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Precise Rates in the Generalized Law of the Iterated Logarithm in $\mathbb{R}^m$

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Abstract Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random vectors with $\mathbb{E}X = (0,\dots,0)_{m\times 1}$ and $\mathrm{Cov}(X,X) = \sigma^2 I_m$, and set $S_n = \sum_{i=1}^n X_i$, $n \geq 1$. For every $d > 0$ and $a_n = o((\log\log n)^{-d})$, the article deals with the precise rates in the genenralized law of the iterated logarithm for a kind of weighted infinite series of $\mathbb{P}(|S_n| \geq (\varepsilon + a_n)\sigma\sqrt{n}(\log\log n)^d)$.

Keywords precise rates; law of iterated logarithm; complete convergence; i.i.d. random vectors

MR(2010) Subject Classification 60F15; 60G50

1. Introduction and main results

Throughout this paper, let $\{X,\ X_n,\ n \geq 1\}$ be a sequence of i.i.d. random vectors with $\mathbb{E}X = (0,\dots,0)_{m\times 1}$ and $\mathrm{Cov}(X,X) = \sigma^2 I_m$, $\sigma < \infty$, and set $S_n = \sum_{i=1}^n X_i$, $n \geq 1$, where I_m is unit $m \times m$ matrix, $\mathrm{Cov}(X_1,X_1) = (\sigma_{uv})_{m\times m}$ is the covariance matrix of $X_1 = (X_{11},\dots,X_{1m})$, $\sigma_{uv} = \mathbb{E}(X_{1u}X_{1v})$. Let N be the standard m-dimensional normal random vector. We denote by C a positive constant which may vary from line to line, and by $\xrightarrow{d}$ convergence in distribution. For $t = (t_1,\dots,t_m) \in \mathbb{R}^m$, $|t| = (\sum_{u=1}^m t_u^2)^{1/2}$ denotes Euclidean norm. Let $\log x = \ln(x \vee \mathrm{e})$, $\log\log x = \ln(\ln(x \vee \mathrm{e}^{\mathrm{e}}))$ and $\lfloor x\rfloor = \sup\{\ell, \ell \leq x, \ell \in \mathbb{Z}^+\}$. The notation $a_n = o(b_n)$ means that $a_n/b_n \to 0$ as $n \to \infty$.

Gut and Spătaru [1] proved that when $m = 1$,

$$\lim_{\varepsilon\searrow 0} \varepsilon^2 \sum_{n=2}^{\infty} \frac{1}{n\log n}\mathbb{P}\{|S_n| \geq \varepsilon\sqrt{n\log\log n}\} = \sigma^2.$$

Zhang [2] deduced that when $m = 1$,

$$\lim_{\varepsilon\searrow 0} \varepsilon^{2b} \sum_{n=3}^{\infty} \frac{(\log\log n)^{b-1}}{n\log n}\mathbb{P}\{|S_n| \geq \varepsilon\sigma\sqrt{2n\log\log n}\} = \frac{\Gamma(b+1/2)}{b\sqrt{\pi}},$$

holds for every $b > 0$. Xiao et al. [3] established that when $m = 1$,

$$\lim_{\varepsilon\searrow 0} \varepsilon^{b/d} \sum_{n=3}^{\infty} \frac{(\log\log n)^{b-1}}{n\log n}\mathbb{P}\{|S_n| \geq (\varepsilon + a_n)\sigma\sqrt{n}(\log\log n)^d\} = \frac{\mathbb{E}|N|^{b/d}}{b}$$

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holds for $b, d > 0$ and $a_n = o((\log\log n)^{-d})$, if $\mathbb{E}X = 0$ and $\mathbb{E}X^2 = \sigma^2$. Furthermore, they also obtained

$$\lim_{\varepsilon\searrow 0} \frac{1}{-\log\varepsilon} \sum_{n=3}^{\infty} \frac{1}{n\log n\log\log n} \mathbb{P}\{|S_n| \geq (\varepsilon + a_n)\sigma\sqrt{n}(\log\log n)^d\} = \frac{1}{d},$$

for $d > 0$, $a_n = o((\log\log n)^{-d})$ and $m = 1$. The law of the iterated logarithm has been extended by many authors, the interested reader could see Huang et al. [4], Jiang and Yang [5], Wu and Wen [6].

It is natural to ask whether or not the results of Xiao et al. [3] still hold in higher dimensional case. In this paper we try to give an affirmative answer to the question. The following are our main results.

Theorem 1.1 *For $b, d > 0$, and $a_n = o((\log\log n)^{-d})$, we have*

$$\lim_{\varepsilon\searrow 0} \varepsilon^{b/d} \sum_{n=3}^{\infty} \frac{(\log\log n)^{b-1}}{n\log n} \mathbb{P}\{|S_n| \geq (\varepsilon + a_n)\sigma\sqrt{n}(\log\log n)^d\} = \frac{2^{b/(2d)}\Gamma(m/2 + b/(2d))}{\Gamma(m/2)b}. \quad (1)$$

Remark 1.2 *In Theorem 1.1, if $m = 1$, $a_n = 0$, $d = 1/2$, put $\sqrt{2}\zeta = \varepsilon$, then (1) reduces to*

$$\lim_{\zeta\searrow 0} (\zeta)^{2b} \sum_{n=3}^{\infty} \frac{(\log\log n)^{b-1}}{n\log n} \mathbb{P}\{|S_n| \geq (\sqrt{2}\zeta)\sigma\sqrt{n}(\log\log n)^{1/2}\} = \frac{\Gamma(1/2 + b)}{\sqrt{\pi}b},$$

which is consistent to the result in Zhang [2].

Theorem 1.3 *For $d > 0$, $a_n = o((\log\log n)^{-d})$, we have*

$$\lim_{\varepsilon\searrow 0} \frac{1}{-\log\varepsilon} \sum_{n=3}^{\infty} \frac{1}{n\log n\log\log n} \mathbb{P}\{|S_n| \geq (\varepsilon + a_n)\sigma\sqrt{n}(\log\log n)^d\} = \frac{1}{d}. \quad (2)$$

The idea of the proof of our main results comes from that of Xiao et al. [3]. We need the following lemma.

Lemma 1.4 *Suppose $\mathbb{E}|X|^\alpha < \infty$, $1 < \alpha \leq 2$. Then for $x, y > 0$,*

$$\mathbb{P}\{|S_n| \geq x\} \leq nm\mathbb{P}\{|X| > y\} + 2mn^{x/(\sqrt{m}y)} \Big(\frac{\mathrm{e}\mathbb{E}|X|^\alpha}{n\mathbb{E}|X|^\alpha + xy^{\alpha-1}/\sqrt{m}}\Big)^{x/(\sqrt{m}y)}. \quad (3)$$

Proof By [3, Lemma 1] (see [7,8]), we have

$$\begin{aligned}
\mathbb{P}\{|S_n| \geq x\} &\leq \sum_{i=1}^{m} \mathbb{P}\{|S_{n,i}| \geq \frac{x}{\sqrt{m}}\} \\
&\leq \sum_{i=1}^{m} \big[n\mathbb{P}\{|X_{1i}| > y\} + 2n^{x/(\sqrt{m}y)} \Big(\frac{\mathrm{e}\mathbb{E}|X_{1i}|^\alpha}{n\mathbb{E}|X_{1i}|^\alpha + xy^{\alpha-1}/\sqrt{m}}\Big)^{x/(\sqrt{m}y)}\big] \\
&\leq nm\mathbb{P}\{|X| > y\} + 2mn^{x/(\sqrt{m}y)} \Big(\frac{\mathrm{e}\mathbb{E}|X|^\alpha}{n\mathbb{E}|X|^\alpha + xy^{\alpha-1}/\sqrt{m}}\Big)^{x/(\sqrt{m}y)},
\end{aligned}$$

where $S_{n,i} = \sum_{j=1}^{n} X_{ji}$, $X_j = (X_{j1,\ldots,jm})$ throughout this paper.

In the following two sections, for $M \geq 3$ and $0 < \varepsilon < 1$, set $b(\varepsilon) = \lfloor\exp\{\exp\{M\varepsilon^{-1/d}\}\}\rfloor$. There is no loss of generality in assuming $\sigma = 1$ for the proof of the two theorems.

2. Proof of Theorem 1.1

Proposition 2.1 *For $b, d > 0$, we have*

$$\lim_{\varepsilon\searrow 0}\varepsilon^{b/d}\sum_{n=3}^{\infty}\frac{(\log\log n)^{b-1}}{n\log n}\mathbb{P}\{|N|\geq\varepsilon(\log\log n)^d\}=\frac{2^{b/(2d)}\Gamma(m/2+b/(2d))}{\Gamma(m/2)b}. \tag{4}$$

Proof By the monotonicity of $\frac{(\log\log y)^{b-1}}{y\log y}\int_{\varepsilon^2(\log\log y)^{2d}}^{\infty}\frac{x^{m/2-1}\mathrm{e}^{-x/2}}{2^{m/2}\Gamma(m/2)}\mathrm{d}x$ for y large enough, we have

$$\begin{aligned}
&\lim_{\varepsilon\searrow 0}\varepsilon^{b/d}\sum_{n=3}^{\infty}\frac{(\log\log n)^{b-1}}{n\log n}\mathbb{P}\{|N|\geq\varepsilon(\log\log n)^d\}\\
&=\lim_{\varepsilon\searrow 0}\varepsilon^{b/d}\int_{\mathrm{e}^{\mathrm{e}}}^{\infty}\frac{(\log\log y)^{b-1}}{y\log y}\mathrm{d}y\int_{\varepsilon^2(\log\log y)^{2d}}^{\infty}\frac{x^{m/2-1}\mathrm{e}^{-x/2}}{2^{m/2}\Gamma(m/2)}\mathrm{d}x\\
&=\lim_{\varepsilon\searrow 0}\frac{1}{2d}\int_{\varepsilon^2}^{\infty}t^{b/(2d)-1}\mathrm{d}t\int_t^{\infty}\frac{x^{m/2-1}\mathrm{e}^{-x/2}}{2^{m/2}\Gamma(m/2)}\mathrm{d}x\\
&=\frac{1}{2d}\lim_{\varepsilon\searrow 0}\int_{\varepsilon^2}^{\infty}\frac{x^{m/2-1}\mathrm{e}^{-x/2}}{2^{m/2}\Gamma(m/2)}\mathrm{d}x\int_{\varepsilon^2}^{x}t^{b/(2d)-1}\mathrm{d}t\\
&=\frac{1}{b}\lim_{\varepsilon\searrow 0}\int_{\varepsilon^2}^{\infty}\frac{x^{m/2-1}\mathrm{e}^{-x/2}}{2^{m/2}\Gamma(m/2)}x^{b/(2d)}\mathrm{d}x-\frac{1}{b}\lim_{\varepsilon\searrow 0}\varepsilon^{b/d}\int_{\varepsilon^2}^{\infty}\frac{x^{m/2-1}\mathrm{e}^{-x/2}}{2^{m/2}\Gamma(m/2)}\mathrm{d}x\\
&=\frac{1}{b}\lim_{\varepsilon\searrow 0}\int_{\varepsilon^2}^{\infty}\frac{x^{m/2-1}\mathrm{e}^{-x/2}}{2^{m/2}\Gamma(m/2)}x^{b/(2d)}\mathrm{d}x\\
&=\frac{2^{b/(2d)}\Gamma(m/2+b/(2d))}{\Gamma(m/2)b}.
\end{aligned}$$

Thus this completes the proof of Proposition 2.1. □

Proposition 2.2 *For $b, d > 0$, we have*

$$\lim_{\varepsilon\searrow 0}\varepsilon^{b/d}\sum_{n\leq b(\varepsilon)}\frac{(\log\log n)^{b-1}}{n\log n}\left|\mathbb{P}\{|S_n|\geq\varepsilon\sqrt{n}(\log\log n)^d\}-\mathbb{P}\{|N|\geq\varepsilon(\log\log n)^d\}\right|=0. \tag{5}$$

Proof Let

$$\Delta_n=\sup_{x\in\mathbb{R}}\left|\mathbb{P}\{|S_n|/\sqrt{n}\geq x\}-\mathbb{P}\{|N|\geq x\}\right|.$$

Then $\Delta_n\to 0$ as $n\to\infty$ since $S_n/\sqrt{n}\xrightarrow{d}N$ (see [9, Theorem 29.5, p.398]). Application of The Toeplitz lemma [3, Proposition 2.2, 10, Lemma 3.2.3, p.120] yields

$$\begin{aligned}
&\varepsilon^{b/d}\sum_{n\leq b(\varepsilon)}\frac{(\log\log n)^{b-1}}{n\log n}\left|\mathbb{P}\{|S_n|\geq\varepsilon\sqrt{n}(\log\log n)^d\}-\mathbb{P}\{|N|\geq\varepsilon(\log\log n)^d\}\right|\\
&\leq\varepsilon^{b/d}\sum_{n\leq b(\varepsilon)}\frac{(\log\log n)^{b-1}}{n\log n}\Delta_n\\
&\leq\frac{CM^b}{(\log\log b(\varepsilon))^b}\sum_{n\leq b(\varepsilon)}\frac{(\log\log n)^{b-1}}{n\log n}\Delta_n\to 0,\ \text{as }\varepsilon\to 0.
\end{aligned}$$

Thus the proof of this proposition is completed. □

Proposition 2.3 *For $b, d > 0$, we have*

$$\lim_{M\to\infty}\limsup_{\varepsilon\searrow 0}\varepsilon^{b/d}\sum_{n>b(\varepsilon)}\frac{(\log\log n)^{b-1}}{n\log n}\mathbb{P}\{|N|\geq\varepsilon(\log\log n)^d\}=0.$$

Proof We could obtain

$$\begin{aligned}&\limsup_{\varepsilon\searrow 0}\varepsilon^{b/d}\sum_{n>b(\varepsilon)}\frac{(\log\log n)^{b-1}}{n\log n}\mathbb{P}\{|N|\geq\varepsilon(\log\log n)^d\}\\ &\leq C\limsup_{\varepsilon\searrow 0}\varepsilon^{b/d}\int_{b(\varepsilon)}^{\infty}\frac{(\log\log y)^{b-1}}{y\log y}\mathrm{d}y\int_{\varepsilon^2(\log\log y)^{2d}}^{\infty}\frac{x^{m/2-1}\mathrm{e}^{-x/2}}{2^{m/2}\Gamma(m/2)}\mathrm{d}x\\ &\leq C\int_{M^{2d}}^{\infty}t^{b/(2d)-1}\mathrm{d}t\int_t^{\infty}\frac{x^{m/2-1}\mathrm{e}^{-x/2}}{2^{m/2}\Gamma(m/2)}\mathrm{d}x\to 0\text{ as }M\to\infty,\end{aligned}$$

since

$$\begin{aligned}&\int_1^{\infty}t^{b/(2d)-1}\mathrm{d}t\int_t^{\infty}\frac{x^{m/2-1}\mathrm{e}^{-x/2}}{2^{m/2}\Gamma(m/2)}\mathrm{d}x=\int_1^{\infty}\frac{x^{m/2-1}\mathrm{e}^{-x/2}}{2^{m/2}\Gamma(m/2)}\mathrm{d}x\int_1^{x}t^{b/(2d)-1}\mathrm{d}t\\ &\leq C\int_1^{\infty}\frac{x^{m/2+b/(2d)-1}\mathrm{e}^{-x/2}}{2^{m/2}\Gamma(m/2)}\mathrm{d}x+C\int_1^{\infty}\frac{x^{m/2-1}\mathrm{e}^{-x/2}}{2^{m/2}\Gamma(m/2)}\mathrm{d}x<\infty,\end{aligned}$$

where the above equality comes from Fubini's theorem. Proposition 2.3 is established. □

Proposition 2.4 *For $b, d > 0$, we have*

$$\lim_{M\to\infty}\limsup_{\varepsilon\searrow 0}\varepsilon^{b/d}\sum_{n>b(\varepsilon)}\frac{(\log\log n)^{b-1}}{n\log n}\mathbb{P}\{|S_n|\geq\varepsilon\sqrt{n}(\log\log n)^d\}=0. \tag{6}$$

Proof When $0 < b < 2d$, we conclude by Markov's inequality that

$$\begin{aligned}&\limsup_{\varepsilon\searrow 0}\varepsilon^{b/d}\sum_{n>b(\varepsilon)}\frac{(\log\log n)^{b-1}}{n\log n}\mathbb{P}\{|S_n|\geq\varepsilon\sqrt{n}(\log\log n)^d\}\\ &\leq C\limsup_{\varepsilon\searrow 0}\varepsilon^{b/d-2}\sum_{n>b(\varepsilon)}\frac{(\log\log n)^{b-1-2d}}{n^2\log n}\sum_{i=1}^{m}\mathbb{E}[|S_{n,i}|^2]\\ &=C\limsup_{\varepsilon\searrow 0}\varepsilon^{b/d-2}\sum_{n>b(\varepsilon)}\frac{(\log\log n)^{b-1-2d}}{n\log n}m\\ &\leq C\limsup_{\varepsilon\searrow 0}\varepsilon^{b/d-2}(\log\log b(\varepsilon))^{b-2d}\\ &\leq CM^{b-2d}\to 0,\ \text{ as } M\to\infty.\end{aligned}$$

For $b \geq 2d$, by Lemma 1.4, we see that

$$\begin{aligned}&\sum_{n>b(\varepsilon)}\frac{(\log\log n)^{b-1}}{n\log n}\mathbb{P}\{|S_n|\geq\varepsilon\sqrt{n}(\log\log n)^d\}\\ &\leq\sum_{n>b(\varepsilon)}\frac{(\log\log n)^{b-1}}{\log n}m\mathbb{P}\{|X|>\varepsilon\sqrt{n}(\log\log n)^d/T\}+\\ &\quad C\sum_{n>b(\varepsilon)}\frac{(\log\log n)^{b-1}}{n\log n}\frac{1}{(\log\log n)^{2dT/\sqrt{m}}\varepsilon^{2T/\sqrt{m}}}:=L_1+L_2,\end{aligned}$$

where T is a positive constant to be specified later. On the one hand, we have

$$\begin{aligned}
\limsup_{\varepsilon\searrow 0}\varepsilon^{b/d}L_2 &\le \limsup_{\varepsilon\searrow 0} C\varepsilon^{b/d-2T/\sqrt{m}}\sum_{n>b(\varepsilon)}\frac{(\log\log n)^{b-1-2dT/\sqrt{m}}}{n\log n}\\
&\le \limsup_{\varepsilon\searrow 0} C\varepsilon^{b/d-2T/\sqrt{m}}(\log\log b(\varepsilon))^{b-2dT/\sqrt{m}}\\
&\le CM^{b-2dT/\sqrt{m}}\to 0,\ \text{as } M\to\infty
\end{aligned}$$

for any $T>\sqrt{m}b/(2d)$. On the other hand, for L_1, without loss of generality assume $T=1$. It is obtained that

$$\begin{aligned}
L_1 &= \sum_{n>b(\varepsilon)}\frac{(\log\log n)^{b-1}}{\log n} m\mathbb{E}\{I\{|X|\ge\varepsilon\sqrt{n}(\log\log n)^d\}\}\\
&= \sum_{n>b(\varepsilon)}\frac{(\log\log n)^{b-1}}{\log n} m\sum_{j=n}^{\infty}\mathbb{E}\{I\{\varepsilon\sqrt{j}(\log\log j)^d\le |X|<\varepsilon\sqrt{j+1}(\log\log(j+1))^d\}\}\\
&= \sum_{j>b(\varepsilon)}\mathbb{E}\{I\{\varepsilon\sqrt{j}(\log\log j)^d\le |X|<\varepsilon\sqrt{j+1}(\log\log(j+1))^d\}\}\sum_{n>b(\varepsilon)}^{j}\frac{(\log\log n)^{b-1}}{\log n}.
\end{aligned}$$

From $(\log\log n)^{b-1}(\log n)^{-1}\to 0$, as $n>b(\varepsilon)\to\infty$, it follows that

$$\begin{aligned}
L_1 &\le C\sum_{j>b(\varepsilon)} j\mathbb{E}\{I\{\varepsilon\sqrt{j}(\log\log j)^d\le |X|<\varepsilon\sqrt{j+1}(\log\log(j+1))^d\}\}\\
&\le C\varepsilon^{-2}\sum_{j>b(\varepsilon)}\mathbb{E}\{|X|^2 I\{\varepsilon\sqrt{j}(\log\log j)^d\le |X|<\varepsilon\sqrt{j+1}(\log\log(j+1))^d\}\}\\
&\le C\varepsilon^{-2}\mathbb{E}\{|X|^2 I\{|X|\ge\varepsilon\sqrt{b(\varepsilon)}(\log\log b(\varepsilon))^d\}\}.
\end{aligned}$$

Consequently, for $b\ge 2d$, we have

$$\limsup_{\varepsilon\searrow 0}\varepsilon^{b/d}L_1\le\limsup_{\varepsilon\searrow 0}C\varepsilon^{b/d-2}\mathbb{E}\{|X|^2 I\{|X|\ge M^d\sqrt{b(\varepsilon)}/2\}\}=0.$$

Therefore (6) holds for every $b,d>0$.

Proposition 2.5 *For* $b,d>0$, *and* $a_n=o((\log\log n)^{-d})$, *we have*

$$\limsup_{\varepsilon\searrow 0}\varepsilon^{b/d}\sum_{n=3}^{\infty}\frac{(\log\log n)^{b-1}}{n\log n}\times \left|\mathbb{P}\{|S_n|\ge(\varepsilon+a_n)\sqrt{n}(\log\log n)^d\}-\mathbb{P}\{|S_n|\ge\varepsilon\sqrt{n}(\log\log n)^d\}\right|=0. \tag{7}$$

Proof Choose n_1 large enough such that $M^d/3\ge|a_n|(\log\log n)^d$ for all $n\ge n_1$. If ε is so small that $b(\varepsilon)\ge n_1$, then

$$M^d/3\ge|a_n|(\log\log n)^d\ge|a_n|(\log\log b(\varepsilon))^d\ge\frac{2}{3}|a_n|M^d\varepsilon^{-1}, n>b(\varepsilon)$$

whence $-|a_n|\ge-\varepsilon/2$, $n>b(\varepsilon)$. It follows that

$$\sum_{n>b(\varepsilon)}\frac{(\log\log n)^{b-1}}{n\log n}\mathbb{P}\{|S_n|\ge(\varepsilon+a_n)\sqrt{n}(\log\log n)^d\}$$

$$\le \sum_{n>b(\varepsilon)} \frac{(\log\log n)^{b-1}}{n\log n}\mathbb{P}\{|S_n|\ge(\varepsilon-|a_n|)\sqrt{n}(\log\log n)^d\}$$
$$\le \sum_{n>b(\varepsilon)} \frac{(\log\log n)^{b-1}}{n\log n}\mathbb{P}\{|S_n|\ge(\varepsilon/2)\sqrt{n}(\log\log n)^d\}.$$

Then from Proposition 2.4, it suffices to get

$$\lim_{M\to\infty}\limsup_{\varepsilon\searrow 0}\varepsilon^{b/d}\sum_{n\le b(\varepsilon)}\frac{(\log\log n)^{b-1}}{n\log n}\times$$
$$\big|\mathbb{P}\{|S_n|\ge(\varepsilon+a_n)\sqrt{n}(\log\log n)^d\}-\mathbb{P}\{|S_n|\ge\varepsilon\sqrt{n}(\log\log n)^d\}\big|=0.$$

By a similar argument of Proposition 2.2, we have

$$\limsup_{\varepsilon\searrow 0}\varepsilon^{b/d}\sum_{n\le b(\varepsilon)}\frac{(\log\log n)^{b-1}}{n\log n}\times$$
$$\big|\mathbb{P}\{|S_n|\ge(\varepsilon+a_n)\sqrt{n}(\log\log n)^d\}-\mathbb{P}\{|N|\ge(\varepsilon+a_n)(\log\log n)^d\}\big|=0.$$

Hence, it remains to get

$$\limsup_{\varepsilon\searrow 0}\varepsilon^{b/d}\sum_{n\le b(\varepsilon)}\frac{(\log\log n)^{b-1}}{n\log n}\big|\mathbb{P}\{|N|\ge(\varepsilon+a_n)(\log\log n)^d\}-\mathbb{P}\{|N|\ge\varepsilon(\log\log n)^d\}\big|=0$$

or an even stronger result

$$\limsup_{\varepsilon\searrow 0}\varepsilon^{b/d}\sum_{n\le b(\varepsilon)}\frac{(\log\log n)^{b-1}}{n\log n}\mathbb{P}\{(\varepsilon-|a_n|)(\log\log n)^d\le|N|\le(\varepsilon+|a_n|)(\log\log n)^d\}=0.$$

But the left hand side of the above equality is no more than

$$\limsup_{\varepsilon\searrow 0}\varepsilon^{b/d}\sum_{n\le b(\varepsilon)}\frac{(\log\log n)^{b-1}}{n\log n}\int_{(\varepsilon-|a_n|)^2(\log\log n)^{2d}}^{(\varepsilon+|a_n|)^2(\log\log n)^{2d}}\frac{x^{m/2-1}\mathrm{e}^{-x/2}}{2^{m/2}\Gamma(m/2)}\mathrm{d}x$$
$$\le C\limsup_{\varepsilon\searrow 0}\varepsilon^{b/d}\sum_{n\le b(\varepsilon)}\frac{(\log\log n)^{b-1}}{n\log n}\int_{(\varepsilon-|a_n|)(\log\log n)^{d}}^{(\varepsilon+|a_n|)(\log\log n)^{d}}\frac{x^{m-1}\mathrm{e}^{-x^2/2}}{2^{m/2}\Gamma(m/2)}2\mathrm{d}x$$
$$\le C\limsup_{\varepsilon\searrow 0}\varepsilon^{b/d}\sum_{n\le b(\varepsilon)}\frac{(\log\log n)^{b-1}}{n\log n}|a_n|(\log\log n)^d$$
$$\le C\limsup_{\varepsilon\searrow 0}\frac{M^b}{(\log\log b(\varepsilon))^b}\sum_{n\le b(\varepsilon)}\frac{(\log\log n)^{b-1}}{n\log n}|a_n|(\log\log n)^d$$
$$=0,$$

by the Toeplitz lemma since $|a_n|(\log\log n)^d\to 0$ as $n\to\infty$. Therefore we have the desired result.

Now by Propositions 2.1–2.5 and the triangle inequality, we obtain the proof of Theorem 1.1. $\Box$

3. Proof of Theorem 1.3

Proposition 3.1 *For $d>0$, we have*

$$\lim_{\varepsilon\searrow 0}\frac{1}{-\log\varepsilon}\sum_{n=3}^{\infty}\frac{1}{n\log n\log\log n}\mathbb{P}\{|N|\ge\varepsilon(\log\log n)^d\}=\frac{1}{d}. \tag{8}$$

Proof We obtain

$$\begin{aligned}
&\lim_{\varepsilon\searrow 0}\frac{1}{-\log\varepsilon}\sum_{n=3}^{\infty}\frac{1}{n\log n\log\log n}\mathbb{P}\{|N|\ge\varepsilon(\log\log n)^d\}\\
&=\lim_{\varepsilon\searrow 0}\frac{1}{-\log\varepsilon}\int_{\mathrm{e}^{\mathrm{e}}}^{\infty}\frac{1}{y\log y\log\log y}\mathrm{d}y\int_{\varepsilon^2(\log\log y)^{2d}}^{\infty}\frac{x^{m/2-1}\mathrm{e}^{-x/2}}{2^{m/2}\Gamma(m/2)}\mathrm{d}x\\
&=\frac{1}{2d}\lim_{\varepsilon\searrow 0}\frac{1}{-\log\varepsilon}\int_{\varepsilon^2}^{\infty}\frac{1}{t}\mathrm{d}t\int_{t}^{\infty}\frac{x^{m/2-1}\mathrm{e}^{-x/2}}{2^{m/2}\Gamma(m/2)}\mathrm{d}x\\
&=\frac{1}{2d}\lim_{\varepsilon\searrow 0}\frac{1}{-\log\varepsilon}\int_{\varepsilon^2}^{\infty}\frac{x^{m/2-1}\mathrm{e}^{-x/2}}{2^{m/2}\Gamma(m/2)}\mathrm{d}x\int_{\varepsilon^2}^{x}\frac{1}{t}\mathrm{d}t\\
&=\frac{1}{2d}\lim_{\varepsilon\searrow 0}\frac{1}{-\log\varepsilon}\int_{\varepsilon^2}^{\infty}\frac{x^{m/2-1}\mathrm{e}^{-x/2}}{2^{m/2}\Gamma(m/2)}\log x\mathrm{d}x+\lim_{\varepsilon\searrow 0}\frac{1}{d}\int_{\varepsilon^2}^{\infty}\frac{x^{m/2-1}\mathrm{e}^{-x/2}}{2^{m/2}\Gamma(m/2)}\mathrm{d}x\\
&=\frac{1}{d}+\frac{1}{2d}\lim_{\varepsilon\searrow 0}\frac{1}{-\log\varepsilon}\int_{\varepsilon^2}^{\infty}\frac{x^{m/2-1}\mathrm{e}^{-x/2}}{2^{m/2}\Gamma(m/2)}\log x\mathrm{d}x\\
&=\frac{1}{d}.
\end{aligned}$$

Indeed, since

$$\int_{\varepsilon^2}^{1}\frac{x^{m/2-1}}{2^{m/2}\Gamma(m/2)}\log x\mathrm{d}x\le\int_{\varepsilon^2}^{1}\frac{x^{m/2-1}\mathrm{e}^{-x/2}}{2^{m/2}\Gamma(m/2)}\log x\mathrm{d}x\le\frac{\mathrm{e}^{-1/2}}{2^{m/2}\Gamma(m/2)}\int_{\varepsilon^2}^{1}x^{m/2-1}\log x\mathrm{d}x,$$

$$\lim_{\varepsilon\searrow 0}\int_{\varepsilon^2}^{1}x^{m/2-1}\log x\mathrm{d}x=\lim_{\varepsilon\searrow 0}\int_{\varepsilon^2}^{1}\frac{2}{m}\log x\mathrm{d}x^{m/2}=-(2/m)^2,$$

and so

$$\lim_{\varepsilon\searrow 0}\frac{1}{-\log\varepsilon}\int_{\varepsilon^2}^{\infty}\frac{x^{m/2-1}\mathrm{e}^{-x/2}}{2^{m/2}\Gamma(m/2)}\log x\mathrm{d}x=0.$$

This establishes (8). □

Proposition 3.2 *For $d>0$, we have*

$$\lim_{\varepsilon\searrow 0}\frac{1}{-\log\varepsilon}\sum_{n\le b(\varepsilon)}\frac{1}{n\log n\log\log n}\left|\mathbb{P}\{|S_n|\ge\varepsilon\sqrt{n}(\log\log n)^d\}-\mathbb{P}\{|N|\ge\varepsilon(\log\log n)^d\}\right|=0.$$

Proof The proof of this proposition is similar to that of [3, Proposition 3.2], so the proof is omitted. □

Proposition 3.3 *For $d>0$, we have*

$$\lim_{\varepsilon\searrow 0}\frac{1}{-\log\varepsilon}\sum_{n>b(\varepsilon)}\frac{1}{n\log n\log\log n}\mathbb{P}\{|N|\ge\varepsilon(\log\log n)^d\}=0.$$

Proof From the proof of Proposition 3.1, we deduce that

$$\lim_{\varepsilon\searrow 0}\frac{1}{-\log\varepsilon}\sum_{n>b(\varepsilon)}\frac{1}{n\log n\log\log n}\mathbb{P}\{|N|\ge\varepsilon(\log\log n)^d\}$$

$$\le C\lim_{\varepsilon\searrow 0}\frac{1}{-\log\varepsilon}\int_{b(\varepsilon)}^{\infty}\frac{1}{y\log y\log\log y}\mathrm{d}y\int_{\varepsilon^2(\log\log y)^{2d}}^{\infty}\frac{x^{m/2-1}\mathrm{e}^{-x/2}}{2^{m/2}\Gamma(m/2)}\mathrm{d}x$$
$$\le C\lim_{\varepsilon\searrow 0}\frac{1}{-\log\varepsilon}\int_{M^{2d}}^{\infty}\frac{1}{t}\mathrm{d}t\int_{t}^{\infty}\frac{x^{m/2-1}\mathrm{e}^{-x/2}}{2^{m/2}\Gamma(m/2)}\mathrm{d}x=0,$$

since

$$\int_{M^{2d}}^{\infty}\frac{1}{t}\mathrm{d}t\int_{t}^{\infty}\frac{x^{m/2-1}\mathrm{e}^{-x/2}}{2^{m/2}\Gamma(m/2)}\mathrm{d}x$$

is integrable. Thus this proves Proposition 3.3. □

Proposition 3.4 *For $d>0$, one has*

$$\lim_{\varepsilon\searrow 0}\frac{1}{-\log\varepsilon}\sum_{n>b(\varepsilon)}\frac{1}{n\log n\log\log n}\mathbb{P}\{|S_n|\ge\varepsilon\sqrt{n}(\log\log n)^d\}=0.$$

Proof The proof of this proposition is similar to that of [3, Proposition 3.4], so the proof is omitted. □

Proposition 3.5 *For $b,d>0$, and $a_n=o((\log\log n)^{-d})$, one has*

$$\limsup_{\varepsilon\searrow 0}\frac{1}{-\log\varepsilon}\sum_{n=3}^{\infty}\frac{1}{n\log n\log\log n}\times$$
$$\big|\mathbb{P}\{|S_n|\ge(\varepsilon+a_n)\sqrt{n}(\log\log n)^d\}-\mathbb{P}\{|S_n|\ge\varepsilon\sqrt{n}(\log\log n)^d\}\big|=0.$$

Proof The proof of this proposition is similar to that of [3, Proposition 3.5], so the proof is omitted. □

Finally, by the triangle inequality and Propositions 3.1–3.5, we obtain the proof of Theorem 1.3.

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References

[1] A. GUT, A. SPĂTARU. *Precise asymptotics in the law of the iterated logarithm*. Ann. Probab., 2000, **28**(4): 1870–1883.
[2] Lixin ZHANG. *Precise rates in the law of the iterated logarithm*. Preprint http://arxiv.org/abs/math.PR/0610519, 2006.
[3] Xiaoyong XIAO, Lixin ZHANG, Hongwei YIN. *Precise rates in the generalized law of the iterated logarithm*. Statist. Probab. Lett., 2013, **83**(83): 616–623.
[4] Wei HUANG, Lixin ZHANG, Ye JIANG. *Precise rate in the law of iterated logarithm for ρ-mixing sequence*. Appl. Math. J. Chinese Univ. Ser. B, 2003, **18**(4): 482–488.
[5] Chaowei JIANG, Xiaorong YANG. *Precise asymptotics in self-normalized sums of iterated logarithm for multidimensionally indexed random variables*. Appl. Math. J. Chinese Univ. Ser. B, 2007, **22**(1): 87–194.
[6] Hongmei WU, Jiwei WEN. *Precise rates in the law of iterated logarithm for R/S statistics*. Appl. Math. J. Chinese Univ. Ser. B, 2006, **21**(4): 461–466.
[7] A. SPĂTARU. *Precise asymptotics in Spitzer'law of large number*. J. Theoret. Probab., 1999, **12**(3): 811–819.
[8] D. H. FUK, S. V. NAGAEV. *Probability inequalities for sums of independent random variables*. Theor. Probab. Appl., 1971, **16**(4): 643–660.
[9] P. BILLINGSLEY. *Probability and Measure*. John Wiley & Sons, Inc., New York, 1986.
[10] W. F. STOUT. *Almost Sure Convergence*. Academic, New York, 1995.