mathrm{cr}_D(H+P_n) \leq Z(6,n)+n. \tag{4}$$

Claim 3.3 $\mathrm{cr}_D(H \cup \bigcup_{i=1}^{n} T^i, P_n^*) = \mathrm{cr}_D(P_n^*) = 0$. *Thus all vertices of P_n^* are placed in the same region.*

By Theorem 2.4, $\mathrm{cr}(H+nK_1) = Z(6,n)+n$. Therefore using (3), we have

$$\begin{aligned}\mathrm{cr}_D(H+P_n) &= \mathrm{cr}_D(H \cup \bigcup_{i=1}^{n} T^i) + \mathrm{cr}_D(H \cup \bigcup_{i=1}^{n} T^i, P_n^*) + \mathrm{cr}_D(P_n^*) \\ &\geq Z(6,n)+n+\mathrm{cr}_D(H \cup \bigcup_{i=1}^{n} T^i, P_n^*) + \mathrm{cr}_D(P_n^*).\end{aligned}$$

This together with assumption (4), implies that $\mathrm{cr}_D(H \cup \bigcup_{i=1}^{n} T^i, P_n^*) = \mathrm{cr}_D(P_n^*) = 0$. So no edges of P_n^* is crossed, and no edges of P_n^* crossed with the edges of $H \cup \bigcup_{i=1}^{n} T^i$. Thus all vertices of P_n^* are placed in the same region.

Next we divide three following different cases to discuss:

Case 1 All vertices of P_n^* are placed in the regions where there are 6 vertices on the boundary.

Consider the drawing of $H \cup T^1$, satisfying $\mathrm{cr}_D(H, T^1) = 0$. By Theorem 2.4, the drawing of $H \cup T^1$ is shown in Figure 5. By Claim 3.3, all t_i $(2 \leq i \leq n)$ are placed in the same region with the vertex t_1. Thus $\mathrm{cr}_D(H \cup T^n, T^i) \geq 5$. Moreover, together with $\mathrm{cr}_D(H \cup T^1) \geq 1$, we have $\mathrm{cr}_D(H+P_n) \geq Z(6,n-1)+5(n-1)+1 > Z(6,n)+n$. This contradicts (4).

Case 2 All vertices of P_n^* are placed in the regions in which there are 5 vertices on the boundary.

Then, for all $i = 1, 2, \ldots, n$, we have $\mathrm{cr}_D(H, T^i) \geq 1$. By Claim 3.3, all t_i $(2 \leq i \leq n)$ are placed in the 5 vertices region. Using Lemma 3.1, we have $\mathrm{cr}_D(\sum_{i=1}^n T^i) \geq C_n^2 \lfloor \frac{5}{2} \rfloor \lfloor \frac{4}{2} \rfloor$. So we have $\mathrm{cr}_D(H + P_n) \geq \mathrm{cr}_D(\sum_{i=1}^n T^i) + \sum_{i=1}^n \mathrm{cr}_D(H, T^i) \geq C_n^2 \lfloor \frac{5}{2} \rfloor \lfloor \frac{4}{2} \rfloor + n > Z(6, n) + n$. This contradicts (4).

Case 3 All vertices of P_n^* are placed in the regions in which no more than 4 vertices are on the boundary.

Then, for all $i = 1, 2, \ldots, n$, we have $\mathrm{cr}_D(H, T^i) \geq 2$. Using (3), we have $\mathrm{cr}_D(H + P_n) \geq \mathrm{cr}_D(\sum_{i=1}^n T^i) + \sum_{i=1}^n \mathrm{cr}_D(H, T^i) \geq Z(6, n) + 2n > Z(6, n) + n$. This contradicts (4).

Together with above three cases, the assumption (4) does not hold. So we have $\mathrm{cr}_D(H + P_n) \geq Z(6, n) + n + 1$. This completes the proof. □

4. The graph $H + C_n$

The graph $H + C_n$ contains both $H + nK_1$ and $H + P_n$ as a subgraph. Let C_n^* denote the subgraph induced on the vertices not belonging to the subgraph H. Let T_i $(1 \leq i \leq 6)$ denote the subgraph induced by n edges of $K_{6,n}$ incident with ith vertex of H. One can easily see that

$$H + C_n = H \cup (\bigcup_{i=1}^{6} T_i) \cup C_n^*. \tag{5}$$

Lemma 4.1 ([5]) *Let D be an optimal drawing of $H + C_n$. Then $\mathrm{cr}_D(C_n^*) = 0$.*

Theorem 4.2 *For $n \geq 3$, we have $\mathrm{cr}_D(H + C_n) = Z(6, n) + n + 3$.*

Proof In Figure 2, it is possible to add n edges from $H + nK_1$, then the edges of C_n^* are crossed only three times. Hence $\mathrm{cr}(H + C_n) \leq Z(6, n) + n + 3$. To prove the reverse inequality, assume that there is a drawing D of $H + C_n$, such that

$$\mathrm{cr}_D(H + C_n) \leq Z(6, n) + n + 2. \tag{6}$$

Since $H + C_n = H \cup (\bigcup_{i=1}^6 T_i) \cup C_n^*$, and $H \cup (\bigcup_{i=1}^6 T_i)$ is isomorphic to $H + nK_1$, from Theorem 2.4, we have $\mathrm{cr}_D(H + nK_1) \geq Z(6, n) + n$. Moreover, using Lemma 4.1 and (5), we have

$$\begin{aligned} \mathrm{cr}_D(H + C_n) &= \mathrm{cr}_D(H + nK_1) + \mathrm{cr}_D(C_n^*, H \cup (\bigcup_{i=1}^{6} T_i)) + \mathrm{cr}_D(C_n^*) \\ &\geq Z(6, n) + n + \mathrm{cr}_D(C_n^*, H \cup (\bigcup_{i=1}^{6} T_i)). \end{aligned} \tag{7}$$

Claim 4.3 $\mathrm{cr}_D(C_n^*, H \cup (\bigcup_{i=1}^6 T_i)) = 2$.

Firstly, assume that $\mathrm{cr}_D(C_n^*, H \cup (\bigcup_{i=1}^6 T_i)) = 0$. By Lemma 3.1, for all T_i and T_j $(1 \leq i < j \leq 6)$, have $\mathrm{cr}_D(T_i, T_j) \geq \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$. So $\mathrm{cr}_D(H + C_n) \geq C_6^2 \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor > Z(6, n) + n + 2$. This contradicts (6).

Secondly, assume that $\mathrm{cr}_D(C_n^*, H \cup (\bigcup_{i=1}^6 T_i)) = 1$. Since H is 2-connected graph ($\mathrm{cr}_D(C_n^*, H) \geq 2$), it is only possible that $\mathrm{cr}_D(C_n^*, \bigcup_{i=1}^6 T_i) = 1$. Then deleting the crossed edges of C_n^* results in the drawing of $H+P_n$ with fewer than $Z(6,n)+n+1$. This contradicts Theorem 3.2.

Thirdly, assume that $\mathrm{cr}_D(C_n^*, H\cup(\bigcup_{i=1}^6 T_i))\geq 3$. By (7), we have $\mathrm{cr}_D(H+C_n)\geq Z(6,n)+n+3$. This contradicts (6).

According to above three kinds of discussion, we have $\mathrm{cr}_D(C_n^*, H\cup(\bigcup_{i=1}^6 T_i))=2$.

Next we divide two following different cases to complete the proof.

Case 1 $\mathrm{cr}_D(C_n^*, H)=2$, $\mathrm{cr}_D(C_n^*, \bigcup_{i=1}^6 T_i)=0$.

Subcase 1.1 There exists a 2-degree vertex of H placed in the inner of C_n^*, and the other five vertices are placed in the external. Since $\mathrm{cr}_D(C_n^*, \bigcup_{i=1}^6 T_i)=0$, using Lemma 3.1, we have $\mathrm{cr}_D(H+C_n)\geq C_5^2\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+2>Z(6,n)+n+2$. This contradicts (6).

Subcase 1.2 There exist two 2-degree vertices of H placed in the inner of C_n^*, and the other four vertices are placed in the external. Without loss of generality, suppose x_5, x_6 are placed in the inner region. Now consider the drawing of H. Moreover, the edges of H do not cross each other more than three times. Otherwise by Lemma 3.1, we have $cr_D(H+C_n)\geq C_4^2\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+2+3>Z(6,n)+n+2$. Therefore, according to the structure of H, the drawing of $H\cup C_n^*$ is as shown in Figure 6.

First we can prove that there exists no edge of C_n^* which are crossed two times. Otherwise deleting the crossed edges of C_n^* results in the drawing of $H+P_n$ with fewer than $Z(6,n)+n+1$. This contradicts Theorem 3.2. So in Figure 6, there exists at least a vertex t_i placed in the boundary of C_n^*.

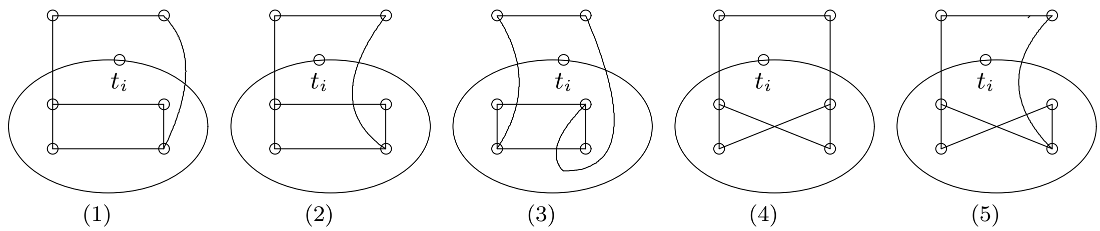


Figure 6 Five good drawings of $H\cup C_n^*$

(i) When $H\cup C_n$ is as shown in Figure 6(1)(3)(5), there hold $\mathrm{cr}_D(H\cup C_n^*, \bigcup_{i=1}^6 T_i)\geq n$. So we have $\mathrm{cr}_D(H+C_n)\geq C_4^2\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+2+n>Z(6,n)+n+2$. This contradicts (6).

(ii) When $H\cup C_n$ is as shown in Figure 6(2)(4), there hold $\mathrm{cr}_D(H\cup C_n^*, \bigcup_{i=1}^6 T_i)\geq 2$. And together with $\mathrm{cr}_D(H)\geq 1$, we have $\mathrm{cr}_D(H+C_n)\geq C_4^2\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+2+2+1>Z(6,n)+n+2$. This contradicts (6).

Case 2 $\mathrm{cr}_D(C_n^*, H)=0$, $\mathrm{cr}_D(C_n^*, \bigcup_{i=1}^6 T_i)=2$.

Subcase 2.1 There exists a vertex, x_6, such that $\mathrm{cr}_D(C_n^*, T_6)=2$. As Lemma 3.1, for $1\leq i\leq 5$, with $\mathrm{cr}_D(T_i,T_j)\geq\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor$. Hence $\mathrm{cr}_D(H+C_n)\geq C_5^2\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+2>Z(6,n)+n+2$. This contradicts (6).

Subcase 2.2 There exist two vertices, x_5, x_6, such that $\mathrm{cr}_D(C_n^*, T_5) = \mathrm{cr}_D(C_n^*, T_6) = 1$.

Subcase 2.2.1 For $n = 3$. There exists no edge of C_3^* which crosses with T_5 and T_6 at the same time. Otherwise deleting the crossed edges of C_3^* results in the drawing of $H + P_3$ with fewer than $Z(6, 3) + 3 + 1$. This contradicts Theorem 3.2. So the edges of T_5 and T_6 crosses with the different edges of C_3^*. According to the structure of C_3^*, the subgraph $T_5 \cup T_6 \cup C_3^*$ is as shown in Figure 7. As H contains a 5-cycle C_5, so regardless of whether or not the edges of H cross each other, we always have $\mathrm{cr}_D(\bigcup_{i=1}^4 T_i, T_5 \cup T_6 \cup H \cup C_3^*) \geq 3$. Hence $\mathrm{cr}_D(H + C_3) \geq \mathrm{cr}_D(\bigcup_{i=1}^4 T_i) + \mathrm{cr}_D(T_5 \cup T_6 \cup H \cup C_3^*) + \mathrm{cr}_D(\bigcup_{i=1}^4 T_i, T_5 \cup T_6 \cup H \cup C_3^*) \geq C_4^2 \lfloor \frac{3}{2} \rfloor \lfloor \frac{2}{2} \rfloor + 3 + 3 > Z(6, 3) + 3 + 2$. This contradicts (6).

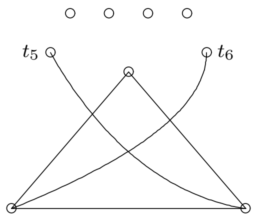


Figure 7 A subdrawing $T_5 \cup T_6 \cup C_3^*$

Subcase 2.2.2 For $n \geq 4$. Based on known conditions, the vertices x_i $(1 \leq i \leq 6)$ are all placed in the same region of C_n^*, say, external region. Then according to the cross of T_i and T_j, we can divide three cases: (i) For $1 \leq i < j \leq 4$, $\mathrm{cr}_D(T_i, T_j) \geq \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$. (ii) For $1 \leq i \leq 4$, $j = 5, 6$, $\mathrm{cr}_D(T_i, T_j) \geq \lfloor \frac{n-1}{2} \rfloor \lfloor \frac{n-2}{2} \rfloor$. (iii) For $i = 5$, $j = 6$, $\mathrm{cr}_D(T_i, T_j) \geq \lfloor \frac{n-2}{2} \rfloor \lfloor \frac{n-3}{2} \rfloor$. So using (5), we have $\mathrm{cr}_D(H + C_n) \geq C_4^2 \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2 \times 4 \lfloor \frac{n-1}{2} \rfloor \lfloor \frac{n-2}{2} \rfloor + \lfloor \frac{n-2}{2} \rfloor \lfloor \frac{n-3}{2} \rfloor + 2 > Z(6, n) + n + 2$ $(n \geq 4)$. This contradicts (6).

So from the above cases, the assumption (6) does not hold. We get $\mathrm{cr}_D(H + C_n) \geq Z(6, n) + n + 3$. This completes the proof. $\Box$

Acknowledgements We thank the referees for their time and comments.

References

[1] J. A. BONDY, U. S. R. MURTY. *Graph Theory with Application*. Macmillan London and Elsevie, New York, 1979.
[2] R. K. GUY. *Crossing numbers of graphs//Graph Theory and Applieations*. LNM 303. Heidelberg: Spring-Verlag, 1972.
[3] D. J. KLEITMAN. *The crossing number of $K_{5,n}$*. J. Comb. Theory, 1970, **9**: 315–323.
[4] H. T. PAK. *The crossing number of $K_{2,4,n}$*. ARS Combinatoria, 2013, **109**: 527–537.
[5] M. KLEŠČ. *The join of graphs and crossing numbers*. Electronic Notes in Discrete Math, 2007, **28**: 349–355.
[6] Ling TANG, Jing WANG, Yuanqiu HUANG. *The crossing number of the join of C_m and P_n*. Int. J. Math. Comb., 2007, **1**(1): 110–116.
[7] M. KLEŠČ. *The crossing numbers of join of the special graph on six vertices with path and cycle*. Discrete Math., 2010, **310**(9): 1475–1481.
[8] Zhidong ZHOU, Shengxiang LV. *On the crossing of the join graph of a path or a cycle*. Adv. in Math., 2014, **43**(1): 69–80. (in Chinese)