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Double Traveling Wave Solutions of the Coupled Nonlinear Klein-Gordon Equations and the Coupled Schrödinger-Boussinesq Equation

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Abstract The new multiple $(\frac{G'}{G})$-expansion method is proposed in this paper to seek the exact double traveling wave solutions of nonlinear partial differential equations. With the aid of symbolic computation, this new method is applied to construct double traveling wave solutions of the coupled nonlinear Klein-Gordon equations and the coupled Schrödinger-Boussinesq equation. As a result, abundant double traveling wave solutions including double hyperbolic tangent function solutions, double tangent function solutions, double rational solutions, and a series of complexiton solutions of these two equations are obtained via this new method. The new multiple $(\frac{G'}{G})$-expansion method not only gets new exact solutions of equations directly and effectively, but also expands the scope of the solution. This new method has a very wide range of application for the study of nonlinear partial differential equations.

Keywords the new multiple $(\frac{G'}{G})$-expansion; the coupled nonlinear Klein-Gordon equations; the coupled Schrödinger-Boussinesq equation; double traveling wave solutions

MR(2010) Subject Classification 35C07; 35C08

1. Introduction

The research of nonlinear science is the frontier and hot spot in the field of natural science at present. Nonlinear partial differential equations (NPDEs) are widely used as models to express many nonlinear phenomena which exist in many fields, such as physics, fluid mechanics, atmospheric science, information science, life science, and water systems science, plasma physics, condensed matter physics. So it plays a vital role to seek exact solution for partial differential equations. During the past decades, many powerful methods to construct solitary wave solutions of nonlinear partial differential equations have been established and developed such as the

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Bäcklund transformation [1], Hirota bilinear transformation [2], Darboux transformation [3], the tanh-coth method [4], the F-expansion method [5], the exp-function method [6], the inverse scattering method [7], the mapping method [8], generalized tanh functions method [9], the auxiliary equation method [10,11], the Jacobi elliptic function expansion method [12], the $(\frac{G'}{G})$-expansion method [13], the first integral method [14], the sine-cosine method [15], and so on. The author also constructed the asymptotic solutions of a series of nonlinear equations by a variety of methods, such as, the generalized variational iteration method [16], the functional mapping method [17], the homotopy mapping method [18–20], the singular perturbation method [21,22] and so on.

Among these methods, the $(\frac{G'}{G})$-expansion method is direct and effective to construct exact solutions of NPDEs [13]. Later, this method was improved and applied to a series of NPDEs successfully [23–25]. However, we find that these methods can only get single traveling wave solutions of NPDEs. So we propose a new multiple $(\frac{G'}{G})$-expansion method to construct double traveling wave solutions of NPDEs in this paper. For illustration, we apply this new method to the coupled nonlinear Klein-Gordon equations and the coupled Schrödinger-Boussinesq equation. As a result, a rich variety of exact solutions which include double hyperbolic tangent function solutions, double tangent function solutions, double rational solutions and complexiton solutions of the above two equations are obtained via the new multiple $(\frac{G'}{G})$-expansion method.

The rest of this letter is organized as follows. In Section 2, we describe the new multiple $(\frac{G'}{G})$-expansion method. In Section 3, we illustrate the applications of this method to the coupled nonlinear Klein-Gordon equations. In Section 4, we apply this method to the coupled Schrödinger-Boussinesq equation. In Section 5, some conclusions are given.

2. Description of the new multiple $(\frac{G'}{G})$-expansion method

In this section, we describe the main steps of the new multiple $(\frac{G'}{G})$-expansion method for double traveling wave solutions of NPDEs. For a given NPDE with independent variables $u=(x_1,x_2,\ldots,x_n,t)$ and dependent variable u:

$$F(u,u_t,u_{x_1},u_{x_2},\ldots,u_{tt},\ldots,u_{x_1x_1},u_{x_2x_2},\ldots)=0, \tag{1}$$

where $u=(x_1,x_2,\ldots,x_n,t)$ is an unknown function, F is a polynomial in $u=(x_1,x_2,\ldots,x_n,t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved.

Step 1. Combining the independent variables $x_1,x_2,\ldots,x_n$ and t into one variable $\varsigma = k_1x_1+k_2x_2+\cdots+k_nx_n-st$, we suppose that

$$\varsigma = k_1x_1+k_2x_2+\cdots+k_nx_n-st,\quad u=u(\varsigma), \tag{2}$$

the travelling wave variable (2) permits us reducing Eq. (1) to an ordinary differential equation (ODE) for $u=u(\varsigma)$

$$H(u,-su',k_1u',k_2u',\ldots,s^2u'',-k_1su'',\ldots,{k_1}^2u'',{k_2}^2u'',\ldots)=0. \tag{3}$$

Step 2. We suppose that the solution of ODE (3) can be expressed by a polynomial in $(\frac{G_1'}{G_1})^i(\frac{G_2'}{G_2})^j$ $(i,j=0,1,2\ldots)$ as follows:

$$u(\varsigma)=a_0+\sum_{k=1}^{n}\sum_{i+j=k}a_{ij}(\frac{G_1'(\xi)}{G_1(\xi)})^i(\frac{G_2'(\eta)}{G_2(\eta)})^j, \tag{4}$$

where a_0, a_{ij} $(i,j=0,1,2)$ are constants to be determined later, $G_1=G_1(\xi)$ and $G_2=G_2(\eta)$ satisfy the following second order nonlinear ordinary differential equations

$$A_1(G_1')^2-B_1G_1G_1''+C_1G_1^2=0,\quad A_2(G_2')^2-B_2G_2G_2''+C_2G_2^2=0, \tag{5}$$

where $A_1, A_2, B_1, B_2, C_1, C_2$ are constants, and $(A_1-A_2)^2+(B_1-B_2)^2+(C_1-C_2)^2\neq 0$.

Step 3. Determine the positive integer n by balancing the highest order derivatives and nonlinear terms in Eq. (3).

Step 4. Substituting ansätz (4) along with Eq. (5) into Eq. (3) and collecting all the terms with the same order of $(\frac{G_1'}{G_1})^i(\frac{G_2'}{G_2})^j$ $(i,j=0,1,2\ldots)$, we convert the left-hand side of Eq. (3) into a polynomial in $(\frac{G_1'}{G_1})^i(\frac{G_2'}{G_2})^j$ $(i,j=0,1,2\ldots)$. Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for a_0, a_{ij} $(i,j=0,1,2\ldots), k_1,\ldots,k_n, s, A_1, A_2, B_1,\ldots,C_2$.

Step 5. Assuming that the constants a_0, a_{ij} $(i,j=0,1,2\ldots), k_1,\ldots,k_n, s, A_1, A_2, B_1,\ldots,C_2$. can be obtained by solving the algebraic equations in Step 4. Since the general solutions of the second order nonlinear ordinary differential equation (5) can be solved, then substituting a_0, a_{ij} $(i,j=0,1,2,\ldots), k_1,\ldots,k_n, s$ and the known general solutions of Eq. (5) into ansätz (4), we can obtain the exact double traveling wave solutions of Eq. (1) immediately.

3. Application to the coupled nonlinear Klein-Gordon equations

The coupled nonlinear Klein-Gordon equations read [26,27]

$$\begin{cases} u_{tt}-c_0^2\nabla^2u+\alpha_1u-\beta_1u^3-\gamma_1uv^2=0,\\ v_{tt}-c_0^2\nabla^2v+\alpha_2v-\beta_2v^3-\gamma_2u^2v=0,\end{cases} \tag{6}$$

which play an important role in modern physics, where

$$\nabla^2=\frac{\partial^2}{\partial^2x}+\frac{\partial^2}{\partial^2y}+\frac{\partial^2}{\partial^2z}, \tag{7}$$

is the Laplace operator.

We seek the travelling wave solutions of Eq. (6) in the form

$$u(x,y,z,t)=u(\varsigma),\quad v(x,y,z,t)=u(\varsigma),\quad \varsigma=kx+ly+pz-\omega t, \tag{8}$$

where $\vec{K}=(k,l,p)$ and ω are the wave vector and angular frequency.

Substituting (8) into (6), we have

$$Au''+\alpha_1u-\beta_1u^3-\gamma_1uv^2=0, \tag{9}$$

$$Av''+\alpha_2v-\beta_2v^3-\gamma_2u^2v=0, \tag{10}$$

where

$$A=\omega^2-K^2c_0^2,\quad K^2=\vec{K}\cdot\vec{K}=k^2+l^2+p^2. \tag{11}$$

By considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in Eqs. (9) and (10), we obtain $m = 1$ for u and $n = 1$ for v. Suppose that the solutions for Eqs. (9) and (10) can be expressed in the following form

$$u = a_0 + a_{10}(\frac{G_1'}{G_1}) + a_{01}(\frac{G_2'}{G_2}), \tag{12}$$

$$u = b_0 + b_{10}(\frac{G_1'}{G_1}) + b_{01}(\frac{G_2'}{G_2}), \tag{13}$$

where $G_1 = G_1(\xi)$, $G_2 = G_2(\eta)$ satisfies Eq. (5), and $a_0, a_{10}, a_{01}, b_0, b_{10}, b_{01}$ are constants to be determined later.

Substituting (12) and (13) along with Eq.(5) into Eqs.(9) and (10), and collecting all terms with the same power of $(\frac{G_1'}{G_1})^i(\frac{G_2'}{G_2})^j$ $(i, j = 0, 1, 2, \ldots)$ together, the left-hand side of Eqs.(9) and (10) is converted into another polynomial $(\frac{G_1'}{G_1})^i(\frac{G_2'}{G_2})^j$ $(i, j = 0, 1, 2, \ldots)$. Equating each coefficient of this polynomial to zero, yields a set of simultaneous algebraic equations for a_0, a_{ij} $(i, j = 0, 1, 2, \ldots), k_1, \ldots, k_n, s, A_1, A_2, B_1, \ldots, C_2$.

Solving the algebraic equations above, yields

$$a_0 = 0,\ a_{10} = \pm\sqrt{\frac{2A(\frac{A_1}{B_1} - 1)^2(\beta_2 - \gamma_1)}{\beta_1\beta_2 - \gamma_1\gamma_2}},\ a_{01} = \pm\sqrt{\frac{2A(\frac{A_1}{B_1} - 1)^2(\beta_2 - \gamma_1)}{\beta_1\beta_2 - \gamma_1\gamma_2}},\ a_{11} = 0,$$

$$b_0 = 0,\ b_{10} = \mp\sqrt{\frac{2A(\frac{A_1}{B_1} - 1)^2(\beta_1 - \gamma_2)}{\beta_1\beta_2 - \gamma_1\gamma_2}},\ b_{01} = \pm\sqrt{\frac{2A(\frac{A_2}{B_2} - 1)^2(\beta_1 - \gamma_2)}{\beta_1\beta_2 - \gamma_1\gamma_2}},\ b_{11} = 0,$$

$$\beta_1 = \frac{\gamma_2\beta_2}{\gamma_1},\ \alpha_1 = \alpha_2 = -2A(\frac{A_1}{B_1} - 1)\frac{C_1}{B_1},\ A_1 = (\frac{A_1}{B_1} - 1)\frac{C_2}{B_2}\frac{B_1^2}{C_1} + B_1.$$

Substituting (14) along with Eq. (5) into Eqs.(12) and (13), from (8), we obtain plentiful double traveling wave solutions consisting of hyperbolic functions, trigonometric functions, rational functions, and their mixture with arbitrary parameters as the follows:

Case 1 When $\begin{cases} \frac{C_1}{B_1}(\frac{A_1}{B_1} - 1) > 0, \\ \frac{C_2}{B_2}(\frac{A_2}{B_2} - 1) > 0, \end{cases}$

$$\begin{aligned} u_1(x,t) = &\pm \sqrt{\frac{2A(\frac{A_1}{B_1} - 1)^2(\beta_2 - \gamma_1)}{\beta_1\beta_2 - \gamma_1\gamma_2}}\sqrt{\frac{C_1}{A_1 - B_1}}\tan(D_1\xi + c_1) \\ &\pm \sqrt{\frac{2A(\frac{A_2}{B_2} - 1)^2(\beta_2 - \gamma_1)}{\beta_1\beta_2 - \gamma_1\gamma_2}}\sqrt{\frac{C_2}{A_2 - B_2}}\tan(D_2\eta + d_1), \\ v_1(x,t) = &\mp \sqrt{\frac{2A(\frac{A_1}{B_1} - 1)^2(\beta_1 - \gamma_2)}{\beta_1\beta_2 - \gamma_1\gamma_2}}\sqrt{\frac{C_1}{A_1 - B_1}}\tan(D_1\xi + c_1) \\ &\pm \sqrt{\frac{2A(\frac{A_2}{B_2} - 1)^2(\beta_1 - \gamma_2)}{\beta_1\beta_2 - \gamma_1\gamma_2}}\sqrt{\frac{C_2}{A_2 - B_2}}\tan(D_2\eta + d_1), \end{aligned}$$

where $D_1 = \frac{C_1}{B_1}\sqrt{\frac{\frac{A_1}{B_1} - 1}{\frac{C_1}{B_1}}}$, $D_2 = \frac{C_2}{B_2}\sqrt{\frac{\frac{A_2}{B_2} - 1}{\frac{C_2}{B_2}}}, c_1, d_1$ are arbitrary constants.

Case 2 When $\begin{cases} \dfrac{C_1}{B_1}(\dfrac{A_1}{B_1}-1)<0, \\ \dfrac{C_2}{B_2}(\dfrac{A_2}{B_2}-1)<0, \end{cases}$

$$\begin{aligned} u_2(x,t) = &\pm\sqrt{\frac{2A(\frac{A_1}{B_1}-1)^2(\beta_2-\gamma_1)}{\beta_1\beta_2-\gamma_1\gamma_2}}\sqrt{\frac{C_1}{B_1-A_1}}\tan h(F_1\xi+c_2) \\ &\pm\sqrt{\frac{2A(\frac{A_2}{B_2}-1)^2(\beta_2-\gamma_1)}{\beta_1\beta_2-\gamma_1\gamma_2}}\sqrt{\frac{C_2}{B_2-A_2}}\tan h(F_2\eta+d_2), \\ v_2(x,t) = &\mp\sqrt{\frac{2A(\frac{A_1}{B_1}-1)^2(\beta_1-\gamma_2)}{\beta_1\beta_2-\gamma_1\gamma_2}}\sqrt{\frac{C_1}{B_1-A_1}}\tan h(F_1\xi+c_2) \\ &\pm\sqrt{\frac{2A(\frac{A_2}{B_2}-1)^2(\beta_1-\gamma_2)}{\beta_1\beta_2-\gamma_1\gamma_2}}\sqrt{\frac{C_2}{B_2-A_2}}\tan h(F_2\eta+d_2), \end{aligned}$$

where $F_1=\frac{C_1}{B_1}\sqrt{\frac{1-\frac{A_1}{B_1}}{\frac{C_1}{B_1}}}$, $F_2=\frac{C_2}{B_2}\sqrt{\frac{1-\frac{A_2}{B_2}}{\frac{C_2}{B_2}}}, c_2, d_2$ are arbitrary constants.

Case 3 When $\begin{cases} \dfrac{C_1}{B_1}(\dfrac{A_1}{B_1}-1)>0, \\ \dfrac{C_2}{B_2}(\dfrac{A_2}{B_2}-1)<0, \end{cases}$

$$\begin{aligned} u_3(x,t) = &\pm\sqrt{\frac{2A(\frac{A_1}{B_1}-1)^2(\beta_2-\gamma_1)}{\beta_1\beta_2-\gamma_1\gamma_2}}\sqrt{\frac{C_1}{A_1-B_1}}\tan(D_1\xi+c_1) \\ &\pm\sqrt{\frac{2A(\frac{A_2}{B_2}-1)^2(\beta_2-\gamma_1)}{\beta_1\beta_2-\gamma_1\gamma_2}}\sqrt{\frac{C_2}{B_2-A_2}}\tan h(F_2\eta+d_2), \\ v_3(x,t) = &\mp\sqrt{\frac{2A(\frac{A_1}{B_1}-1)^2(\beta_1-\gamma_2)}{\beta_1\beta_2-\gamma_1\gamma_2}}\sqrt{\frac{C_1}{A_1-B_1}}\tan(D_1\xi+c_1) \\ &\pm\sqrt{\frac{2A(\frac{A_2}{B_2}-1)^2(\beta_1-\gamma_2)}{\beta_1\beta_2-\gamma_1\gamma_2}}\sqrt{\frac{C_2}{B_2-A_2}}\tan h(F_2\eta+d_2), \end{aligned}$$

where $D_1=\frac{C_1}{B_1}\sqrt{\frac{\frac{A_1}{B_1}-1}{\frac{C_1}{B_1}}}$, $F_2=\frac{C_2}{B_2}\sqrt{\frac{1-\frac{A_2}{B_2}}{\frac{C_2}{B_2}}}, c_1, d_2$ are arbitrary constants.

Case 4 When $\begin{cases} \dfrac{C_1}{B_1}(\dfrac{A_1}{B_1}-1)<0, \\ \dfrac{C_2}{B_2}(\dfrac{A_2}{B_2}-1)>0, \end{cases}$

$$\begin{aligned} u_4(x,t) = &\pm\sqrt{\frac{2A(\frac{A_1}{B_1}-1)^2(\beta_2-\gamma_1)}{\beta_1\beta_2-\gamma_1\gamma_2}}\sqrt{\frac{C_1}{B_1-A_1}}\tan h(F_1\xi+c_2) \\ &\pm\sqrt{\frac{2A(\frac{A_2}{B_2}-1)^2(\beta_2-\gamma_1)}{\beta_1\beta_2-\gamma_1\gamma_2}}\sqrt{\frac{C_2}{A_2-B_2}}\tan(D_2\eta+d_1), \end{aligned}$$

$$v_4(x,t)=\mp\sqrt{\frac{2A(\frac{A_1}{B_1}-1)^2(\beta_1-\gamma_2)}{\beta_1\beta_2-\gamma_1\gamma_2}}\sqrt{\frac{C_1}{B_1-A_1}}\tan h(F_1\xi+c_2)$$
$$\pm\sqrt{\frac{2A(\frac{A_2}{B_2}-1)^2(\beta_1-\gamma_2)}{\beta_1\beta_2-\gamma_1\gamma_2}}\sqrt{\frac{C_2}{A_2-B_2}}\tan(D_2\eta+d_1),$$

where $F_1=\frac{C_1}{B_1}\sqrt{\frac{1-\frac{A_1}{B_1}}{\frac{C_1}{B_1}}}$, $D_2=\frac{C_2}{B_2}\sqrt{\frac{\frac{A_2}{B_2}-1}{\frac{C_2}{B_2}}}$, c_1,d_1 are arbitrary constants.

Case 5 When $\begin{cases}\frac{C_1}{B_1}(\frac{A_1}{B_1}-1)>0,\\ \frac{C_2}{B_2}(\frac{A_2}{B_2}-1)=0\ (\frac{C_2}{B_2}=0),\end{cases}$

$$u_5(x,t)=\pm\sqrt{\frac{2A(\frac{A_1}{B_1}-1)^2(\beta_2-\gamma_1)}{\beta_1\beta_2-\gamma_1\gamma_2}}\sqrt{\frac{C_1}{A_1-B_1}}\tan(D_1\xi+c_1)$$
$$\pm\sqrt{\frac{2A(\frac{A_2}{B_2}-1)^2(\beta_2-\gamma_1)}{\beta_1\beta_2-\gamma_1\gamma_2}}\frac{-1}{(\frac{A_2}{B_2}-1)\eta+d_3},$$
$$v_5(x,t)=\mp\sqrt{\frac{2A(\frac{A_1}{B_1}-1)^2(\beta_1-\gamma_2)}{\beta_1\beta_2-\gamma_1\gamma_2}}\sqrt{\frac{C_1}{A_1-B_1}}\tan(D_1\xi+c_1)$$
$$\pm\sqrt{\frac{2A(\frac{A_2}{B_2}-1)^2(\beta_1-\gamma_2)}{\beta_1\beta_2-\gamma_1\gamma_2}}\frac{-1}{(\frac{A_2}{B_2}-1)\eta+d_3},$$

where $D_1=\frac{C_2}{B_2}\sqrt{\frac{\frac{A_2}{B_2}-1}{\frac{C_2}{B_2}}}$, c_1,d_3 are arbitrary constants.

Case 6 When $\begin{cases}\frac{C_1}{B_1}(\frac{A_1}{B_1}-1)>0,\\ \frac{C_2}{B_2}(\frac{A_2}{B_2}-1)=0\ (\frac{A_2}{B_2}-1=0),\end{cases}$

$$u_6(x,t)=\pm\sqrt{\frac{2A(\frac{A_1}{B_1}-1)^2(\beta_2-\gamma_1)}{\beta_1\beta_2-\gamma_1\gamma_2}}\sqrt{\frac{C_1}{A_1-B_1}}\tan(D_1\xi+c_1)$$
$$\pm\sqrt{\frac{2A(\frac{A_2}{B_2}-1)^2(\beta_2-\gamma_1)}{\beta_1\beta_2-\gamma_1\gamma_2}}(\frac{C_2}{B_2}\eta+d_4),$$
$$v_6(x,t)=\mp\sqrt{\frac{2A(\frac{A_1}{B_1}-1)^2(\beta_1-\gamma_2)}{\beta_1\beta_2-\gamma_1\gamma_2}}\sqrt{\frac{C_1}{A_1-B_1}}\tan(D_1\xi+c_1)$$
$$\pm\sqrt{\frac{2A(\frac{A_2}{B_2}-1)^2(\beta_1-\gamma_2)}{\beta_1\beta_2-\gamma_1\gamma_2}}(\frac{C_2}{B_2}\eta+d_4),$$

where $D_1=\frac{C_1}{B_1}\sqrt{\frac{\frac{A_1}{B_1}-1}{\frac{C_1}{B_1}}}$, c_1,d_4 are arbitrary constants.

Case 7 When $\begin{cases} \frac{C_1}{B_1}(\frac{A_1}{B_1}-1)<0, \\ \frac{C_2}{B_2}(\frac{A_2}{B_2}-1)=0 \ (\frac{C_2}{B_2}=0), \end{cases}$

$$\begin{aligned} u_7(x,t) = &\pm \sqrt{\frac{2A(\frac{A_1}{B_1}-1)^2(\beta_2-\gamma_1)}{\beta_1\beta_2-\gamma_1\gamma_2}}\sqrt{\frac{C_1}{B_1-A_1}}\tan h(F_1\xi+c_2) \\ &\pm \sqrt{\frac{2A(\frac{A_2}{B_2}-1)^2(\beta_2-\gamma_1)}{\beta_1\beta_2-\gamma_1\gamma_2}}\frac{-1}{(\frac{A_2}{B_2}-1)\eta+d_3}, \\ v_7(x,t) = &\mp \sqrt{\frac{2A(\frac{A_1}{B_1}-1)^2(\beta_1-\gamma_2)}{\beta_1\beta_2-\gamma_1\gamma_2}}\sqrt{\frac{C_1}{B_1-A_1}}\tan h(F_1\xi+c_2) \\ &\pm \sqrt{\frac{2A(\frac{A_2}{B_2}-1)^2(\beta_1-\gamma_2)}{\beta_1\beta_2-\gamma_1\gamma_2}}\frac{-1}{(\frac{A_2}{B_2}-1)\eta+d_3}, \end{aligned}$$

where $F_1 = \frac{C_1}{B_1}\sqrt{\frac{1-\frac{A_1}{B_1}}{\frac{C_1}{B_1}}}$, c_2, d_3 are arbitrary constants.

Case 8 When $\begin{cases} \frac{C_1}{B_1}(\frac{A_1}{B_1}-1)<0, \\ \frac{C_2}{B_2}(\frac{A_2}{B_2}-1)=0 \ (\frac{A_2}{B_2}-1=0), \end{cases}$

$$\begin{aligned} u_8(x,t) = &\pm \sqrt{\frac{2A(\frac{A_1}{B_1}-1)^2(\beta_2-\gamma_1)}{\beta_1\beta_2-\gamma_1\gamma_2}}\sqrt{\frac{C_1}{B_1-A_1}}\tan h(F_1\xi+c_2) \\ &\pm \sqrt{\frac{2A(\frac{A_2}{B_2}-1)^2(\beta_2-\gamma_1)}{\beta_1\beta_2-\gamma_1\gamma_2}}(\frac{C_2}{B_2}\eta+d_4), \\ v_8(x,t) = &\mp \sqrt{\frac{2A(\frac{A_1}{B_1}-1)^2(\beta_1-\gamma_2)}{\beta_1\beta_2-\gamma_1\gamma_2}}\sqrt{\frac{C_1}{B_1-A_1}}\tan h(F_1\xi+c_2) \\ &\pm \sqrt{\frac{2A(\frac{A_2}{B_2}-1)^2(\beta_1-\gamma_2)}{\beta_1\beta_2-\gamma_1\gamma_2}}(\frac{C_2}{B_2}\eta+d_4), \end{aligned}$$

where $F_1 = \frac{C_1}{B_1}\sqrt{\frac{1-\frac{A_1}{B_1}}{\frac{C_1}{B_1}}}$, c_2, d_4 are arbitrary constants.

Case 9 When $\begin{cases} \frac{C_1}{B_1}(\frac{A_1}{B_1}-1)=0 \ (\frac{C_1}{B_1}=0), \\ \frac{C_2}{B_2}(\frac{A_2}{B_2}-1)>0, \end{cases}$

$$\begin{aligned} u_9(x,t) = &\pm \sqrt{\frac{2A(\frac{A_1}{B_1}-1)^2(\beta_2-\gamma_1)}{\beta_1\beta_2-\gamma_1\gamma_2}}\frac{-1}{(\frac{A_1}{B_1}-1)\xi+c_3} \\ &\pm \sqrt{\frac{2A(\frac{A_2}{B_2}-1)^2(\beta_2-\gamma_1)}{\beta_1\beta_2-\gamma_1\gamma_2}}\sqrt{\frac{C_2}{A_2-B_2}}\tan(D_2\eta+d_1), \end{aligned}$$

$$v_9(x,t) = \mp\sqrt{\frac{2A(\frac{A_1}{B_1}-1)^2(\beta_1-\gamma_2)}{\beta_1\beta_2-\gamma_1\gamma_2}}\frac{-1}{(\frac{A_1}{B_1}-1)\xi+c_3}$$
$$\pm\sqrt{\frac{2A(\frac{A_2}{B_2}-1)^2(\beta_1-\gamma_2)}{\beta_1\beta_2-\gamma_1\gamma_2}}\sqrt{\frac{C_2}{A_2-B_2}}\tan(D_2\eta+d_1),$$

where $D_2=\frac{C_2}{B_2}\sqrt{\frac{\frac{A_2}{B_2}-1}{\frac{C_2}{B_2}}}, c_3, d_1$ are arbitrary constants.

Case 10 When $\begin{cases}\dfrac{C_1}{B_1}(\dfrac{A_1}{B_1}-1)=0\ (\dfrac{A_1}{B_1}-1=0),\\ \dfrac{C_2}{B_2}(\dfrac{A_2}{B_2}-1)>0,\end{cases}$

$$u_{10}(x,t) = \pm\sqrt{\frac{2A(\frac{A_1}{B_1}-1)^2(\beta_2-\gamma_1)}{\beta_1\beta_2-\gamma_1\gamma_2}}(\frac{C_1}{B_1}\xi+c_4)$$
$$\pm\sqrt{\frac{2A(\frac{A_2}{B_2}-1)^2(\beta_2-\gamma_1)}{\beta_1\beta_2-\gamma_1\gamma_2}}\sqrt{\frac{C_2}{A_2-B_2}}\tan(D_2\eta+d_1),$$
$$v_{10}(x,t) = \mp\sqrt{\frac{2A(\frac{A_1}{B_1}-1)^2(\beta_1-\gamma_2)}{\beta_1\beta_2-\gamma_1\gamma_2}}(\frac{C_1}{B_1}\xi+c_4)$$
$$\pm\sqrt{\frac{2A(\frac{A_2}{B_2}-1)^2(\beta_1-\gamma_2)}{\beta_1\beta_2-\gamma_1\gamma_2}}\sqrt{\frac{C_2}{A_2-B_2}}\tan(D_2\eta+d_1),$$

where $D_2=\frac{C_2}{B_2}\sqrt{\frac{\frac{A_2}{B_2}-1}{\frac{C_2}{B_2}}}, c_4, d_1$ are arbitrary constants.

Case 11 When $\begin{cases}\dfrac{C_1}{B_1}(\dfrac{A_1}{B_1}-1)=0\ (\dfrac{C_1}{B_1}=0),\\ \dfrac{C_2}{B_2}(\dfrac{A_2}{B_2}-1)<0,\end{cases}$

$$u_{11}(x,t) = \pm\sqrt{\frac{2A(\frac{A_1}{B_1}-1)^2(\beta_2-\gamma_1)}{\beta_1\beta_2-\gamma_1\gamma_2}}\frac{-1}{(\frac{A_1}{B_1}-1)\xi+c_3}$$
$$\pm\sqrt{\frac{2A(\frac{A_2}{B_2}-1)^2(\beta_2-\gamma_1)}{\beta_1\beta_2-\gamma_1\gamma_2}}\sqrt{\frac{C_2}{A_2-B_2}}\tan h(F_2\eta+d_2),$$
$$v_{11}(x,t) = \mp\sqrt{\frac{2A(\frac{A_1}{B_1}-1)^2(\beta_1-\gamma_2)}{\beta_1\beta_2-\gamma_1\gamma_2}}\frac{-1}{(\frac{A_1}{B_1}-1)\xi+c_3}$$
$$\pm\sqrt{\frac{2A(\frac{A_2}{B_2}-1)^2(\beta_1-\gamma_2)}{\beta_1\beta_2-\gamma_1\gamma_2}}\sqrt{\frac{C_2}{B_2-C_2}}\tan h(F_2\eta+d_2),$$

where $F_2=\frac{C_2}{B_2}\sqrt{\frac{1-\frac{A_2}{B_2}}{\frac{C_2}{B_2}}}, c_3, d_2$ are arbitrary constants.

Case 12 When $\begin{cases} \dfrac{C_1}{B_1}(\dfrac{A_1}{B_1}-1)=0\ (\dfrac{A_1}{B_1}-1=0), \\ \dfrac{C_2}{B_2}(\dfrac{A_2}{B_2}-1)<0, \end{cases}$

$$\begin{aligned} u_{12}(x,t) = &\pm\sqrt{\frac{2A(\frac{A_1}{B_1}-1)^2(\beta_2-\gamma_1)}{\beta_1\beta_2-\gamma_1\gamma_2}}(\frac{C_1}{B_1}\xi+c_4) \\ &\pm\sqrt{\frac{2A(\frac{A_2}{B_2}-1)^2(\beta_2-\gamma_1)}{\beta_1\beta_2-\gamma_1\gamma_2}}\sqrt{\frac{C_2}{B_2-A_2}}\tan h(F_2\eta+d_2), \\ v_{12}(x,t) = &\mp\sqrt{\frac{2A(\frac{A_1}{B_1}-1)^2(\beta_1-\gamma_2)}{\beta_1\beta_2-\gamma_1\gamma_2}}(\frac{C_1}{B_1}\xi+c_4) \\ &\pm\sqrt{\frac{2A(\frac{A_2}{B_2}-1)^2(\beta_1-\gamma_2)}{\beta_1\beta_2-\gamma_1\gamma_2}}\sqrt{\frac{C_2}{B_2-A_2}}\tan h(F_2\eta+d_2), \end{aligned}$$

where $F_2=\frac{C_2}{B_2}\sqrt{\frac{1-\frac{A_2}{B_2}}{\frac{C_2}{B_2}}}, c_4, d_2$ are arbitrary constants.

Case 13 When $\begin{cases} \dfrac{C_1}{B_1}(\dfrac{A_1}{B_1}-1)=0\ (\dfrac{C_1}{B_1}=0), \\ \dfrac{C_2}{B_2}(\dfrac{A_2}{B_2}-1)=0\ (\dfrac{C_2}{B_2}=0), \end{cases}$

$$\begin{aligned} u_{13}(x,t) = &\pm\sqrt{\frac{2A(\frac{A_1}{B_1}-1)^2(\beta_2-\gamma_1)}{\beta_1\beta_2-\gamma_1\gamma_2}}\frac{-1}{(\frac{A_1}{B_1}-1)\xi+c_3} \\ &\pm\sqrt{\frac{2A(\frac{A_2}{B_2}-1)^2(\beta_2-\gamma_1)}{\beta_1\beta_2-\gamma_1\gamma_2}}\frac{-1}{(\frac{A_2}{B_2}-1)\eta+d_3}, \\ v_{13}(x,t) = &\mp\sqrt{\frac{2A(\frac{A_1}{B_1}-1)^2(\beta_1-\gamma_2)}{\beta_1\beta_2-\gamma_1\gamma_2}}\frac{-1}{(\frac{A_1}{B_1}-1)\xi+c_3} \\ &\pm\sqrt{\frac{2A(\frac{A_2}{B_2}-1)^2(\beta_1-\gamma_2)}{\beta_1\beta_2-\gamma_1\gamma_2}}\frac{-1}{(\frac{A_2}{B_2}-1)\eta+d_3}, \end{aligned}$$

where c_3, d_3 are arbitrary constants.

Case 14 When $\begin{cases} \dfrac{C_1}{B_1}(\dfrac{A_1}{B_1}-1)=0\ (\dfrac{C_1}{B_1}=0), \\ \dfrac{C_2}{B_2}(\dfrac{A_2}{B_2}-1)=0\ (\dfrac{A_2}{B_2}-1=0), \end{cases}$

$$\begin{aligned} u_{14}(x,t) = &\pm\sqrt{\frac{2A(\frac{A_1}{B_1}-1)^2(\beta_2-\gamma_1)}{\beta_1\beta_2-\gamma_1\gamma_2}}\frac{-1}{(\frac{A_1}{B_1}-1)\xi+c_3} \\ &\pm\sqrt{\frac{2A(\frac{A_2}{B_2}-1)^2(\beta_2-\gamma_1)}{\beta_1\beta_2-\gamma_1\gamma_2}}(\frac{C_2}{B_2}\eta+d_4), \\ v_{14}(x,t) = &\mp\sqrt{\frac{2A(\frac{A_1}{B_1}-1)^2(\beta_1-\gamma_2)}{\beta_1\beta_2-\gamma_1\gamma_2}}\frac{-1}{(\frac{A_1}{B_1}-1)\xi+c_3} \end{aligned}$$

$$\pm\sqrt{\frac{2A(\frac{A_2}{B_2}-1)^2(\beta_1-\gamma_2)}{\beta_1\beta_2-\gamma_1\gamma_2}}(\frac{C_2}{B_2}\eta+d_4),$$

where c_3, d_4 are arbitrary constants.

Case 15 When $\begin{cases} \frac{C_1}{B_1}(\frac{A_1}{B_1}-1)=0\ (\frac{A_1}{B_1}-1=0), \\ \frac{C_2}{B_2}(\frac{A_2}{B_2}-1)=0\ (\frac{A_2}{B_2}-1=0), \end{cases}$

$$\begin{aligned} u_{15}(x,t)=&\pm\sqrt{\frac{2A(\frac{A_1}{B_1}-1)^2(\beta_2-\gamma_1)}{\beta_1\beta_2-\gamma_1\gamma_2}}(\frac{C_1}{B_1}\xi+c_4) \\ &\pm\sqrt{\frac{2A(\frac{A_2}{B_2}-1)^2(\beta_2-\gamma_1)}{\beta_1\beta_2-\gamma_1\gamma_2}}(\frac{C_2}{B_2}\eta+d_4), \\ v_{15}(x,t)=&\mp\sqrt{\frac{2A(\frac{A_1}{B_1}-1)^2(\beta_1-\gamma_2)}{\beta_1\beta_2-\gamma_1\gamma_2}}(\frac{C_1}{B_1}\xi+c_4) \\ &\pm\sqrt{\frac{2A(\frac{A_2}{B_2}-1)^2(\beta_1-\gamma_2)}{\beta_1\beta_2-\gamma_1\gamma_2}}(\frac{C_2}{B_2}\eta+d_4), \end{aligned}$$

where c_4, d_4 are arbitrary constants.

Case 16 When $\begin{cases} \frac{C_1}{B_1}(\frac{A_1}{B_1}-1)=0\ (\frac{A_1}{B_1}-1=0), \\ \frac{C_2}{B_2}(\frac{A_2}{B_2}-1)=0\ (\frac{C_2}{B_2}=0), \end{cases}$

$$\begin{aligned} u_{16}(x,t)=&\pm\sqrt{\frac{2A(\frac{A_1}{B_1}-1)^2(\beta_2-\gamma_1)}{\beta_1\beta_2-\gamma_1\gamma_2}}(\frac{C_1}{B_1}\xi+c_4) \\ &\pm\sqrt{\frac{2A(\frac{A_2}{B_2}-1)^2(\beta_2-\gamma_1)}{\beta_1\beta_2-\gamma_1\gamma_2}}\frac{-1}{(\frac{A_2}{B_2}-1)\eta+d_3}, \\ v_{16}(x,t)=&\mp\sqrt{\frac{2A(\frac{A_1}{B_1}-1)^2(\beta_1-\gamma_2)}{\beta_1\beta_2-\gamma_1\gamma_2}}(\frac{C_1}{B_1}\xi+c_4) \\ &\pm\sqrt{\frac{2A(\frac{A_2}{B_2}-1)^2(\beta_1-\gamma_2)}{\beta_1\beta_2-\gamma_1\gamma_2}}\frac{-1}{(\frac{A_2}{B_2}-1)\eta+d_3}, \end{aligned}$$

where c_4, d_3 are arbitrary constants.

4. Application to the coupled Schrödinger-Boussinesq equation

We consider the coupled nonlinear Schrödinger-Boussinesq equations

$$\begin{cases} iu_t+u_{xx}+\beta_1 u=vu, \\ 3v_{tt}-v_{xxxx}+3(v^2)_{xx}+\beta_2 v_{xx}=(|u|^2)_{xx}, \end{cases} \tag{15}$$

where β_1, β_2 are real constants and $v(x,t)$ is a real function while $u(x,t)$ is a complex function. The Eq. (15) is known to describe various physical processes in Laser and plasma, such as formation, Langmuir field amplitude and intense electromagnetic waves and modulation instabilities [28,29].

In order to look for the exact traveling wave solutions of Eq. (15), we suppose that

$$u(x,t) = \varphi(x,t)e^{i\tau}, \quad \tau = kx + lt + \xi_o, \tag{16}$$

where $\varphi(x,t)$ is a real function, k, l, ξ_0 are real constants. By substituting (16) into (15), we obtain

$$\varphi_t + 2k\varphi_x = 0, \tag{17}$$

$$\varphi_{xx} - (l + k^2 - \beta_1)\varphi = v\varphi, \tag{18}$$

$$3v_{tt} - v_{xxxx} + 3(v^2)_{xx} + \beta_2 v_{xx} = (\varphi^2)_{xx}, \tag{19}$$

According to Eq. (17), we suppose that

$$\varphi(x,t) = \varphi(\varsigma) = \varphi(x - 2kt + \xi_1), \tag{20}$$

where ξ_1 is the arbitrary constant. By substituting (20) into (18), we obtain

$$v(x,t) = \frac{\varphi''(\varsigma)}{\varphi(\varsigma)} - (l + k^2 - \beta_1). \tag{21}$$

We can suppose that

$$v(x,t) = \psi(\varsigma) = \psi(x - 2kt + \xi_1). \tag{22}$$

By substituting (20) and (22) into (18) and (19) and setting integral constant to zero, we obtain ODE

$$\begin{cases} \varphi''(\varsigma) - (l + k^2 - \beta_1)\varphi(\varsigma) - \psi(\varsigma)\varphi(\varsigma) = 0, \\ -\psi''(\varsigma) + (12k^2 + \beta_2)\psi(\varsigma) + 3\psi^2(\varsigma) - \varphi^2(\varsigma) = 0. \end{cases} \tag{23}$$

By considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in Eq. (23), we obtain $m = 2$ for φ and $n = 2$ for ψ. Suppose that the solutions for Eq. (23) can be expressed in the following form

$$\varphi = a_0 + a_{10}(\frac{G_1'}{G_1}) + a_{01}(\frac{G_2'}{G_2}) + a_{20}(\frac{G_1'}{G_1})^2 + a_{11}(\frac{G_1'}{G_1})(\frac{G_2'}{G_2}) + a_{02}(\frac{G_2'}{G_2})^2, \tag{24}$$

$$\psi = a_0 + b_{10}(\frac{G_1'}{G_1}) + b_{01}(\frac{G_2'}{G_2}) + b_{20}(\frac{G_1'}{G_1})^2 + b_{11}(\frac{G_1'}{G_1})(\frac{G_2'}{G_2}) + b_{02}(\frac{G_2'}{G_2})^2, \tag{25}$$

where $G_1 = G_1(\xi), G_2 = G_2(\eta)$ satisfies Eq. (5), and $a_0, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_0, b_{10}, b_{01}, b_{20}, b_{11}, b_{02}$ are constants to be determined later.

Substituting (24) and (25) along with Eq.(5) into Eq.(23), and collecting all terms with the same power of $(\frac{G_1'}{G_1})^i(\frac{G_2'}{G_2})^j$ $(i,j = 0,1,2,\ldots)$ together, the left-hand side of Eq.(23) is converted into another polynomial $(\frac{G_1'}{G_1})^i(\frac{G_2'}{G_2})^j$ $(i,j = 0,1,2,\ldots)$. Equating each coefficient of this polynomial to zero, yields a set of simultaneous algebraic equations for a_0, a_{ij} $(i,j = 0,1,2,\ldots), k_1,\ldots,k_n, s, A_1, A_2, B_1,\ldots,C_2$.

Solving the algebraic equations above, yields

$$a_0 = 0,\ a_{10} = 0,\ a_{01} = 0,\ a_{20} = \pm 6\sqrt{2}(\frac{A_1}{B_1} - 1)^2,\ a_{02} = \pm 6\sqrt{2}(\frac{A_2}{B_2} - 1)^2,\ a_{11} = 0,$$

$$b_0=0,\ b_{10}=0,\ b_{01}=0,\ b_{20}=6(\frac{A_1}{B_1}-1)^2,\ b_{02}=6(\frac{A_2}{B_2}-1)^2,\ b_{11}=0,$$
$$l=\frac{22}{3}(\frac{A_1}{B_1}-1)\frac{C_1}{B_1}+\beta_1+\frac{1}{12}\beta_2,\ A_1=(\frac{A_1}{B_1}-1)\frac{C_2}{B_2}\frac{B_1^2}{C_1}+B_1. \tag{26}$$

Substituting (26) along with Eq. (5) into (24) and (25), from (16), we obtain plentiful double traveling wave solutions consisting of hyperbolic functions, trigonometric functions, rational functions, and their mixture with arbitrary parameters as follows:

Case 1 When $\begin{cases}\frac{C_1}{B_1}(\frac{A_1}{B_1}-1)>0,\\ \frac{C_2}{B_2}(\frac{A_2}{B_2}-1)>0,\end{cases}$

$$\begin{aligned}u_1(x,t)=&[\pm 6\sqrt{2}(\frac{A_1}{B_1}-1)^2\frac{C_1}{A_1-B_1}\tan^2(D_1\xi+c_1)\\&\pm 6\sqrt{2}(\frac{A_2}{B_2}-1)^2\frac{C_2}{A_2-B_2}\tan^2(D_2\eta+d_1)]e^{i(kx+(\frac{22}{3}(\frac{A_1}{B_1}-1)\frac{C_1}{B_1}+\beta_1+\frac{1}{12}\beta_2)t+\xi_0)},\\ v_1(x,t)=&6(\frac{A_1}{B_1}-1)^2\frac{C_1}{A_1-B_1}\tan^2(D_1\xi+c_1)+6(\frac{A_2}{B_2}-1)^2\frac{C_2}{A_2-B_2}\tan^2(D_2\eta+d_1),\end{aligned}$$

where $D_1=\frac{C_1}{B_1}\sqrt{\frac{\frac{A_1}{B_1}-1}{\frac{C_1}{B_1}}}$, $D_2=\frac{C_2}{B_2}\sqrt{\frac{\frac{A_2}{B_2}-1}{\frac{C_2}{B_2}}}$, c_1,d_1 are arbitrary constants.

Case 2 When $\begin{cases}\frac{C_1}{B_1}(\frac{A_1}{B_1}-1)<0,\\ \frac{C_2}{B_2}(\frac{A_2}{B_2}-1)<0,\end{cases}$

$$\begin{aligned}u_2(x,t)=&[\pm 6\sqrt{2}(\frac{A_1}{B_1}-1)^2\frac{C_1}{B_1-A_1}\tan h^2(F_1\xi+c_2)\\&\pm 6\sqrt{2}(\frac{A_2}{B_2}-1)^2\frac{C_2}{B_2-A_2}\tan h^2(F_2\eta+d_2)]e^{i(kx+(\frac{22}{3}(\frac{A_1}{B_1}-1)\frac{C_1}{B_1}+\beta_1+\frac{1}{12}\beta_2)t+\xi_0)},\\ v_2(x,t)=&6(\frac{A_1}{B_1}-1)^2\frac{C_1}{B_1-A_1}\tan h^2(F_1\xi+c_2)+6(\frac{A_2}{B_2}-1)^2\frac{C_2}{B_2-A_2}\tan h^2(F_2\eta+d_2),\end{aligned}$$

where $F_1=\frac{C_1}{B_1}\sqrt{\frac{1-\frac{A_1}{B_1}}{\frac{C_1}{B_1}}}$, $F_2=\frac{C_2}{B_2}\sqrt{\frac{1-\frac{A_2}{B_2}}{\frac{C_2}{B_2}}}$, c_2,d_2 are arbitrary constants.

Case 3 When $\begin{cases}\frac{C_1}{B_1}(\frac{A_1}{B_1}-1)>0,\\ \frac{C_2}{B_2}(\frac{A_2}{B_2}-1)<0,\end{cases}$

$$\begin{aligned}u_3(x,t)=&[\pm 6\sqrt{2}i(\frac{A_1}{B_1}-1)^2\frac{C_1}{A_1-B_1}\tan^2(D_1\xi+c_1)\\&\pm 6\sqrt{2}i(\frac{A_2}{B_2}-1)^2\frac{C_2}{B_2-A_2}\tan h^2(F_2\eta+d_2)]e^{i(kx+(\frac{22}{3}(\frac{A_1}{B_1}-1)\frac{C_1}{B_1}+\beta_1+\frac{1}{12}\beta_2)t+\xi_0)},\\ v_3(x,t)=&6(\frac{A_1}{B_1}-1)^2\frac{C_1}{A_1-B_1}\tan^2(D_1\xi+c_1)+6(\frac{A_2}{B_2}-1)^2\frac{C_2}{B_2-A_2}\tan h^2(F_2\eta+d_2),\end{aligned}$$

where $D_1=\frac{C_1}{B_1}\sqrt{\frac{\frac{A_1}{B_1}-1}{\frac{C_1}{B_1}}}$, $F_2=\frac{C_2}{B_2}\sqrt{\frac{1-\frac{A_2}{B_2}}{\frac{C_2}{B_2}}}$, c_1,d_2 are arbitrary constants.

Case 4 When $\begin{cases} \dfrac{C_1}{B_1}(\dfrac{A_1}{B_1}-1)<0, \\ \dfrac{C_2}{B_2}(\dfrac{A_2}{B_2}-1)>0, \end{cases}$

$$\begin{aligned} u_4(x,t)=&[\pm 6\sqrt{2}i(\frac{A_1}{B_1}-1)^2\frac{C_1}{B_1-A_1}\tan h^2(F_1\xi+c_2) \\ &\pm 6\sqrt{2}i(\frac{A_2}{B_2}-1)^2\frac{C_2}{A_2-B_2}\tan^2(D_2\eta+d_1)]e^{i(kx+(\frac{22}{3}(\frac{A_1}{B_1}-1)\frac{C_1}{B_1}+\beta_1+\frac{1}{12}\beta_2)t+\xi_0)}, \\ v_4(x,t)=&6(\frac{A_1}{B_1}-1)^2\frac{C_1}{B_1-A_1}\tan h^2(F_1\xi+c_2)+6(\frac{A_2}{B_2}-1)^2\frac{C_2}{A_2-B_2}\tan^2(D_2\eta+d_1), \end{aligned}$$

where $F_1=\frac{C_1}{B_1}\sqrt{\frac{1-\frac{A_1}{B_1}}{\frac{C_1}{B_1}}}$, $D_2=\frac{C_2}{B_2}\sqrt{\frac{\frac{A_2}{B_2}-1}{\frac{C_2}{B_2}}}, c_1, d_1$ are arbitrary constants.

Case 5 When $\begin{cases} \dfrac{C_1}{B_1}(\dfrac{A_1}{B_1}-1)>0, \\ \dfrac{C_2}{B_2}(\dfrac{A_2}{B_2}-1)=0\ (\dfrac{C_2}{B_2}=0), \end{cases}$

$$\begin{aligned} u_5(x,t)=&[\pm 6\sqrt{2}(\frac{A_1}{B_1}-1)^2\frac{C_1}{A_1-B_1}\tan^2(D_1\xi+c_1) \\ &\pm 6\sqrt{2}(\frac{A_2}{B_2}-1)^2\frac{1}{((\frac{A_2}{B_2}-1)\eta+d_3)^2}]e^{i(kx+(\frac{22}{3}(\frac{A_1}{B_1}-1)\frac{C_1}{B_1}+\beta_1+\frac{1}{12}\beta_2)t+\xi_0)}, \\ v_5(x,t)=&6(\frac{A_1}{B_1}-1)^2\frac{C_1}{A_1-B_1}\tan^2(D_1\xi+c_1)+6(\frac{A_2}{B_2}-1)^2\frac{1}{((\frac{A_2}{B_2}-1)\eta+d_3)^2}, \end{aligned}$$

where $D_1=\frac{C_1}{B_1}\sqrt{\frac{\frac{A_1}{B_1}-1}{\frac{C_1}{B_1}}}, c_1, d_3$ are arbitrary constants.

Case 6 When $\begin{cases} \dfrac{C_1}{B_1}(\dfrac{A_1}{B_1}-1)>0, \\ \dfrac{C_2}{B_2}(\dfrac{A_2}{B_2}-1)=0\ (\dfrac{A_2}{B_2}-1=0), \end{cases}$

$$\begin{aligned} u_6(x,t)=&[\pm 6\sqrt{2}(\frac{A_1}{B_1}-1)^2\frac{C_1}{A_1-B_1}\tan^2(D_1\xi+c_1) \\ &\pm 6\sqrt{2}(\frac{A_2}{B_2}-1)^2(\frac{C_2}{B_2}\eta+d_4)^2]e^{i(kx+(\frac{22}{3}(\frac{A_1}{B_1}-1)\frac{C_1}{B_1}+\beta_1+\frac{1}{12}\beta_2)t+\xi_0)}, \\ v_6(x,t)=&6(\frac{A_1}{B_1}-1)^2\frac{C_1}{A_1-B_1}\tan^2(D_1\xi+c_1)+6(\frac{A_2}{B_2}-1)^2(\frac{C_2}{B_2}\eta+d_4)^2, \end{aligned}$$

where $D_1=\frac{C_1}{B_1}\sqrt{\frac{\frac{A_1}{B_1}-1}{\frac{C_1}{B_1}}}, c_1, d_4$ are arbitrary constants.

Case 7 When $\begin{cases} \dfrac{C_1}{B_1}(\dfrac{A_1}{B_1}-1)<0, \\ \dfrac{C_2}{B_2}(\dfrac{A_2}{B_2}-1)=0\ (\dfrac{C_2}{B_2}=0), \end{cases}$

$$u_7(x,t)=[\pm 6\sqrt{2}(\frac{A_1}{B_1}-1)^2\frac{C_1}{B_1-A_1}\tan h^2(F_1\xi+c_2)$$

$$\pm 6\sqrt{2}(\frac{A_2}{B_2}-1)^2\frac{1}{((\frac{A_2}{B_2}-1)\eta+d_3)^2}]e^{i(kx+(\frac{22}{3}(\frac{A_1}{B_1}-1)\frac{C_1}{B_1}+\beta_1+\frac{1}{12}\beta_2)t+\xi_0)},$$

$$v_7(x,t)=6(\frac{A_1}{B_1}-1)^2\frac{C_1}{B_1-A_1}\tan h^2(F_1\xi+c_2)+6(\frac{A_2}{B_2}-1)^2\frac{1}{((\frac{A_2}{B_2}-1)\eta+d_3)^2},$$

where $F_1=\frac{C_1}{B_1}\sqrt{\frac{1-\frac{A_1}{B_1}}{\frac{C_1}{B_1}}}$, c_2, d_3 are arbitrary constants.

Case 8 When $\begin{cases}\frac{C_1}{B_1}(\frac{A_1}{B_1}-1)<0,\\ \frac{C_2}{B_2}(\frac{A_2}{B_2}-1)=0\ (\frac{A_2}{B_2}-1=0),\end{cases}$

$$\begin{aligned}u_8(x,t)=&[\pm 6\sqrt{2}(\frac{A_1}{B_1}-1)^2\frac{C_1}{B_1-A_1}\tan h^2(F_1\xi+c_2)\\ &\pm 6\sqrt{2}(\frac{A_2}{B_2}-1)^2(\frac{C_2}{B_2}\eta+d_4)^2]e^{i(kx+(\frac{22}{3}(\frac{A_1}{B_1}-1)\frac{C_1}{B_1}+\beta_1+\frac{1}{12}\beta_2)t+\xi_0)},\\ v_8(x,t)=&6(\frac{A_1}{B_1}-1)^2\frac{C_1}{B_1-A_1}\tan h^2(F_1\xi+c_2)+6(\frac{A_2}{B_2}-1)^2(\frac{C_2}{B_2}\eta+d_4)^2,\end{aligned}$$

where $F_1=\frac{C_1}{B_1}\sqrt{\frac{1-\frac{A_1}{B_1}}{\frac{C_1}{B_1}}}$, c_2, d_4 are arbitrary constants.

Case 9 When $\begin{cases}\frac{C_1}{B_1}(\frac{A_1}{B_1}-1)=0\ (\frac{C_1}{B_1}=0),\\ \frac{C_2}{B_2}(\frac{A_2}{B_2}-1)>0,\end{cases}$

$$\begin{aligned}u_9(x,t)=&[\pm 6\sqrt{2}(\frac{A_1}{B_1}-1)^2\frac{1}{((\frac{A_1}{B_1}-1)\xi+c_3)^2}\\ &\pm 6\sqrt{2}(\frac{A_2}{B_2}-1)^2\frac{C_2}{A_2-B_2}\tan^2(D_2\eta+d_1)]e^{i(kx+(\frac{22}{3}(\frac{A_1}{B_1}-1)\frac{C_1}{B_1}+\beta_1+\frac{1}{12}\beta_2)t+\xi_0)},\\ v_9(x,t)=&6(\frac{A_1}{B_1}-1)^2\frac{1}{((\frac{A_1}{B_1}-1)\xi+c_3)^2})+6(\frac{A_2}{B_2}-1)^2\frac{C_2}{A_2-B_2}\tan^2(D_2\eta+d_1),\end{aligned}$$

where $D_2=\frac{C_2}{B_2}\sqrt{\frac{\frac{A_2}{B_2}-1}{\frac{C_2}{B_2}}}$, c_3, d_1 are arbitrary constants.

Case 10 When $\begin{cases}\frac{C_1}{B_1}(\frac{A_1}{B_1}-1)=0\ (\frac{A_1}{B_1}-1=0),\\ \frac{C_2}{B_2}(\frac{A_2}{B_2}-1)>0,\end{cases}$

$$\begin{aligned}u_{10}(x,t)=&[\pm 6\sqrt{2}(\frac{A_1}{B_1}-1)^2(\frac{C_1}{B_1}\xi+c_4)^2\\ &\pm 6\sqrt{2}(\frac{A_2}{B_2}-1)^2\frac{C_2}{A_2-B_2}\tan^2(D_2\eta+d_1)]e^{i(kx+(\frac{22}{3}(\frac{A_1}{B_1}-1)\frac{C_1}{B_1}+\beta_1+\frac{1}{12}\beta_2)t+\xi_0)},\\ v_{10}(x,t)=&6(\frac{A_1}{B_1}-1)^2(\frac{C_1}{B_1}\xi+c_4)^2+6(\frac{A_2}{B_2}-1)^2\frac{C_2}{A_2-B_2}\tan^2(D_2\eta+d_1),\end{aligned}$$

where $D_2 = \frac{C_2}{B_2}\sqrt{\frac{\frac{A_2}{B_2}-1}{\frac{C_2}{B_2}}}, c_4, d_1$ are arbitrary constants.

Case 11 When $\begin{cases} \dfrac{C_1}{B_1}(\dfrac{A_1}{B_1}-1)=0\ (\dfrac{C_1}{B_1}=0), \\ \dfrac{C_2}{B_2}(\dfrac{A_2}{B_2}-1)<0, \end{cases}$

$$\begin{aligned} u_{11}(x,t) =& [\pm 6\sqrt{2}(\frac{A_1}{B_1}-1)^2\frac{1}{((\frac{A_1}{B_1}-1)\xi+c_3)^2} \\ & \pm 6\sqrt{2}(\frac{A_2}{B_2}-1)^2\frac{C_2}{B_2-A_2}\tan h^2(F_2\eta+d_2)]e^{i(kx+(\frac{22}{3}(\frac{A_1}{B_1}-1)\frac{C_1}{B_1}+\beta_1+\frac{1}{12}\beta_2)t+\xi_0)}, \\ v_{11}(x,t) =& 6(\frac{A_1}{B_1}-1)^2\frac{1}{((\frac{A_1}{B_1}-1)\xi+c_3)^2}+6(\frac{A_2}{B_2}-1)^2\frac{C_2}{B_2-A_2}\tan h^2(F_2\eta+d_2), \end{aligned}$$

where $F_2 = \frac{C_2}{B_2}\sqrt{\frac{1-\frac{A_2}{B_2}}{\frac{C_2}{B_2}}}, c_3, d_2$ are arbitrary constants.

Case 12 When $\begin{cases} \dfrac{C_1}{B_1}(\dfrac{A_1}{B_1}-1)=0\ (\dfrac{A_1}{B_1}-1=0), \\ \dfrac{C_2}{B_2}(\dfrac{A_2}{B_2}-1)<0, \end{cases}$

$$\begin{aligned} u_{12}(x,t) =& [\pm 6\sqrt{2}(\frac{A_1}{B_1}-1)^2(\frac{C_1}{B_1}\xi+c_4)^2 \\ & \pm 6\sqrt{2}(\frac{A_2}{B_2}-1)^2\frac{C_2}{B_2-A_2}\tan h^2(F_2\eta+d_2)]e^{i(kx+(\frac{22}{3}(\frac{A_1}{B_1}-1)\frac{C_1}{B_1}+\beta_1+\frac{1}{12}\beta_2)t+\xi_0)}, \\ v_{12}(x,t) =& 6(\frac{A_1}{B_1}-1)^2(\frac{C_1}{B_1}\xi+c_4)^2+6(\frac{A_2}{B_2}-1)^2\frac{C_2}{B_2-A_2}\tan h^2(F_2\eta+d_2), \end{aligned}$$

where $F_2 = \frac{C_2}{B_2}\sqrt{\frac{1-\frac{A_2}{B_2}}{\frac{C_2}{B_2}}}, c_4, d_2$ are arbitrary constants.

Case 13 When $\begin{cases} \dfrac{C_1}{B_1}(\dfrac{A_1}{B_1}-1)=0\ (\dfrac{C_1}{B_1}=0), \\ \dfrac{C_2}{B_2}(\dfrac{A_2}{B_2}-1)=0\ (\dfrac{C_2}{B_2}=0), \end{cases}$

$$\begin{aligned} u_{13}(x,t) =& [\pm 6\sqrt{2}(\frac{A_1}{B_1}-1)^2\frac{1}{((\frac{A_1}{B_1}-1)\xi+c_3)^2} \\ & \pm 6\sqrt{2}(\frac{A_2}{B_2}-1)^2\frac{1}{((\frac{A_2}{B_2}-1)\eta+d_3)^2}]e^{i(kx+(\frac{22}{3}(\frac{A_1}{B_1}-1)\frac{C_1}{B_1}+\beta_1+\frac{1}{12}\beta_2)t+\xi_0)}, \\ v_{13}(x,t) =& 6(\frac{A_1}{B_1}-1)^2\frac{1}{((\frac{A_1}{B_1}-1)\xi+c_3)^2}+6(\frac{A_2}{B_2}-1)^2\frac{1}{((\frac{A_2}{B_2}-1)\eta+d_3)^2}, \end{aligned}$$

where c_3, d_3 are arbitrary constants.

Case 14 When $\begin{cases} \dfrac{C_1}{B_1}(\dfrac{A_1}{B_1}-1)=0\ (\dfrac{C_1}{B_1}=0), \\ \dfrac{C_2}{B_2}(\dfrac{A_2}{B_2}-1)=0\ (\dfrac{A_2}{B_2}-1=0), \end{cases}$

$$\begin{aligned} u_{14}(x,t)=&[\pm 6\sqrt{2}(\frac{A_1}{B_1}-1)^2\frac{1}{((\frac{A_1}{B_1}-1)\xi+c_3)^2} \\ &\pm 6\sqrt{2}(\frac{A_2}{B_2}-1)^2(\frac{C_2}{B_2}\eta+d_4)^2]e^{i(kx+(\frac{22}{3}(\frac{A_1}{B_1}-1)\frac{C_1}{B_1}+\beta_1+\frac{1}{12}\beta_2)t+\xi_0)}, \\ v_{14}(x,t)=&6(\frac{A_1}{B_1}-1)^2\frac{1}{((\frac{A_1}{B_1}-1)\xi+c_3)^2}+6(\frac{A_2}{B_2}-1)^2(\frac{C_2}{B_2}\eta+d_4)^2, \end{aligned}$$

where c_3, d_4 are arbitrary constants.

Case 15 When $\begin{cases} \dfrac{C_1}{B_1}(\dfrac{A_1}{B_1}-1)=0\ (\dfrac{A_1}{B_1}-1=0), \\ \dfrac{C_2}{B_2}(\dfrac{A_2}{B_2}-1)=0\ (\dfrac{A_2}{B_2}-1=0), \end{cases}$

$$\begin{aligned} u_{15}(x,t)=&[\pm 6\sqrt{2}(\frac{A_1}{B_1}-1)^2(\frac{C_1}{B_1}\xi+c_4)^2 \\ &\pm 6\sqrt{2}(\frac{A_2}{B_2}-1)^2 2(\frac{C_2}{B_2}\eta+d_4)^2]e^{i(kx+(\frac{22}{3}(\frac{A_1}{B_1}-1)\frac{C_1}{B_1}+\beta_1+\frac{1}{12}\beta_2)t+\xi_0)}, \\ v_{15}(x,t)=&6(\frac{A_1}{B_1}-1)^2(\frac{C_1}{B_1}\xi+c_4)^2+6(\frac{A_2}{B_2}-1)^2(\frac{C_2}{B_2}\eta+d_4)^2, \end{aligned}$$

where c_4, d_4 are arbitrary constants.

Case 16 When $\begin{cases} \dfrac{C_1}{B_1}(\dfrac{A_1}{B_1}-1)=0\ (\dfrac{A_1}{B_1}-1=0), \\ \dfrac{C_2}{B_2}(\dfrac{A_2}{B_2}-1)=0\ (\dfrac{C_2}{B_2}=0), \end{cases}$

$$\begin{aligned} u_{16}(x,t)=&[\pm 6\sqrt{2}(\frac{A_1}{B_1}-1)^2(\frac{C_1}{B_1}\xi+c_4)^2 \\ &\pm 6\sqrt{2}(\frac{A_2}{B_2}-1)^2\frac{1}{((\frac{A_2}{B_2}-1)\eta+d_3)^2}]e^{i(kx+(\frac{22}{3}(\frac{A_1}{B_1}-1)\frac{C_1}{B_1}+\beta_1+\frac{1}{12}\beta_2)t+\xi_0)}, \\ v_{16}(x,t)=&6(\frac{A_1}{B_1}-1)^2(\frac{C_1}{B_1}\xi+c_4)^2+6(\frac{A_2}{B_2}-1)^2\frac{1}{((\frac{A_2}{B_2}-1)\eta+d_3)^2}, \end{aligned}$$

where c_4, d_3 are arbitrary constants.

5. Conclusion

In summary, a new multiple $(\frac{G'}{G})$-expansion method has been first proposed and then applied to the coupled nonlinear Klein-Gordon equations and the coupled Schrödinger-Boussinesq equation. With the aid of symbolic computation, a rich variety of solutions are obtained. These solutions include double hyperbolic tangent function solutions, double tangent function solutions, double rational solutions, and complexiton solutions. Such complexiton solutions possess combination of hyperbolic tangent and tangent function solutions, combination of hyperbolic tangent and rational function solutions, combination of tangent and rational function solutions.

The double traveling wave solutions of NPDEs are significant. We use nonlinear ordinary differential equation $A_i(G_i')^2 - B_iG_iG_i'' + C_iG_i^2 = 0$ $(i = 1, 2)$ in this new method and get new double traveling wave solutions in the form $u(\varsigma) = a_0 + \sum_{k=1}^{n}\sum_{i+j=k} a_{ij}(\frac{G_1'(\xi)}{G_1(\xi)})^i(\frac{G_2'(\eta)}{G_2(\eta)})^j$. However, Wang [13] used linear ordinary differential equation $G'' + \lambda G + \mu G = 0$ as auxiliary equation and the solutions presented in the form $u(\varsigma) = a_0 + \sum_{i=1}^{m} a_i(\frac{G'(\zeta)}{G(\zeta)})$. This traditional $(\frac{G'}{G})$-expansion can only get single traveling wave solutions of NPDEs. So the new multiple $(\frac{G'}{G})$-expansion method is very effective and powerful to handle various nonlinear partial differential equations which frequently arise in mathematical physics, engineering sciences and many scientific real time application fields, and provides more theoretical basis to reveal physical properties, vibration inner laws and exterior factors.

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