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Right Bi-Giraud Recollements of Abelian Categories

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Abstract We introduce the notion of a right bi-Giraud recollement for abelian categories. We show that right bi-Giraud recollements are bijective to cohereditary and hereditary torsion pairs. We obtain such torsion pairs in the module category via certain idempotent ideals.

Keywords right bi-Giraud recollements; cohereditary and hereditary torsion pairs; idempotent ideals

MR(2010) Subject Classification 13D30; 16D25; 18E35

1. Introduction

In [1], there is a one-to-one correspondence between equivalence classes of recollements of a triangulated category $\mathcal{C}$ and its TTF triples. And in [1–3], there is also a bijection between right recollements and stable t-structures of triangulated categories $\mathcal{C}$. For right (left) recollements of triangulated categories, readers can refer to [4].

It is natural to study similar questions to triangulated categories in abelian categories. In [5], it was shown that such a correspondence holds in the category of right R-modules, Mod$R$, for a unitary ring R, and that recollements of an abelian category are in bijection with its bilocalising TTF-classes. So we ask naturally what is about the right recollements of abelian categories. We shall show that there is a bijection between right Bi-Giraud recollements of an abelian category and its cohereditary and hereditary torsion pairs of $\mathcal{C}$, which perfectly coincides with the situation in triangulated category. Now we state our main result (Theorem 3.4) as follows:

Theorem 1.1 *There is a bijection between $\mathcal{R}(\mathcal{C}',\ \mathcal{C},\ \mathcal{C}'')$ right Bi-Giraud recollements of abelian categories and $(\mathcal{T},\mathcal{F})$ cohereditary and hereditary torsion pairs of an abelian category $\mathcal{C}$.*

In view of the theorem above, it motivates us to consider a method of gaining cohereditary and hereditary torsion pairs for a given abelian category. We obtain the following result (Theorem 4.3).

Theorem 1.2 *Let R be a two-sided noetherian ring and $I = I^2$ an idempotent ideal of R. Then $(\mathcal{T}_I,\mathcal{F}_I)$ is a cohereditary and hereditary torsion pair of Mod-R if and only if I is projective as a left R-module, where $\mathcal{T}_I := \{M \in \mathrm{Mod}R | MI = M\}$; $\mathcal{F}_I := \{M \in \mathrm{Mod}R | MI = 0\}$.*

Here, we want to point out why we give a direct proof for [6, Theorem 4.1] as Theorem 3.3 in this paper. On the one hand, the original proof in [6] involves lots of preparations in [6,

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Sections 2 and 3]. In fact, we find that it can be directly and self-containedly proved through our observation of Lemma 3.2. On the other hand, it is easy for readers to know how we establish the connections between right recollements and torsion pairs.

2. Right Bi-Giraud recollements of abelian categories

In this section, we define right Bi-Giraud recollement and discuss its some properties. First, let us recall that a right recollement $(\mathcal{C}',\ \mathcal{C},\ \mathcal{C}'',\ i_*,\ i^!,\ j^*,\ j_*)$ of abelian categories is a diagram of functors

$$\mathcal{C}' \underset{i^!}{\overset{i_*}{\rightleftarrows}} \mathcal{C} \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathcal{C}''$$

Diagram 1 Functors diagram

satisfying the following conditions:

(i) $(i_*, i^!)$ and (j^*, j_*) are adjoint pairs;

(ii) i_* and j_* are fully faithful; and

(iii) $\mathrm{Im} i_* = \mathrm{Ker} j^*$.

In [6], a coreflective subcategroy of an abelian category is called Co-Giraud if the coreflector preserves cokernels. Dually, a reflective subcategroy of an abelian categrory is called Giraud if the reflector preserves kernels. Accordingly, we make the following

Definition 2.1 *A right recollement of abelian categories is called Bi-Giraud if the two functors $i^!, j^*$ are exact.*

Now, we give some equivalent conditions of right Bi-Giraud recollement of abelian categroies.

Theorem 2.2 *Let $\mathcal{C}'$, $\mathcal{C}$, $\mathcal{C}''$ be abelian categories. If a diagram of functors*

$$\mathcal{C}' \underset{i^!}{\overset{i_*}{\rightleftarrows}} \mathcal{C} \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathcal{C}''$$

Diagram 2 Functors diagram

satisfies the following conditions:

(i) $(i_*, i^!)$ and (j^*, j_*) are adjoint pairs;

(ii) i_* and j_* are fully faithful; and

(iii) $i^!, j^*$ are exact,

then the following conditions are equivalent:

(1) (R1) $j^* i_* = 0$;

(R2) for any $C \in \mathcal{C}$, there exists a short exact sequence

$$0 \longrightarrow i_* i^! C \longrightarrow C \longrightarrow j_* j^* C \longrightarrow 0.$$

(1)′ (R1) $i^! j_* = 0$; (R2)′=(R2).

(2) (R3) $\mathrm{Im} i_* = \mathrm{Ker} j^*$.

(2)′ (R3)′ $\mathrm{Im} j_* = \mathrm{Ker} i^!$.

Proof (1) $\Rightarrow$ (2). It is only to show that Ker$j^* \subseteq$ Imi_*. For any $C \in \mathcal{C}$ with $j^*C = 0$, since there exists the short exact sequence

$$0 \longrightarrow i_*i^!C \longrightarrow C \longrightarrow j_*j^*C \longrightarrow 0,$$

$j_*j^*C = 0$ implying $i_*i^!C \cong C$, and $C \in$ Imi_*.

(2) $\Rightarrow$ (1)$'$. First, we prove (R1)$'$ $i^!j_* = 0$. It is easy to be seen from the following formulas

$$\mathrm{Hom}_{\mathcal{C}'}(i^!j_*, i^!j_*) \cong \mathrm{Hom}_{\mathcal{C}}(i_*i^!j_*, j_*) \cong \mathrm{Hom}_{\mathcal{C}''}(j^*i_*i^!j_*, -) = 0.$$

Next, we show the condition (R2) holds. For any $C \in \mathcal{C}$, we consider the following exact sequence

$$0 \longrightarrow \mathrm{Ker}\eta_C \longrightarrow C \xrightarrow{\eta_C} j_*j^*C \longrightarrow \mathrm{Coker}\eta_C \longrightarrow 0, \tag{2.1}$$

where η is the unit of the adjoint pair (j^*, j_*). Now, applying the exact functor j^* to this exact sequence, we get exact sequence

$$0 \longrightarrow j^*\mathrm{Ker}\eta_C \longrightarrow j^*C \xrightarrow{j^*\eta_C} j^*j_*j^*C \longrightarrow j^*\mathrm{Coker}\eta_C \longrightarrow 0.$$

By j_* fully faithful, $j^*\eta_C$ is isomorphic which results in $j^*\mathrm{Ker}\eta_C = 0$, and $j^*\mathrm{Coker}\eta_C = 0$. Under the assumption Im$i_* =$ Kerj^*, $\mathrm{Ker}\eta_C = i_*C_1'$ and $\mathrm{Coker}\eta_C = i_*C_2'$ for some $C_1', C_2' \in \mathcal{C}'$. We now can rewrite (2.1) as follows

$$0 \longrightarrow i_*C_1' \longrightarrow C \xrightarrow{\eta_C} j_*j^*C \longrightarrow i_*C_2' \longrightarrow 0. \tag{2.2}$$

Now, applying the exact functor $i^!$ to (2.2), we get

$$0 \longrightarrow i^!i_*C_1' \longrightarrow i^!C \xrightarrow{i^!\eta_C} i^!j_*j^*C = 0 \longrightarrow i^!i_*C_2' \longrightarrow 0.$$

So by i_* fully faithful,

$$0 = i^!i_*C_2' \cong C_2', \quad \mathrm{Coker}\eta_C = i_*C_2' = 0; \quad i^!i_*C_1' \cong i^!C, \quad i_*C_1' \cong i_*i^!i_*C_1' \cong i_*i^!C$$

which results in that (2.2) can be replaced by the following short exact sequence

$$0 \longrightarrow i_*i^!C \longrightarrow C \longrightarrow j_*j^*C \longrightarrow 0.$$

(1)$'$ $\Rightarrow$ (2)$'$. Similar to (1)$\Rightarrow$ (2).

(2)$'$ $\Rightarrow$ (1). Similar to (2)$\Rightarrow$ (1)$'$.

From now on, we always suppose that a right recollement $(\mathcal{C}',\ \mathcal{C},\ \mathcal{C}'',\ i_*,\ i^!,\ j^*,\ j_*)$ of abelian categroies is Bi-Giraud and denoted by $\mathcal{R}(\mathcal{C}',\ \mathcal{C},\ \mathcal{C}'')$. On the knowledge of cohereditary and hereditary torsion pair of abelian category, readers can refer to the next section of this paper.

Proposition 2.3 *Let $\mathcal{R}(\mathcal{C}',\ \mathcal{C},\ \mathcal{C}'')$ be a right Bi-Giraud recollement. Then*

(1) i_* and j_* are exact;

(2) $(i_*\mathcal{C}', j_*\mathcal{C}'')$ is a cohereditary and hereditary torsion pair of $\mathcal{C}$;

(3) $\mathcal{C}/\mathcal{C}' \cong \mathcal{C}''$ and $\mathcal{C}/\mathcal{C}'' \cong \mathcal{C}'$.

Proof (1) We only prove that i_* is exact and the exactness of j_* can be proved similarly. In fact, it is only to need i_* that preserves monomorphism, due to the right exactness of i_*. For any short exact sequence in $\mathcal{C}'$

$$0 \longrightarrow C_1' \xrightarrow{f} C' \xrightarrow{g} C_2' \longrightarrow 0,$$

we can get the exact sequence

$$0 \longrightarrow \mathrm{Ker} i_* f \longrightarrow i_* C_1' \xrightarrow{i_* f} i_* C' \xrightarrow{i_* g} i_* C_2' \longrightarrow 0, \tag{2.3}$$

by acting i_*. Again acting the exact functor j^* on (2.3), we have

$$0 \longrightarrow j^* \mathrm{Ker} i_* f \longrightarrow j^* i_* C_1' = 0 \xrightarrow{j^* i_* f} j^* i_* C' = 0 \xrightarrow{j^* i_* g} j^* i_* C_2' = 0 \longrightarrow 0,$$

implying $j^* \mathrm{Ker} i_* f = 0$. So there exists $C_0' \in \mathcal{C}'$ such that $\mathrm{Ker} i_* f = i_* C_0'$, which renders that (2.3) can be replaced by

$$0 \longrightarrow i_* C_0' \longrightarrow i_* C_1' \xrightarrow{i_* f} i_* C' \xrightarrow{i_* g} i_* C_2' \longrightarrow 0. \tag{2.4}$$

Further, by the exactness of $i^!$ and i_* being fully faithful , we have

$$0 \longrightarrow i^! i_* C_0' \longrightarrow i^! i_* C_1' \cong C_1' \xrightarrow{i^! i_* f} i^! i_* C' \cong C' \xrightarrow{i^! i_* g} i^! i_* C_2' \cong C_2' \longrightarrow 0,$$

which indicates that $C_0' \cong i^! i_* C_0' = 0$ and $i_* C_0' = 0$. Comparing with (2.4), we get that i_* is exact.

(2) $\mathrm{Hom}(i_*\mathcal{C}', j_*\mathcal{C}'') \cong \mathrm{Hom}(j^* i_*\mathcal{C}', \mathcal{C}'') = 0$, and (R2) infer that $(i_*\mathcal{C}', j_*\mathcal{C}'')$ is a torsion pair of $\mathcal{C}$. By i_*, j_* exact fucntors and the arguments of (1), we can easily get that $(i_*\mathcal{C}', j_*\mathcal{C}'')$ is cohereditary and hereditary.

(3) We only prove that $\mathcal{C}/\mathcal{C}' \cong \mathcal{C}''$, and $\mathcal{C}/\mathcal{C}'' \cong \mathcal{C}'$ can be proved similarly. By (2), $\mathcal{C}'$ can be seen as serre's subcategory of $\mathcal{C}$, which makes us get the exact sequence of abelian categories

$$0 \longrightarrow \mathcal{C}' \xrightarrow{i_*} \mathcal{C} \xrightarrow{F} \mathcal{C}/\mathcal{C}' \longrightarrow 0$$

where F is the quotient functor satisfying F is exact and $\mathrm{Ker} F = \mathcal{C}'$. Since $\mathrm{Im} i_* = \mathrm{Ker} j^*$, there exists uniquely a functor $G : \mathcal{C}/\mathcal{C}' \longrightarrow \mathcal{C}''$ such that $j^* = GF$. Next, we only need to check that $G \circ F j_* \cong Id_{\mathcal{C}''}$ and $F j_* \circ G \cong Id_{\mathcal{C}/\mathcal{C}'}$. Clearly, $G \circ F j_* \cong GF \circ j_* \cong j^* j_* \cong Id_{\mathcal{C}''}$. Now, taking any object $C \in \mathcal{C}/\mathcal{C}'$, $F j_* \circ GC = F j_* j^* C = j_* j^* C \in \mathcal{C}/\mathcal{C}'$. In fact, $j_* j^* C \cong C$ in $\mathcal{C}/\mathcal{C}'$, in view of the following exact sequence

$$0 \longrightarrow i_* i^! C \longrightarrow C \longrightarrow j_* j^* C \longrightarrow 0.$$

3. Cohereditary and hereditary torsion pair of abelian category

In this section, most of the contents are from [6]. One of our main works in this section is to prove [6, Theorem 4.1] in a direct way. And the other is to give our main theorem of this paper.

Recall that a pair $(\mathcal{T}, \mathcal{F})$ of abelian category $\mathcal{C}$ is called torsion pair if (i) $\mathrm{Hom}(\mathcal{T}, \mathcal{F}) = 0$ and (ii) for any C in $\mathcal{C}$, there is a short exact sequence

$$0 \longrightarrow T \longrightarrow C \longrightarrow F \longrightarrow 0$$

with $T \in \mathcal{T}, F \in \mathcal{F}$, where $\mathcal{T}$ is called torsion class and $\mathcal{F}$ is called torsion-free class. A torsion pair $(\mathcal{T},\mathcal{F})$ is cohereditary (resp., hereditary) if $\mathcal{F}$ (resp., $\mathcal{T}$) is closed under factor objects (resp., subobjects).

Recall that a functor $t:\mathcal{C}\to\mathcal{C}$ is called an idempotent radical functor of $\mathcal{C}$ if (i) there exists a natural transformation $\Phi: t\to Id_{\mathcal{C}}$ such that $\Phi_C: tC\to C$ is a monomorphism for any $C\in\mathcal{C}$; (ii) $t(\mathrm{Coker}\Phi_C)=0$ and (iii) $t^2=t$. The definition of idempotent coradical functor r can be dually given.

It is well known that there is a bijective correspondence between torsion pairs and idempotent radical functors (resp., idempotent coradical functors) of $\mathcal{C}$ such that $\mathcal{T}=\{C\in\mathcal{C}|tC=C\}$ and $\mathcal{F}=\{C\in\mathcal{C}|tC=0\}$ (resp., $\mathcal{T}=\{C\in\mathcal{C}|rC=0\}$ and $\mathcal{F}=\{C\in\mathcal{C}|rC=C\}$). Readers can refer to [7–9] for more knowledge on torsion theory.

Let $i_*:\mathcal{T}\to\mathcal{C}$ and $j_*:\mathcal{F}\to\mathcal{C}$ be inclusion functors. And let $i^!:\mathcal{C}\to\mathcal{T}$ induced by $t:\mathcal{C}\to\mathcal{C}$ and $j^*:\mathcal{C}\to\mathcal{F}$ induced by $r:\mathcal{C}\to\mathcal{C}$. Then we have the following diagram of funtors

$$\begin{tikzcd} \mathcal{T} \arrow[r, shift left, "i_*"] & \mathcal{C} \arrow[l, shift left, "i^!"] \arrow[r, shift left, "j^*"] & \mathcal{F} \arrow[l, shift left, "j_*"] \end{tikzcd}$$

Diagram 3 Functors diagram

satisfying the following conditions:

(i) $(i_*,i^!)$ and (j^*,j_*) are adjoint pairs;

(ii) i_* and j_* are fully faithful; and

(iii) $\mathrm{Im}i_*=\mathrm{Ker}j^*$.

But in general, this diagram is not a right Bi-Giraud recolloments, even not a right recollements, because $\mathcal{T},\mathcal{F}$ are not abelian categories. In order to turn the diagram above into a right Bi-Giraud recollement, we need to make the following preparations.

In the following, we only state and prove what is on cohereditary torsion pair. On hereditary torsion pair, it has dual results.

Lemma 3.1 ([6]) *(1) Let $(\mathcal{T},\mathcal{F})$ be cohereditary torsion pair of $\mathcal{C}$. If $f:X\to M$ is epimorphic with $M\in\mathcal{T}$, then $f\Phi_X:tX\to M$ is also epimorphic.*

(2) Let $(\mathcal{T},\mathcal{F})$ be torsion pair of $\mathcal{C}$. Then $(\mathcal{T},\mathcal{F})$ is cohereditary if and only if t preserves epimorphisms.

Recall an object $C\in\mathcal{C}$ is called codivisible (resp., divisible) with respect to torsion pair $(\mathcal{T},\mathcal{F})$ if $\mathrm{Hom}(C,-)$ $(\mathrm{resp.},\mathrm{Hom}(-,C))$ is exact on all short exact sequences

$$0\longrightarrow X'\longrightarrow X\longrightarrow X''\longrightarrow 0$$

in $\mathcal{C}$ with $X'\in\mathcal{F}$ $(\mathrm{resp.}, X''\in\mathcal{T})$.

Notations With respect to torsion pair $(\mathcal{T},\mathcal{F})$ of $\mathcal{C}$,

$$\mathcal{C}':=\{C\in\mathcal{C}|C\text{ torsion and codivisible objects}\};$$

$$\mathcal{C}'':=\{C\in\mathcal{C}|C\text{ torsionfree and divisible objects}\}.$$

Lemma 3.2 *Let $(\mathcal{T},\mathcal{F})$ be a torsion pair of $\mathcal{C}$. If a short exact sequence*

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

has the middle term $X \in \mathcal{C}'$, then $X' \in \mathcal{T}$ if and only if $X'' \in \mathcal{C}'$.

Proof Suppose that $X' \in \mathcal{T}$. It is clear that $X'' \in \mathcal{T}$ because $X \in \mathcal{T}$. Now, we need to show that X'' is codivisible with respect to $(\mathcal{T},\mathcal{F})$. Taking a short exact sequence $0 \to F \to Y \to Z \to 0$ with $F \in \mathcal{F}$, we have the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & (X'',F) \arrow[r] \arrow[d] & (X'',Y) \arrow[r] \arrow[d] & (X'',Z) \arrow[d] & \\
0 \arrow[r] & (X,F) \arrow[r] \arrow[d] & (X,Y) \arrow[r] \arrow[d] & (X,Z) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (X',F) \arrow[r] & (X',Y) \arrow[r] & (X',Z) &
\end{tikzcd}$$

Diagram 4 Commutative diagram

where each row and column is exact, and $\mathrm{Hom}(X',F) = 0$. So $\mathrm{Hom}(X',Y) \to \mathrm{Hom}(X',Z)$ is monomorphic, implying $\mathrm{Hom}(X'',Y) \to \mathrm{Hom}(X'',Z)$ is epimorphic. This shows that X'' is codivisible.

Conversely, assume that $X'' \in \mathcal{C}'$. We need to prove $\mathrm{Hom}(X',F) = 0$ for any $F \in \mathcal{F}$, which is equivalent to prove $\mathrm{Im}g = 0$ for any $g \in \mathrm{Hom}(X',F)$. Now, consider the following push-out diagram

$$\begin{tikzcd}
0 \arrow[r] & X' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Im}g \arrow[r] & M \arrow[r] & X'' \arrow[r] & 0
\end{tikzcd}$$

Diagram 5 Push-out diagram

It is easy to see that the second exact row is splitting, since $\mathrm{Im}g \in \mathcal{F}$ and X'' is codivisible, which results in $\mathrm{Im}g \in \mathcal{T}$ and $\mathrm{Im}g = 0$.

Now, we give our direct proof of [6, Theorem 4.1].

Theorem 3.3 *Let $(\mathcal{T},\mathcal{F})$ be a torsion pair of $\mathcal{C}$. Then the following are equivalent:*

(1) $(\mathcal{T},\mathcal{F})$ is cohereditary and hereditary;

(2) The idempotent radical (resp., coradical) functor t (resp., r) is exact;

(3) $\mathcal{T} = \mathcal{C}'$ (resp., $\mathcal{F} = \mathcal{C}''$).

Proof (1) $\Rightarrow$ (2). It is the direct corollary of Lemma 3.1.

(2) $\Rightarrow$ (3). In fact, we only need $\mathcal{T} \subseteq \mathcal{C}'$, i.e., every $T \in \mathcal{T}$ is codivisible. Given an exact sequence $0 \to F \to Y \to Z \to 0$ with $F \in \mathcal{F}$, and a morphism $h : T \to Z$. Then we have the following commutative diagram

$$\begin{tikzcd}
 & & & T \arrow[d] & \\
0 \arrow[r] & tF \arrow[r] \arrow[d] & tM \arrow[r] \arrow[d] & t\mathrm{Im}h \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & \mathrm{Im}h \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F \arrow[r] & Y \arrow[r] & Z \arrow[r] & 0
\end{tikzcd}$$

Diagram 6 Commutative diagram

where each row is exact (the exactness of the first row is due to the exactness of t and the second row is the pull-back of the third row). Since $F \in \mathcal{F}$ and $\mathrm{Im}h \in \mathcal{T}$, we get $tF = 0$ and $tM \cong t\mathrm{Im}h = \mathrm{Im}h$. This shows that T is codivisible.

(3) $\Rightarrow$ (1). It follows from Lemma 3.2. $\square$

Until now, we can give our main theorem in this paper.

Theorem 3.4 *There is a bijection between $\mathcal{R}(\mathcal{C}', \mathcal{C}, \mathcal{C}'')$ right Bi-Giraud recollements of abelian categories and $(\mathcal{T}, \mathcal{F})$ cohereditary and hereditary torsion pairs of an abelian category $\mathcal{C}$.*

Proof It follows clearly from Proposition 2.3 and Theorem 3.3. $\square$

4. A method producing cohereditary and hereditary torsion pairs

In this section, we suppose that R is a two-sided noetherian ring with unit, Mod$R$ is the right R-modules category, and $I = I^2$ is an idempotent ideal of R.

Notations

$$\mathcal{T}_I := \{M \in \mathrm{Mod}R | MI = M\};$$

$$\mathcal{F}_I := \{M \in \mathrm{Mod}R | MI = 0\}.$$

It is easy to observe that

Proposition 4.1 *$(\mathcal{T}_I, \mathcal{F}_I)$ is a cohereditary torsion pair of* Mod$R$.

Proof First, taking $M \in \mathcal{T}_I$, $N \in \mathcal{F}_I$ and a morphism $f : M \to N$, then

$$\mathrm{Im}f = f(M) = f(MI) = f(M)I = 0$$

implying $f = 0$ and $\mathrm{Hom}(\mathcal{T}_I, \mathcal{F}_I) = 0$.

Next, for any $M \in$ Mod-R, there is a short exact sequence $0 \to MI \to M \to M/MI \to 0$. By I being an idempotent ideal of R, it is clear that $M = MI \in \mathcal{T}_I$ and $(M/MI)I = 0 \in \mathcal{F}_I$. Until now, we have shown that $(\mathcal{T}_I, \mathcal{F}_I)$ is a torsion pair of Mod$R$.

At last, it is obvious that $\mathcal{F}_I$ is closed under factor objects. $\square$

Remark 4.2 In fact, the result in Proposition 4.1 also holds without the assumption that

R is a two-sided noetherian ring. And $\mathcal{T}_I$ is still a torsion class of Mod$R$; $\mathcal{F}_I$ is closed under subobjects and factor objects under the weaker condition that I is an ideal of R. In other words, The assumption that I is an idempotent ideal of R guarantees that $\mathcal{F}_I$ is closed under extensions.

From this Proposition above, we know that an idempotent ideal I of R can produce a cohereditary torsion pair. The following theorem answers that an idempotent ideal I can produce a cohereditary and hereditary torsion pair when I as a left R-module is projective.

Theorem 4.3 *Let R be a two-sided noetherian ring and $I = I^2$ an idempotent ideal of R. Then $\mathcal{T}_I$ is closed under subobjects if and only if I is projective as a left R-module.*

Proof $\Longrightarrow$. Let $\mathcal{T}_I$ be closed under subobjects. By R being noetherian, I is a finitely generated left R-module. So it only needs to prove that I is flat. For any short exact sequence in Mod$R$

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0,$$

we can get the following exact sequence

$$0 \longrightarrow \mathrm{Ker}(f \otimes I) \longrightarrow L \otimes I \longrightarrow M \otimes I \longrightarrow N \otimes I \longrightarrow 0.$$

We need to show that $\mathrm{Ker}(f \otimes I) = 0$. On the one hand, $(L \otimes I)I = L \otimes I^2 = L \otimes I \in \mathcal{T}_I$, which implies $\mathrm{Ker}(f \otimes I) \in \mathcal{T}_I$ by hypothesis. On the other hand, for any element $\sum l_i \otimes r_i \in \ker(f \otimes I)$, we have $0 = (f \otimes I)(\sum l_i \otimes r_i) = \sum f(l_i) \otimes r_i \in M \otimes I$ and $0 = \sum f(l_i)r_i = f(\sum l_i r_i) \in MI$. So $\sum l_i r_i = 0$ by $f : L \to M$ monomorphism. Moreover, for any $r \in I$,

$$(\sum l_i \otimes r_i)r = \sum l_i \otimes r_i r = (\sum l_i r_i) \otimes r = 0$$

which infers that $\mathrm{Ker}(f \otimes I)I = 0$, i.e., $\mathrm{Ker}(f \otimes I) \in \mathcal{F}_I$.

$\Longleftarrow$. If I is projective as a left R-module, $M \otimes I \cong MI$ for any $M \in$ Mod$R$. Taking any $M \in \mathcal{T}_I$ and short exact sequence$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$, we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L \otimes I & \longrightarrow & M \otimes I & \longrightarrow & N \otimes I & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & LI & \longrightarrow & MI & \longrightarrow & NI & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

Diagram 7 Commutative diagram

where each row is exact. By the five lemma, we can get $LI = L \in \mathcal{T}_I$. □

Combining Proposition 4.1 with Theorem 4.3, it follows that

Theorem 4.4 *Let R be a two-sided noetherian ring and $I = I^2$ an idempotent ideal of R. Then $(\mathcal{T}_I, \mathcal{F}_I)$ is cohereditary and hereditary if and only if I is projective as a left R-module.*

Now, removing the restriction of R being a two-sided noetherian ring, and only demanding that R is a ring with unit, we can obtain the following fact:

Proposition 4.5 *Let R be a ring and I an ideal of R. Then I is an idempotent ideal if and only if $\mathcal{T}_I = \mathrm{Gen}(I_R)$, where $\mathrm{Gen}(I_R) = \{M \in \mathrm{Mod}R | \sum \mathrm{Im}(f) = M, f \in \mathrm{Hom}(I, M)\}$.*

Proof The part of if is obvious.

For the part of only if, let I be an idempotent ideal. Then $I = I^2 \in \mathcal{T}_I$, and it is easy to check that $\mathrm{Gen}(I_R) \subseteq \mathcal{T}_I$. Before using $\mathcal{T}_I \subseteq \mathrm{Gen}(I_R)$, we consider morphisms $\psi_{r_0} : I \to I$ for any $r_0 \in R$ by assigning $r_0 r$ to r for any $r \in I$ and morphisms $f_m : I \to M \in \mathrm{Mod}R$ for any $m \in M$ by assigning mr to r for any $r \in I$. It is easy to verify that ψ_{r_0} and f_m are right R-module homomorphisms, rendering $f_m \psi_{r_0} \in \mathrm{Hom}(I, M)$.

Now, for any $M \in \mathcal{T}_I$, $M = MI$. So for any element $m \in M$, there exist $m_i \in M$, $r_i \in I$, $i = 1, \dots, t$ such that $m = m_1 r_1 + \cdots + m_t r_t$, i.e.,

$$m = \sum m_i r_i = \sum f_{m_i}(r_i) = \sum f_{m_i} \psi_{1_R}(r_i) \in \sum \mathrm{Im} f,$$

where f takes over $\mathrm{Hom}(I, M)$. Thus $M \in \mathrm{Gen}(I_R)$. $\square$

Next example explains that there are non-trivial right bi-Giraud recollements and not all right recollements are right bi-Giraud.

Example 4.6 Let R and S be rings and ${}_S M_R$ ($\neq 0$) an S-R-bimodule. Then we can construct the triangular matrix ring $\Lambda = \left(\begin{smallmatrix} R & 0 \\ M & S \end{smallmatrix}\right)$ and describe the right Λ-modules as triples (X_R, Y_S, f) where X_R is a right R-module, Y_S a right S-module and $f : Y \otimes_S M \to X$ a right R-module morphism. The morphisms between two objects (X_R, Y_S, f) and (X'_R, Y'_S, f') are pair of morphisms (α, β) where $\alpha : X \to X'$ is an R-morphism and $\beta : Y \to Y'$ is an S-morphism, such that the following diagram commutes

$$\begin{tikzcd} Y \otimes_S M \arrow[r, "\beta\otimes M"] \arrow[d, "f"'] & Y' \otimes_S M \arrow[d, "f'"] \\ X \arrow[r, "\alpha"'] & X'. \end{tikzcd}$$

Diagram 8 Commutative diagram

Now, we begin to define the following functors:

(1) The functor $T_R : \mathrm{Mod}R \to \mathrm{Mod}\Lambda$ is defined by $T_R(X) = (X, 0, 0)$ on the objects $X \in \mathrm{Mod}R$ and given an R-morphism $\alpha : X \to X'$ then $T_R(\alpha) = (\alpha, 0)$.

(2) The functor $H_R : \mathrm{Mod}R \to \mathrm{Mod}\Lambda$ is defined by $H_R(X) = (X, \mathrm{Hom}_R(M, X), \varepsilon_X)$ on the objects $X \in \mathrm{Mod}R$ where $\varepsilon_X : \mathrm{Hom}_R(M, X) \otimes_S M \to X$ is standard, and given an R-morphism $\alpha : X \to X'$ then $H_R(\alpha) = (\alpha, \mathrm{Hom}_R(M, \alpha))$.

(3) The functor $U_R : \mathrm{Mod}\Lambda \to \mathrm{Mod}R$ is defined by $U_R(X, Y, f) = X$ on the objects $(X, Y, f) \in \mathrm{Mod}\Lambda$ and given an Λ-morphism $(\alpha, \beta) : (X, Y, f) \to (X', Y', f')$ then $U_R(\alpha, \beta) = \alpha$.

(4) The functor $T_S : \mathrm{Mod}S \to \mathrm{Mod}\Lambda$ is defined by $T_S(Y) = (Y \otimes_S M, Y, \mathrm{Id}_{Y \otimes_S M})$ on the objects $Y \in \mathrm{Mod}S$ and given an S-morphism $\beta : Y \to Y'$ then $T_S(\beta) = (\beta \otimes M, \beta)$.

(5) The functor $H_S : \mathrm{Mod}S \to \mathrm{Mod}\Lambda$ is defined by $H_S(Y) = (0, Y, 0)$ on the objects $Y \in \mathrm{Mod}S$ and given an S-morphism $\beta : Y \to Y'$ then $H_S(\beta) = (0, \beta)$.

(6) The functor U_S : Mod$\Lambda \to$ Mod$S$ is defined by $U_S(X,Y,f) = Y$ on the objects $(X,Y,f) \in$ ModΛ and given an Λ-morphism $(\alpha,\beta) : (X,Y,f) \to (X',Y',f')$ then $U_S(\alpha,\beta) = \beta$.

It is easy to know that

(i) The functors T_R, T_S and H_R, H_S are fully faithful;

(ii) The pairs $(T_R, U_R), (T_S, U_S)$ and $(U_R, H_R), (U_S, H_S)$ are adjoint pairs of functors.

(iii) The functors U_R and U_S are exact.

(iv) We have KerU_R = Mod$\Lambda/\Lambda e_1\Lambda \cong$ Mod$S$, KerU_S = Mod$\Lambda/\Lambda e_2\Lambda \cong$ Mod$R$, Mod$e_1\Lambda e_1$ = Mod$R$ and Mod$e_2\Lambda e_2$ = Mod$S$, where $e_1 = \left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$ and $e_2 = \left(\begin{smallmatrix}0 & 0\\ 0 & 1\end{smallmatrix}\right)$ are idempotent elements of Λ.

So by Definition 2.1, we get right (left) bi-Giraud recollements

$$\begin{tikzcd}\mathrm{Mod}S \arrow[r, shift right, "H_S"'] & \mathrm{Mod}\Lambda \arrow[l, shift right, "U_S"'] \arrow[r, shift right, "U_R"'] & \mathrm{Mod}R \arrow[l, shift right, "T_R"']\end{tikzcd}; \qquad \begin{tikzcd}\mathrm{Mod}R \arrow[r, shift left, "T_R"] & \mathrm{Mod}\Lambda \arrow[l, shift left, "U_R"] \arrow[r, shift left, "U_S"] & \mathrm{Mod}S. \arrow[l, shift left, "H_S"]\end{tikzcd}$$

At the same time, we also point out that not all right (left) recollements are bi-Giraud as follows:

$$\begin{tikzcd}\mathrm{Mod}S \arrow[r, shift left, "H_S"] & \mathrm{Mod}\Lambda \arrow[l, shift left] \arrow[r, shift left, "U_R"] & \mathrm{Mod}R \arrow[l, shift left, "H_R"]\end{tikzcd}; \qquad \begin{tikzcd}\mathrm{Mod}R \arrow[r, shift right, "T_R"'] & \mathrm{Mod}\Lambda \arrow[l, shift right] \arrow[r, shift right, "U_S"'] & \mathrm{Mod}S. \arrow[l, shift right, "T_S"']\end{tikzcd}$$

We have proved that these two are right (left) bi-Giraud recollements if and only if the S-R-bimodule ${}_SM_R = 0$.

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