Journal of Mathematical Research with Applications
Nov., 2017, Vol. 37, No. 6, pp. 710–724
DOI:10.3770/j.issn:2095-2651.2017.06.007
Http://jmre.dlut.edu.cn

Dual Toeplitz Operators on the Unit Ball

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Abstract In this paper, we study some properties of dual Toeplitz operators on the orthogonal complement of Bergman space of the unit ball. We first completely characterize the boundedness and compactness of dual Toeplitz operators. Then we obtain spectral properties of dual Toeplitz operators. Finally, we show that there are no quasinormal dual Toeplitz operators with bounded holomorphic or anti-holomorphic symbols.

Keywords Dual Toeplitz operators; unit ball; compactness; spectrum; quasinormal

MR(2010) Subject Classification 47B35; 46E22

1. Introduction

For a fixed integer n, let B_n denote the unit ball in $\mathbb{C}^n$. The Bergman space $A^2(B_n)$ is the Hilbert space of holomorphic functions on the unit ball B_n that are square integrable with respect to normalized volume measure $\mathrm{d}V$.

The reproducing kernel on $A^2(B_n)$ is given by

$$K_w(z)=\frac{1}{(1-\langle z,w\rangle)^{n+1}},$$

where $z=(z_1,\dots,z_n), w=(w_1,\dots,w_n)\in B_n$ and $\langle z,w\rangle=z_1\overline{w}_1+\cdots+z_n\overline{w}_n$. If $\langle\cdot,\cdot\rangle_2$ denotes the inner product in $L^2(B_n,\mathrm{d}V)$, then $\langle h,K_w\rangle_2=h(w)$, for every $h\in A^2(B_n)$ and $w\in B_n$.

Let P be the Bergman orthogonal projection from $L^2(B_n,\mathrm{d}V)$ onto $A^2(B_n,\mathrm{d}V)$, which is given by

$$(Pg)(w)=\langle g,K_w\rangle_2=\int_{B_n}g(z)\frac{1}{(1-\langle w,z\rangle)^{n+1}}\mathrm{d}V(z),$$

for every $g\in L^2(B_n)$ and $w\in B_n$. In this paper, we use $\|\cdot\|$ to denote the norm in $L^2(B_n,\mathrm{d}V)$. Given $f\in L^\infty(B_n,\mathrm{d}V)$, the Toeplitz operator T_f is defined by

$$T_f(h)(w)=P(fh)(w)=\int_{B_n}\frac{f(z)h(z)}{(1-\langle w,z\rangle)^{n+1}}\mathrm{d}V(z),$$

Received April 6, 2017; Accepted September 15, 2017
Supported by the National Natural Science Foundation of China (Grant Nos. 11501075; 11271059), the Natural Science Foundation of Liaoning Education Department (Grant No. L2015084) and the Natural Science Foundation of Guangxi Education Department (Grant No. KY2015LX518).
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for $h \in A^2(B_n)$ and $w \in B_n$. The dual Toeplitz operator S_f with symbol f is defined by

$$S_f u = (I - P)(fu)$$

for $u \in (A^2(B_n))^{\perp}$. It is clear that $S_f : (A^2(B_n))^{\perp} \to (A^2(B_n))^{\perp}$ is a bounded linear operator. In what follows, let Q denote $I - P$.

Although dual Toeplitz operators differ in many ways from Toeplitz operators, they do have some of the same properties. The purpose of this paper is to study some properties of dual Toeplitz operators on the Bergman space of the unit ball. Our results for dual Toeplitz operators may offer some insight into the study of similar questions for Toeplitz operators on the Bergman space. The reader can obtain corresponding details about dual Toeplitz operators in [1–7].

There is a natural and fundamental question: What is the relationship between the properties of an operator and its symbol? We shall recall some classical results. Brown and Halmos [8] showed that the only compact Toeplitz operator on the Hardy space is the zero operator and a Toeplitz operator is bounded on the Hardy space if and only if its symbol is bounded. This is false for Toeplitz operators on the Bergman space. Axler and Zheng [9] completely characterized compact Toeplitz operators on the Bergman space. Le [10] obtained that if the symbol of a Toeplitz operator is continuous on the closed unit ball, then the Toeplitz operator is compact if and only if its symbol is zero on the unit sphere. A characterization of compact dual Toeplitz operators on the orthogonal complement of Bergman space has been obtained by Karel Stroethoff and Zheng in [11]. In Section 3, we continue to investigate the boundedness and compactness of dual Toeplitz operators on the Bergman space of the unit ball.

The symbol map on the Toeplitz algebra in the Hardy space has been an important tool in the study of Fredholm properties of Toeplitz operators and the structure of the Toeplitz algebra [12, Chapter 7]. Analogous to the symbol map in the classical Hardy space setting, in Section 4, we obtained the symbol map on the Bergman space of the unit ball. As an application of our symbol map we obtain a necessary condition on symbols of a finite number of dual Toeplitz operators whose product is the zero operator.

The spectral properties of dual Toeplitz operators on the orthogonal complement of the Bergman space have been introduced and well elaborated by Karel Stroethoff and Zheng [11]. Further investigations from a spectral point of view has been done by Guediri [13]. In Section 5, we mainly study the spectral properties of dual Toeplitz operators on the higher dimensional space. In addition, we obtain a necessary and sufficient condition for the inverse of a dual Toeplitz operator to be a dual Toeplitz operator.

In the final section of the paper we discuss quasinormal dual Toeplitz operator on the orthogonal complement of Bergman space of the unit ball. We use the operator $\mathcal{L}_w$ to prove that there are no quasinormal dual Toeplitz operators with bounded holomorphic or anti-holomorphic symbols. The purpose we defined quasinormal Toeplitz operator was to answer an open question whether every subnormal Toeplitz operator is either normal or analytic on the Hardy space. The original problem was reduced to whether every quasinormal Toeplitz operator is either normal or analytic, it was completely solved by the authors in [14]. Guediri [13] showed that there are no quasinormal dual Toeplitz operators with bounded analytic or co-analytic symbols on the Bergman space. The reader can obtain more corresponding details in [15–18].

2. Preliminaries

Both Toeplitz operators and dual Toeplitz operators are closely related to Hankel operators. For a bounded measurable function f on B_n, the Hankel operator H_f is the operator $A^2(B_n) \to (A^2(B_n))^{\perp}$ defined by

$$H_f h = (I - P)(fh) = Q(fh), \quad h \in A^2(B_n).$$

Under the decomposition $L^2(B_n) = A^2(B_n) \oplus (A^2(B_n))^{\perp}$, for $f \in L^{\infty}(B_n, \mathrm{d}V)$, the multiplication operator M_f is represented as

$$M_f = \begin{bmatrix} T_f & H^*_{\overline{f}} \\ H_f & S_f \end{bmatrix}.$$

The identity $M_{fg} = M_f M_g$ implies the following basic relations between those operators:

$$T_{fg} = T_f T_g + H^*_{\overline{f}} H_g, \tag{1}$$

$$S_{fg} = S_f S_g + H_f H^*_{\overline{g}}, \tag{2}$$

$$H_{fg} = H_f T_g + S_f H_g. \tag{3}$$

Lemma 2.1 *If f and g are in $L^{\infty}(B_n, \mathrm{d}V)$, α and β are in $\mathbb{C}$, then*

$$S^*_f = S_{\overline{f}}, \quad S_{\alpha f + \beta g} = \alpha S_f + \beta S_g.$$

If f is a bounded holomorphic function on B_n and g is a bounded measurable function on B_n, then the following identities hold:

$$S_{fg} = S_f S_g, \quad S_{g\overline{f}} = S_g S_{\overline{f}}, \tag{4}$$

$$S_f H_g = H_g T_f, \quad H^*_g S_{\overline{f}} = T_{\overline{f}} H^*_g. \tag{5}$$

Lu and Yang characterized the following condition for the product of two dual Toeplitz operators to be a dual Toeplitz operator [19, Theorem 3.3].

Lemma 2.2 *If f and g are in $L^{\infty}(B_n, \mathrm{d}V)$, then $S_f S_g$ is a dual Toeplitz operator if and only if f is holomorphic on B_n or g is anti-holomorphic on B_n, in which case $S_f S_g = S_{fg}$.*

Suppose f and g are in $L^2(B_n)$. Consider the operator $f \otimes g$ defined by

$$(f \otimes g)h = \langle h, g \rangle_2 f$$

for $h \in L^2(B_n)$. We also have

$$A(f \otimes g)B^* = (Af) \otimes (Bg),$$

where A and B are bounded linear operators.

For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, where $\mathbb{N}$ denote the set of all non-negative integers, we write

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \quad \alpha! = \alpha_1! \cdots \alpha_n!$$

and

$$z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}, \quad z = (z_1, \dots, z_n) \in \mathbb{C}^n.$$

For $i \in \mathbb{N}$, we have

$$\langle z, w \rangle^i = \sum_{|\gamma|=i} \frac{i!}{\gamma!} z^\gamma \overline{w}^\gamma.$$

By the binomial rule, we obtain

$$\begin{aligned} K_w^{-1}(z) = (1 - \langle z, w \rangle)^{n+1} &= \sum_{i=0}^{n+1} \frac{(n+1)!}{i!(n+1-i)!} (-1)^i \langle z, w \rangle^i \\ &= \sum_{i=0}^{n+1} \sum_{|\gamma|=i} \frac{(-1)^i (n+1)!}{i!(n+1-i)!} \frac{i!}{\gamma!} z^\gamma \overline{w}^\gamma \\ &= \sum_{i=0}^{n+1} \sum_{|\gamma|=i} \lambda_{i,\gamma} z^\gamma \overline{w}^\gamma, \quad \text{where} \quad \lambda_{i,\gamma} = \frac{(-1)^i (n+1)!}{(n+1-i)!\gamma!}. \end{aligned}$$

Using the reproducing property of K_w, we have

$$\|K_w\|^2 = \langle K_w, K_w \rangle_2 = K_w(w) = \frac{1}{(1 - |w|^2)^{n+1}},$$

thus the normalized reproducing kernel is given by

$$k_w(z) = \frac{(1 - |w|^2)^{\frac{n+1}{2}}}{(1 - \langle z, w \rangle)^{n+1}}.$$

Let $w \in B_n - \{0\}$. Automorphism φ_w is defined by

$$\varphi_w(z) = \frac{w - P_w(z) - \sqrt{1 - |w|^2} Q_w(z)}{1 - \langle z, w \rangle},$$

where P_w is the orthogonal projection from $\mathbb{C}^n$ onto the one dimensional subspace $[w]$ generated by w, and Q_w is the orthogonal projection from $\mathbb{C}^n$ onto $\mathbb{C}^n \ominus [w]$. When $w = 0$, we simply define $\varphi_w(z) = -z$. The reader can get more details about automorphism φ_w in [20]. Define an operator U_w on $A^2(B_n)$ by $U_w h = (h \circ \varphi_w) k_w$, $h \in A^2(B_n)$. Then U_w is unitary. Furthermore, $T_{f \circ \varphi_w} U_w = U_w T_f$. We have the following fact about the normalized reproducing kernel [19, Lemma 2.3].

Proposition 2.3 *For all $w \in B_n$, we have*

$$k_w \otimes k_w = \sum_{i=0}^{n+1} \sum_{|\gamma|=i} \lambda_{i,\gamma} T_{\varphi_w^\gamma} T_{\overline{\varphi_w}^\gamma}.$$

For a bounded linear operator T on $(A^2(B_n))^\perp$ and $w \in B_n$, we define the operator $\mathcal{L}_w(T)$ by

$$\mathcal{L}_w(T) = \sum_{i=0}^{n+1} \sum_{|\gamma|=i} \lambda_{i,\gamma} S_{\varphi_w^\gamma} T S_{\overline{\varphi_w}^\gamma}.$$

The operator $\mathcal{L}_w(T)$ gives rise to an interesting characterization of dual Toeplitz operators on the orthogonal complement of the Bergman space of the unit ball.

Proposition 2.4 *Let S_f be a dual Toeplitz operator on $(A^2(B_n))^\perp$, for $f \in L^\infty(B_n)$. Then*

$$\mathcal{L}_w(S_f) = S_{\frac{f}{\|K_{\varphi_w}\|^2}}, \quad \text{for all } w \in B_n.$$

Proof Fix $w \in B_n$, we have

$$\mathcal{L}_w(S_f) = \sum_{i=0}^{n+1} \sum_{|\gamma|=i} \lambda_{i,\gamma} S_{\varphi_w^\gamma} S_f S_{\overline{\varphi_w}^\gamma} = \sum_{i=0}^{n+1} \sum_{|\gamma|=i} \lambda_{i,\gamma} S_{|\varphi_w^\gamma|^2 f} = S_\psi,$$

where

$$\begin{aligned}\psi(z) &= f(z) \sum_{i=0}^{n+1} \sum_{|\gamma|=i} \lambda_{i,\gamma} |\varphi_w^\gamma(z)|^2 \\ &= f(z)(1 - \langle \varphi_w(z), \varphi_w(z) \rangle)^{n+1} = \frac{f(z)}{\|K_{\varphi_w(z)}\|^2}. \ \Box\end{aligned}$$

We also have the relationship between the operator $\mathcal{L}_w(T)$ and Hankel operators. It can be obtained by (5) and Proposition 2.3.

Proposition 2.5 *Let f and g be in $L^\infty(B_n)$. Then we have*

$$\mathcal{L}_w(H_f H_{\overline{g}}^*) = (H_f k_w) \otimes (H_{\overline{g}} k_w).$$

3. Bounded and compact dual Toeplitz operators

In this section, we will characterize the bounded and compact dual Toeplitz operators. The dual Toeplitz operator S_f is densely defined by the formula $S_f h = Q(fh)$, where f is in $L^2(B_n)$ and h is in $A^2(B_n) \bigcap L^\infty(B_n)$. For $w \in B_n$ and $0 < s < 1 - |w|$, let $g_{w,s}$ be the function on B_n defined by $g_{w,s} = (\overline{z_1 - w_1})\chi_{w+sB_n}(z)$, for $z \in B_n$. For any multi-index $\alpha \in \mathbb{N}^n$, we have

$$\int_{B_n} z^\alpha \overline{g_{w,s}} \mathrm{d}V(z) = \int_{sB_n} (z+w)^\alpha z_1 \mathrm{d}V(z) = 0.$$

Thus $g_{w,s} \in (A^2(B_n))^\perp$ and $(\overline{z_1 - w_1}) g_{w,s} \in (A^2(B_n))^\perp$. Let $u_{w,s} = \frac{g_{w,s}}{\|g_{w,s}\|}$. The function $u_{w,s}$ is the unit vector in the space $(A^2(B_n))^\perp$.

Lemma 3.1 *The function $u_{w,s} \to 0$ weakly in $(A^2(B_n))^\perp$, as $s \to 0^+$.*

Proof Let $\psi \in L^2(B_n, \mathrm{d}V)$. The Cauchy-Schwarz inequality gives

$$|\langle \psi, g_{w,s} \rangle_2| = \Big| \int_{w+sB_n} \psi(z) \overline{g_{w,s}(z)} \mathrm{d}V(z) \Big| \le \|g_{w,s}\| \Big(\int_{w+sB_n} |\psi(z)|^2 \mathrm{d}V(z) \Big)^{\frac{1}{2}},$$

so that

$$|\langle \psi, u_{w,s} \rangle_2| \le \Big(\int_{w+sB_n} |\psi(z)|^2 \mathrm{d}V(z) \Big)^{\frac{1}{2}}.$$

Thus $u_{w,s} \to 0$ weakly in $L^2(B_n, \mathrm{d}V)$ as $s \to 0^+$, which gives the stated result, since $(A^2(B_n))^\perp \subset L^2(B_n, \mathrm{d}V)$. $\Box$

Lemma 3.2 *Let $f \in L^2(B_n, \mathrm{d}V)$. For each $w \in B_n$, $\lim_{s \to 0^+} \|H_{\overline{f}}^* u_{w,s}\| = 0$.*

Proof Fix $w \in B_n$. For each $z \in B_n$ we have

$$\begin{aligned}|H_{\overline{f}}^* u_{w,s}(z)| &= \Big|\int_{B_n} f(\lambda) u_{w,s}(\lambda)\overline{K_z(\lambda)}\mathrm{d}V(\lambda)\Big| \\ &\le \|u_{w,s}\|\Big(\int_{w+sB_n} |f(\lambda)\overline{K_z(\lambda)}|^2 \mathrm{d}V(\lambda)\Big)^{\frac{1}{2}} \\ &= \Big(\int_{w+sB_n} |f(\lambda)K_z(\lambda)|^2 \mathrm{d}V(\lambda)\Big)^{\frac{1}{2}}.\end{aligned}$$

Integrating with respect to z gives

$$\begin{aligned}\|H_{\overline{f}}^* u_{w,s}(z)\|^2 &= \Big|\int_{B_n} |H_{\overline{f}}^* u_{w,s}(z)|^2 \mathrm{d}V(z)\Big| \\ &\le \int_{w+sB_n} |f(\lambda)|^2 \Big\{\int_{B_n} |K_z(\lambda)|^2 \mathrm{d}V(z)\Big\}\mathrm{d}V(\lambda).\end{aligned}$$

Using $\int_{B_n} |K_z(\lambda)|^2 \mathrm{d}V(z) = \int_{B_n} |K_\lambda(z)|^2 \mathrm{d}V(z) = \langle K_\lambda(z), K_\lambda(z)\rangle_2 = \frac{1}{(1-|\lambda|^2)^{n+1}}$, we get

$$\begin{aligned}\|H_{\overline{f}}^* u_{w,s}(z)\|^2 &\le \int_{w+sB_n} \frac{|f(\lambda)|^2}{(1-|\lambda|^2)^{n+1}} \mathrm{d}V(\lambda) \\ &\le \frac{1}{(1-(|w|+s))^{n+1}} \int_{w+sB_n} |f(\lambda)|^2 \mathrm{d}V(\lambda).\end{aligned}$$

Since $\int_{w+sB_n} |f(\lambda)|^2 \mathrm{d}V(\lambda) \to 0$ as $s \to 0^+$, we can obtain the stated result. $\square$

Lemma 3.3 *Let $f \in L^2(B_n, \mathrm{d}V)$. For a.e. $w \in B_n$, $|f(w)| = \lim_{s\to 0^+} \|S_f u_{w,s}\|$.*

Proof Note that $M_f u = S_f u + H_{\overline{f}}^* u$, and $S_f u \perp H_{\overline{f}}^* u$ for every bounded $u \in (A^2(B_n))^\perp$. Thus

$$\|M_f u\|^2 = \|S_f u\|^2 + \|H_{\overline{f}}^* u\|^2.$$

Taking $u = u_{w,s}$ in the above equality, by Lemma 3.2 we have

$$\lim_{s\to 0^+} \|M_f u_{w,s}\|^2 = \lim_{s\to 0^+} \|S_f u_{w,s}\|^2.$$

We claim that

$$\lim_{s\to 0^+} \|M_f u_{w,s}\|^2 = \lim_{s\to 0^+} \frac{\int_{|z-w|<s} |f(z)|^2 |z_1 - w_1|^2 \mathrm{d}V(z)}{\int_{|z-w|<s} |z_1 - w_1|^2 \mathrm{d}V(z)} = |f(w)|^2,$$

for a.e. $w \in B_n$. Clearly this claim will prove the stated result. Using the fact that

$$\int_{|z-w|<s} |z_1 - w_1|^2 \mathrm{d}V(z) = s^{2n+2} \int_{B_n} |z_1|^2 \mathrm{d}V(z) = \frac{s^{2n+2}}{n+1}$$

and

$$V(B(w,s)) = \int_{|z-w|<s} \mathrm{d}V(z) = s^{2n} \int_{B_n} \mathrm{d}V(z) = s^{2n},$$

we have

$$\begin{aligned}\Big|\frac{\int_{|z-w|<s} |f(z)|^2 |z_1 - w_1|^2 \mathrm{d}V(z)}{\int_{|z-w|<s} |z_1 - w_1|^2 \mathrm{d}V(z)} - |f(w)|^2\Big| &\le \frac{(n+1)s^2 \int_{|z-w|<s} ||f(z)|^2 - |f(w)|^2| \mathrm{d}V(z)}{s^{2n+2}} \\ &= \frac{(n+1)\int_{|z-w|<s} ||f(z)|^2 - |f(w)|^2| \mathrm{d}V(z)}{V(B(w,s))}.\end{aligned}$$

Let $A=\{w\in B_n:\lim_{s\to 0^+}\frac{\int_{B(w,s)}||f(z)|^2-|f(w)|^2|\mathrm{d}V(z)}{V(B(w,s))}=0\}$. It is a classical theorem of Lebesgue that the complement of the above set in B_n has volume measure 0 (see [21, Theorem 8.8]). $\square$

Theorem 3.4 *Let $f\in L^2(B_n,\mathrm{d}V)$. Then S_f is bounded if and only if $f\in L^\infty(B_n)$, in which case $\|S_f\|=\|f\|_\infty$.*

Proof If $f\in L^\infty(B_n)$, then S_f is bounded with $\|S_f\|\le\|f\|_\infty$. To prove the "only if" part, suppose that S_f is bounded. Then $\|S_fu_{w,s}\|\le\|S_f\|$, for all $w\in B_n$ and $0<s<1$. It follows from Lemma 3.3 that $\|f\|_\infty\le\|S_f\|$. $\square$

Theorem 3.5 *For f in $L^\infty(B_n)$, S_f is compact if and only if $f=0$ a.e. on B_n.*

Proof Since $u_{w,s}\to 0$ weakly in $(A^2(B_n))^\perp$, if S_f is compact, then for each $w\in B_n$ we have $\|S_fu_{w,s}\|\to 0$ as $s\to 0^+$, and it follows from Lemma 3.3 that $f=0$ a.e. on B_n. $\square$

Theorem 3.6 *Let f and g in $L^\infty(B_n)$. If S_fS_g is a compact perturbation of a dual Toeplitz operator S_h, then $f(w)g(w)=h(w)$ for almost all $w\in B_n$, and $H_fH_{\overline{g}}^*$ is compact.*

Proof Since $S_fS_g-S_h$ is compact, then using (2), we see that the operator

$$S_{fg-h}-H_fH_{\overline{g}}^*=S_fS_g-S_h$$

is compact. The function $u_{w,s}\to 0$ weakly in $(A^2(B_n))^\perp$, as $s\to 0^+$. Thus

$$\|(S_{fg-h}-H_fH_{\overline{g}}^*)u_{w,s}\|\to 0.$$

By Lemma 3.2 we also have

$$\|H_fH_{\overline{g}}^*u_{w,s}\|\to 0.$$

Thus

$$\|S_{fg-h}u_{w,s}\|\to 0.$$

Applying Lemma 3.3 we see that

$$\|S_{fg-h}u_{w,s}\|\to|f(w)g(w)-h(w)|,$$

for almost all $w\in B_n$. Since $f(w)g(w)-h(w)=0$ for a.e. $w\in B_n$, we have that $S_{fg-h}=0$. Hence $H_fH_{\overline{g}}^*$ is compact. $\square$

4. Symbol map on the dual Toeplitz algebra

The symbol map on the Toeplitz algebra in the Hardy space setting was described in [12, Chapter 7]. Stroethoff and Zheng [11] obtained the existence of a symbol map on the dual Toeplitz algebra in the Bergman space of unit disk. In this section, we will show the existence of a symbol map on the dual Toeplitz algebra on the Bergman space of unit ball.

Lemma 4.1 *If the operator S is in the closed ideal generated by the semicommutators of all bounded dual Toeplitz operators, then*

$$\|Su_{w,s}\|\to 0$$

for all $w \in B_n$ as $s \to 0^+$.

Proof If operator S is in the closed ideal generated by the semicommutators of all bounded dual Toeplitz operators, then S can be approximated by a finite sum of finite products of dual Toeplitz operators or operators of the form $S_{fg} - S_f S_g$. Noting that

$$S_{fg} - S_f S_g = H_f H^*_{\overline{g}},$$

Lemma 3.2 gives that

$$\|(S_{fg} - S_f S_g) u_{w,s}\| \to 0,$$

for all $w \in B_n$ as $s \to 0^+$. To prove the stated result, it suffices to show that for $f, g, h_1, \ldots, h_n \in L^\infty(B_n)$,

$$\|(S_{fg} - S_f S_g) S_{h_1} \cdots S_{h_n} u_{w,s}\| \to 0,$$

for all $w \in B_n$ as $s \to 0^+$. This can be proved using induction. Let $h = h_1$, repeatedly using (2) we have

$$\begin{aligned}(S_{fg} - S_f S_g) S_h &= S_{fg} S_h - S_f S_g S_h = S_{fg} S_h - S_f (S_{gh} - H_g H^*_{\overline{h}}) \\ &= S_{fg} S_h - S_f S_{gh} + S_f H_g H^*_{\overline{h}} \\ &= (S_{fgh} - S_f S_{gh}) - (S_{fgh} - S_{fg} S_h) + S_f H_g H^*_{\overline{h}} \\ &= H_f H^*_{\overline{gh}} - H_{fg} H^*_{\overline{h}} + S_f H_g H^*_{\overline{h}}.\end{aligned}$$

Using Lemma 3.2 we conclude that $\|(S_{fg} - S_f S_g) S_h u_{w,s}\| \to 0$, for all $w \in B_n$ as $s \to 0^+$. The case $n > 2$ can be proved similarly. The induction step follows from the observation that

$$\begin{aligned}(S_{fg} - S_f S_g) S_{h_1} \cdots S_{h_n} =& (S_{fg} - S_f S_g) S_{h_1} \cdots S_{h_{n-2}} S_{h_{n-1} h_n} \\ & - (S_{fg} - S_f S_g) S_{h_1} \cdots S_{h_{n-2}} H_{h_{n-1}} H^*_{\overline{h_n}},\end{aligned}$$

for $n > 2$. □

Proposition 4.2 *For $f_1, f_2, \ldots, f_n \in L^\infty(B_n)$ the operator*

$$S_{f_1} S_{f_2} \cdots S_{f_n} - S_{f_1 f_2 \cdots f_n}$$

belongs to the closed ideal generated by the semicommutators of all bounded dual Toeplitz operators.

Proof Writing

$$S_{f_1} S_{f_2} \cdots S_{f_n} - S_{f_1 f_2 \cdots f_n} = S_{f_1} (S_{f_2} \cdots S_{f_n} - S_{f_2 \cdots f_n}) + S_{f_1} S_{f_2 \cdots f_n} - S_{f_1 f_2 \cdots f_n},$$

the statement follows by induction. □

Let $\mathcal{B}((A^2)^\perp)$ be the set of bounded linear operators on $(A^2(B_n))^\perp$. If $\mathcal{F}$ is a subset of $L^\infty(B_n)$, then we write $\mathcal{I}(\mathcal{F})$ for the smallest closed subalgebra of $\mathcal{B}((A^2)^\perp)$ containing $\{S_f : f \in \mathcal{F}\}$. The dual Toeplitz algebra is $\mathcal{I}(L^\infty(B_n))$. Let $\mathcal{D}$ be the semicommutator ideal of the dual Toeplitz algebra $\mathcal{I}(L^\infty(B_n))$. We will show the existence of a symbol map from the dual Toeplitz algebra $\mathcal{I}(L^\infty(B_n))$ to $L^\infty(B_n)$.

Theorem 4.3 *There is a contractive C^*-homomorphism ρ from the dual Toeplitz algebra $\mathcal{I}(L^\infty(B_n))$ to $L^\infty(B_n)$ such that $\rho(S_f) = f$, for all $f \in L^\infty(B_n)$.*

Proof First we define ρ on finite sums of finite products of dual Toeplitz operators. If $S = \sum_{i=1}^n S_{f_{i1}} S_{f_{i2}} \cdots S_{f_{in_i}}$, we define

$$\rho(S) = \sum_{i=1}^n f_{i1} f_{i2} \cdots f_{in_i}.$$

We must prove that ρ is well-defined. Suppose that S has another representation:

$$S = \sum_{i=1}^m S_{g_{i1}} S_{g_{i2}} \cdots S_{g_{im_i}}.$$

Let $F = \sum_{i=1}^n f_{i1} f_{i2} \cdots f_{in_i}$ and $G = \sum_{i=1}^m g_{i1} g_{i2} \cdots g_{im_i}$. We only need to show that $F(w) = G(w)$ a.e. on B_n. $S_F - S_G$ is in $\mathcal{D}$, since $S - S_F$ and $S - S_G$ are in the semicommutator ideal $\mathcal{D}$. By Lemma 4.1 we have

$$\lim_{s \to 0^+} \|(S_F - S_G) u_{w,s}\| \to 0$$

for a.e. $w \in B_n$. On the other hand, Lemma 3.3 gives that

$$|F(w) - G(w)| = \lim_{s \to 0^+} \|(S_F - S_G) u_{w,s}\|.$$

Thus $F(w) = G(w)$ a.e. on B_n. So that ρ is well-defined.

For each $S \in \mathcal{I}(L^\infty(B_n))$ and a given positive integer n there is a finite sum F_n of finite products of dual Toeplitz operators such that

$$\|S - F_n\| < \frac{1}{n}.$$

By the first part of the proof, $\rho(F_n)$ is well-defined. The sequence $\{\rho(F_n)\}$ in $L^\infty(B_n)$ is a Cauchy sequence, since

$$\|\rho(F_n) - \rho(F_m)\|_\infty \leq \|F_n - F_m\|.$$

We define $\rho(S)$ to be the limit of the Cauchy sequence $\{\rho(F_n)\}$ in $L^\infty(B_n)$. It is easily seen that $\rho(S)$ does not depend on the chosen sequence $\{F_n\}$.

The mapping ρ is clearly linear, and it is seen that $\rho(S^*) = \overline{\rho(S)}$. To prove that ρ is contractive, it suffices to show that $\|\rho(S)\|_\infty \leq \|S\|$ if S is a finite sum of finite products of dual Toeplitz operators. Writing $F = \rho(S)$, the operator $D = S - S_F$ is in the semicommutator ideal $\mathcal{D}$, so that by Lemma 4.1, $\lim_{s \to 0^+} \|D u_{w,s}\| = 0$. Using Lemma 3.3 it follows that

$$\|S\| = \|S_F + D\| \geq \lim_{s \to 0^+} \|(S_F + D) u_{w,s}\| = |F(w)|,$$

for a.e. $w \in B_n$, proving that indeed $\|\rho(S)\|_\infty = \|F\|_\infty \leq \|S\|$.

To prove that ρ is a C^*-algebra homomorphism, it suffices to prove that $\rho(ST) = \rho(S)\rho(T)$, for operator S and T which are finite products of dual Toeplitz operators. Clearly it will be sufficient to show that

$$\rho(S_{f_1} \cdots S_{f_n}) = \rho(S_{f_1}) \cdots \rho(S_{f_n}),$$

for $f_1, \ldots, f_n$ in $L^\infty(B_n)$. $\square$

We call ρ the symbol map on the dual Toeplitz algebra $\mathcal{I}(L^\infty(B_n))$. Define the mapping $\xi : L^\infty(B_n) \to \mathcal{B}((A^2)^\perp)$ by $\xi(f) = S_f$, for $f \in L^\infty(B_n)$.

Theorem 4.4 *If $\mathcal{D}$ is the semicommutator ideal of the dual Toeplitz algebra $\mathcal{I}(L^\infty(B_n))$, then the mapping $\overline{\xi}$ induced from $L^\infty(B_n)$ to $\mathcal{I}(L^\infty(B_n))/\mathcal{D}$ by ξ is a $*$-isometric isomorphism. Thus there is a short exact sequence*

$$(0) \to \mathcal{D} \to \mathcal{I}(L^\infty(B_n)) \xrightarrow{\rho} L^\infty(B_n) \to (0)$$

for which ξ is an isometric cross section.

Proof The mapping $\overline{\xi}$ is obviously linear and contractive. To show that $\overline{\xi}$ is multiplicative, observe that for f and g in $L^\infty(B_n)$,

$$\xi(f)\xi(g) - \xi(fg) = S_f S_g - S_{fg}$$

is in the semicommutator $\mathcal{D}$. Thus $\overline{\xi}$ is multiplicative on $L^\infty(B_n)$.

To complete the proof, we show that $\|S_f + K\| \geq \|S_f\|$, for $f \in L^\infty(B_n)$ and $K \in \mathcal{D}$, and hence $\overline{\xi}$ is an isometry. Note that $\|S_f\| = \|f\|_\infty$. So it suffices to show that $\|S_f + K\| \geq \|f\|_\infty$. Since K is in the semicommutator, by Lemma 4.1 we have

$$\lim_{s\to 0^+} \|K u_{w,s}\| = 0,$$

for all $w \in B_n$. By Lemma 3.3 we also have

$$|f(w)| = \lim_{s\to 0^+} \|S_f u_{w,s}\|,$$

for a.e. $w \in B_n$. Thus

$$\|S_f + K\| \geq \lim_{s\to 0^+} \|(S_f + K)u_{w,s}\| = |f(w)|,$$

for a.e. $w \in B_n$. So this gives that $\|S_f + K\| \geq \|f\|_\infty$, which completes the proof. □

Theorem 4.5 *The semicommutator ideal $\mathcal{D}$ contains the ideal $\mathcal{K}$ of compact operators on $(A^2(B_n))^\perp$.*

Proof First we show that $\mathcal{D}$ contains the rank one operator $\overline{z}^\alpha \otimes \overline{z}^\beta$, for all α and β in $\mathbb{N}^n$.

As a special case of Proposition 2.5 we have

$$\begin{aligned}\overline{z}^\alpha \otimes \overline{z}^\beta &= H_{\overline{z}^\alpha}(1\otimes 1)H^*_{\overline{z}^\beta} = \mathcal{L}_0(H_{\overline{z}^\alpha}H^*_{\overline{z}^\beta}) \\ &= \sum_{i=0}^{n+1}\sum_{|\gamma|=i} \lambda_{i,\gamma} S_{\varphi_0^\gamma} H_{\overline{z}^\alpha} H^*_{\overline{z}^\beta} S_{\overline{\varphi_0}^\gamma} \\ &= \sum_{i=0}^{n+1}\sum_{|\gamma|=i} \lambda_{i,\gamma} S_{(-z)^\gamma} H_{\overline{z}^\alpha} H^*_{\overline{z}^\beta} S_{(-\overline{z})^\gamma}.\end{aligned}$$

By (2),

$$H_{\overline{z}^\alpha} H^*_{\overline{z}^\beta} = S_{\overline{z}^\alpha z^\beta} - S_{\overline{z}^\alpha} S_{z^\beta} \in \mathcal{D}.$$

It follows from Lemma 2.2 and Proposition 4.2 that $\overline{z}^\alpha \otimes \overline{z}^\beta \in \mathcal{D}$.

Next, we will show that the set $\mathcal{D}$ is irreducible in $(A^2(B_n))^\perp$. Let $\mathcal{N}$ be a closed linear subspace of $(A^2(B_n))^\perp$ which is reducing for $\mathcal{D}$. We have to show that $\mathcal{N} = (A^2(B_n))^\perp$. We first prove the following claim.

Claim. The functions $\overline{z}_i \in \mathcal{N}$, $1 \le i \le n$.

Since $\mathcal{N}$ is nonzero, it contains a nonzero function φ. Since the linear combinations of the functions $z^\alpha \overline{z}^\beta$ are dense in $L^2(B_n, \mathrm{d}V)$, φ cannot be orthogonal to all $z^\alpha\overline{z}^\beta$, and thus there exist multi-index α and β such that $\langle \varphi, z^\alpha \overline{z}^\beta\rangle_2 \neq 0$. Since $\varphi \in (A^2(B_n))^\perp$ is orthogonal to the function z^α, we must have $\beta \succ 0$, thus there exists some $\beta_j > 0$. Note that

$$\langle \varphi, z^\alpha \overline{z}^\beta\rangle_2 \overline{z}_i = \langle \overline{z}^\alpha z^{\beta'} \varphi, \overline{z}_j\rangle_2 \overline{z}_i = \langle S_{\overline{z}^\alpha z^{\beta'}}\varphi, \overline{z}_j\rangle_2 \overline{z}_i = (\overline{z}_i \otimes \overline{z}_j) S_{\overline{z}^\alpha z^{\beta'}}\varphi,$$

where $\beta' = (\beta_1, \ldots, \beta_{j-1}, \beta_j - 1, \beta_{j+1}, \ldots, \beta_n) \succeq 0$. By the first part of the proof, $\overline{z}_i \otimes \overline{z}_j \in \mathcal{D}$. Since $\mathcal{D}$ is an ideal, $(\overline{z}_i \otimes \overline{z}_j) S_{\overline{z}^\alpha z^{\beta'}} \in \mathcal{D}$. Because $\mathcal{N}$ is reducing for every operator in $\mathcal{D}$, we have $\langle \varphi, z^\alpha\overline{z}^\beta\rangle_2 \overline{z}_i \in \mathcal{N}$. Since $\langle \varphi, z^\alpha\overline{z}^\beta\rangle_2 \neq 0$, we conclude that $\overline{z}_i \in \mathcal{N}$, and our claim is proved.

Now let ψ be a function in $(A^2(B_n))^\perp$ which is orthogonal to $\mathcal{N}$. If $\alpha \succeq 0$ and $\beta \succ 0$, where we can assume $\beta_j > 0$, then $(\overline{z}_i \otimes \overline{z}_j) S_{\overline{z}^\alpha z^{\beta'}} \in \mathcal{D}$, and since $\mathcal{N}$ is reducing for $\mathcal{D}$ it follows that

$$\langle \psi, z^\alpha \overline{z}^\beta\rangle_2 \overline{z}_i = (\overline{z}_i \otimes \overline{z}_j) S_{\overline{z}^\alpha z^{\beta'}}\psi$$

is orthogonal to $\mathcal{N}$. Since $\overline{z}_i \in \mathcal{N}$, we must have $\langle \psi, z^\alpha\overline{z}^\beta\rangle_2 = 0$. Note that this is also true if $\beta = 0$, since $\psi \in (A^2(B_n))^\perp$. So ψ is orthogonal to functions $z^\alpha\overline{z}^\beta$, for all α and β in $\mathbb{N}^n$. We conclude that $\psi = 0$ a.e. on B_n, and hence $\mathcal{N} = (A^2(B_n))^\perp$. This completes the proof that $\mathcal{D}$ is irreducible. $\square$

Note that $\mathcal{D}$ contains the nonzero compact operator $\overline{z}^\alpha \otimes \overline{z}^\beta = H_{\overline{z}^\alpha}(1 \otimes 1) H^*_{\overline{z}^\beta}$. By [11, Theorem 5.39], $\mathcal{D}$ contains the ideal $\mathcal{K}$ of compact operators on $(A^2(B_n))^\perp$.

Theorem 4.6 *The C^*-algebra $\mathcal{I}(C(\overline{B_n}))$ contains the ideal $\mathcal{K}$ of compact operators on $(A^2(B_n))^\perp$ as its semicommutator ideal, and the sequence*

$$(0) \to \mathcal{K} \to \mathcal{I}(C(\overline{B_n})) \to C(\overline{B_n}) \to (0)$$

is short exact; that is, the quotient algebra $\mathcal{I}(C(\overline{B_n}))/\mathcal{K}$ is $*$-isometrically isomorphic to $C(\overline{B_n})$.

Proof Write $\mathcal{S}$ to denote the semicommutator ideal in the dual Toeplitz algebra $\mathcal{I}(C(\overline{B_n}))$. By the proof of the previous theorem, $\mathcal{K}$ is contained in $\mathcal{S}$. For two continuous functions f and g on $\overline{B_n}$, by (2) the semicommutator

$$S_{fg} - S_f S_g = H_f H^*_{\overline{g}}$$

is compact. Since $\mathcal{S}$ is generated by semicommutators of dual Toeplitz operators with symbols in $C(\overline{B_n})$, it follows that $\mathcal{S}$ is contained in $\mathcal{K}$. Hence $\mathcal{K}$ equals the semicommutator ideal $\mathcal{S}$. $\square$

The symbol mapping can be used to obtain the following generalization of Theorem 3.5.

Theorem 4.7 *Let $f_1, \ldots, f_n \in L^\infty(B_n)$. If the product $S_{f_1} S_{f_2} \cdots S_{f_n}$ is compact, then $f_1(w) \cdots f_n(w) = 0$ for almost all w in B_n.*

Proof If $S_{f_1}S_{f_2}\cdots S_{f_n}$ is compact, by Theorem 4.5, $S_{f_1}S_{f_2}\cdots S_{f_n}$ is in the semicommutator ideal $\mathcal{D}$. Using Proposition 4.2 we see that $S_{f_1f_2\cdots f_n}$ is in $\mathcal{D}$. It follows that $\rho(S_{f_1f_2\cdots f_n}) = f_1f_2\cdots f_n = 0$ a.e. on B_n. $\Box$

5. Spectral properties of dual Toeplitz operators

In this section we discuss the spectrum and essential spectrum of dual Toeplitz operators on the orthogonal complement of Bergman space of the unit ball.

Proposition 5.1 *Let f be a function in $L^\infty(B_n)$. If S_f is invertible, then f is invertible in $L^\infty(B_n)$.*

Proof Assume that for some $\delta > 0$ we have $\|S_f u\| \geq \delta$, for all $u \in (A^2(B_n))^\perp$ with $\|u\| = 1$. By Lemma 3.3, for a.e. $w \in B_n$ we have

$$|f(w)| = \lim_{s\to 0^+} \|S_f u_{w,s}\| \geq \delta.$$

This completes the proof. $\Box$

Theorem 5.2 *Let f be a function in $L^\infty(B_n)$. If S_f is invertible, then S_f^{-1} is a dual Toeplitz operator if and only if f is holomorphic or $\overline{f}$ is holomorphic.*

Proof For the if part, assume that S_f is invertible. Using Proposition 5.1 it follows that f is invertible in $L^\infty(B_n)$. We first suppose that f is holomorphic. It follows from (4) that $S_f S_{\frac{1}{f}} = I$. Therefore, by uniqueness of the inverse, we infer that the inverse of S_f must be a dual Toeplitz operator $S_{\frac{1}{f}}$. Similarly, if $\overline{f}$ is holomorphic, by (4), we see that $S_{\frac{1}{f}} S_f = I$, hence $S_f^{-1} = S_{\frac{1}{f}}$, which is a dual Toeplitz operator as well.

For the only if part, suppose that S_f^{-1} is a dual Toeplitz operator S_g for some bounded symbol g. On the one hand, since $S_f^{-1}S_f = S_gS_f = I = S_1$ is a dual Toeplitz operator, then either $\overline{f}$ is holomorphic or g is holomorphic by Lemma 2.2. On the other hand, since $S_fS_f^{-1} = S_fS_g = I = S_1$, again using Lemma 2.2, we obtain that either f is holomorphic or $\overline{g}$ is holomorphic. Now, if f is holomorphic, then we complete the proof. If f is not holomorphic, then $\overline{g}$ must be holomorphic and nonconstant (if g is a constant λ, then $S_g = S_f^{-1} = \lambda I$, which means that $S_f = \frac{1}{\lambda}I$, i.e., $f = \frac{1}{\lambda}$, which is holomorphic), it follows that g is not holomorphic. By "On the one hand" conclusion, we must have that $\overline{f}$ is holomorphic, which completes the proof. $\Box$

If f is a measurable function on B_n, then the essential range $\mathcal{R}(f)$ of f is the set of all λ in $\mathbb{C}$ for which $\{z \in B_n : |f(z) - \lambda| < \varepsilon\}$ has positive measure for every $\varepsilon > 0$. We have the following spectrum inclusion theorem, completely analogous to the spectrum inclusion theorem of Hartman and Wintner for Toeplitz operators on the Hardy space [11, Corollary 7.7].

Theorem 5.3 *If f is in $L^\infty(B_n)$, then $\mathcal{R}(f) \subset \sigma(S_f)$.*

Proof Since $S_f - \lambda = S_{f-\lambda}$ for λ in $\mathbb{C}$, using Proposition 5.1 it follows that $\mathcal{R}(f) \subset \sigma(S_f)$. $\Box$

Corollary 5.4 *The mapping $\xi : L^\infty(B_n) \to \mathcal{B}((A^2(B_n))^\perp)$ defined by $\xi(f) = S_f$ is isometric, for $f \in L^\infty(B_n)$.*

Proof By Theorem 5.3 and [12, Proposition 2.28], we have

$$\begin{aligned}\|f\|_\infty \geq \|S_f\| \geq r(S_f) &= \sup\{|\lambda| : \lambda \in \sigma(S_f)\}\\ &\geq \sup\{|\lambda| : \lambda \in \mathcal{R}(f)\} = \|f\|_\infty,\end{aligned}$$

for f in $L^\infty(B_n)$. □

Recall that an operator T is called nilpotent if some positive integral power T^n is zero and the least such power is the index of nilpotence. An operator T is called quasinilpotent if $\sigma(T) = \{0\}$ (or equivalently if $\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = 0$).

Corollary 5.5 *There are no non-zero quasinilpotents (and hence no nilpotents) dual Toeplitz operators.*

Proof From Corollary 5.4, if S_f is quasinilpotent, then

$$\|S_f\| = \|f\|_\infty = r(S_f) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = 0,$$

it follows that $f \equiv 0$. □

In the sequel, we also have the following useful additional fact about dual Toeplitz operators on the Bergman space of the unit ball.

Corollary 5.6 *Let f be in $L^\infty(B_n)$. Then $S_f \geq 0$ if and only if $f \geq 0$.*

Proof If $f \geq 0$, we have $\langle S_f g, g\rangle_2 = \langle Q(fg), g\rangle_2 = \langle fg, g\rangle_2 = \int_{B_n} f(z)|g(z)|^2 \mathrm{d}V(z) \geq 0$, for $g \in (A^2(B_n))^\perp$. Conversely, suppose that $S_f \geq 0$, its spectrum lies in $\mathbb{R}^+$, by Corollary 5.4, we obtain $\mathcal{R}(f) \subset \sigma(S_f) \subset \mathbb{R}^+$, it follows that $f \geq 0$. □

Stroethoff and Zheng [11] proved that the spectrum of a dual Toeplitz operator on the Bergman space is contained in the closed convex hull of essential range of its symbol. The same argument as the proof of [11, Theorem 9.3] shows that this is also true for dual Toeplitz operator on the Bergman space of the unit ball.

Theorem 5.7 *Since f is in $L^\infty(B_n)$ we have $\sigma(S_f) \subset h(\mathcal{R}(f))$, where $h(\mathcal{R}(f))$ is the closed convex hull of $\mathcal{R}(f)$.*

A bounded operator S on $(A^2(B_n))^\perp$ is Fredholm if and only if the operator $S + \mathcal{K}$ is invertible in the Calkin algebra $\mathcal{B}((A^2(B_n))^\perp)/\mathcal{K}$. The following proposition states that a dual Toeplitz operator can only be Fredholm if its symbol is invertible.

Proposition 5.8 *If f is a function in $L^\infty(B_n)$ such that S_f is a Fredholm operator, then f is invertible in $L^\infty(B_n)$.*

Proof If S_f is Fredholm, then $S_f + \mathcal{K}$ is invertible in the Calkin algebra $\mathcal{B}((A^2(B_n))^\perp)/\mathcal{K}$. Since $\mathcal{I}(L^\infty(B_n))/\mathcal{K}$ is a closed self-adjoint subalgebra of $\mathcal{B}((A^2(B_n))^\perp)/\mathcal{K}$, it follows from [12, Theorem 4.28] that $S_f + \mathcal{K}$ is invertible in $\mathcal{I}(L^\infty(B_n))/\mathcal{K}$. By Theorem 4.5, $\mathcal{K} \subset \mathcal{D}$, so

$S_f + \mathcal{D}$ is invertible in the Calkin algebra $\mathcal{I}(L^\infty(B_n))/\mathcal{D}$. It follows that $f = \rho(S_f)$ is invertible in $L^\infty(B_n)$. $\square$

The essential spectrum of a bounded linear operator S on $(A^2(B_n))^\perp$, denoted by $\sigma_e(S)$, is the spectrum of $S + \mathcal{K}$ in the Calkin algebra $\mathcal{B}((A^2(B_n))^\perp)/\mathcal{K}$. We have the following inclusion theorem for the essential spectrum of a dual Toeplitz operator.

Theorem 5.9 *If f is in $L^\infty(B_n)$, then $\mathcal{R}(f) \subset \sigma_e(S_f)$.*

Proof Since $S_f - \lambda = S_{f-\lambda}$ for λ in $\mathbb{C}$, using Proposition 5.8 it follows that $\mathcal{R}(f) \subset \sigma_e(S_f)$. $\square$

Theorem 5.10 *If f is in $L^\infty(B_n)$ such that both Hankel operators H_f and $H_{\overline{f}}$ are compact, then $\mathcal{R}(f) = \sigma_e(S_f)$.*

Proof By the previous theorem it suffices to prove that $\sigma_e(S_f) \subset \mathcal{R}(f)$. If $\lambda \in \mathbb{C} \setminus \mathcal{R}(f)$, then for some $\varepsilon > 0$ we have $|f(z) - \lambda| \geq \varepsilon$, for a.e. z in B_n. Thus $g = \frac{1}{f-\lambda}$ is in $L^\infty(B_n)$. By (2)

$$S_{f-\lambda} S_g = I - H_f H^*_{\overline{g}} \quad \text{and} \quad S_g S_{f-\lambda} = I - H_g H^*_{\overline{f}}.$$

Since both $H_f H^*_{\overline{g}}$ and $H_g H^*_{\overline{f}}$ are compact, $S_{f-\lambda} + \mathcal{K}$ is invertible in the Calkin algebra, so that $\lambda \in \mathbb{C} \setminus \sigma_e(S_f)$. $\square$

6. Quasinormal dual Toeplitz operators

The operator T is quasinormal if T commutes with T^*T. As known to all, the quasinormal operator is subnormal. Guediri [22] showed that there are no quasinormal dual Toeplitz operators with bounded holomorphic or anti-holomorphic symbols on the sphere, which adumbrates a famous conjecture of Halmos. Similarly to [22], we can get the following Theorem.

Theorem 6.1 *Let f be a bounded holomorphic function on B_n. If S_f is quasinormal, then the symbol function f must be constant.*

For bounded anti-holomorphic symbols, we also have the following result.

Proposition 6.2 *Let $\overline{f}$ be a bounded holomorphic function on B_n. If S_f is quasinormal, then the symbol function f must be constant.*

Remark 6.3 If f is constant, then S_f is normal and quasinormal. Theorems 6.1 and Proposition 6.2 can also be expressed as follows: If f is a bounded holomorphic or anti-holomorphic function. S_f is quasinormal if and only if f is a constant. Then we can show there are no quasinormal dual Toeplitz operators with bounded holomorphic or anti-holomorphic symbols.

For a bounded pluriharmonic function f, $f = g_1 + \overline{g}_2$, where g_1 and $\overline{g}_2$ are holomorphic, if $g_1 = g, g_2 = \lambda g$, where $\lambda \in \mathbb{C}$, g is a bounded holomorphic function, we can get the following result.

Proposition 6.4 *Suppose that $f = g + \lambda\overline{g}$, where g is nonconstant and bounded holomorphic function, $\lambda \in \mathbb{C}$. If S_f is quasinormal, then S_f is normal and λ must be unimodular.*

Acknowledgements We thank the referees for their time and comments.

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