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Ordering Quasi-Tree Graphs on n Vertices by Their Spectral Radii

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Abstract A connected graph $G=(V,E)$ is called a quasi-tree graph, if there exists a vertex $v_0\in V(G)$ such that $G-v_0$ is a tree. Liu and Lu [Linear Algebra Appl. 428 (2008) 2708-2714] determined the maximal spectral radius together with the corresponding graph among all quasi-tree graphs on n vertices. In this paper, we extend their result, and determine the second to the fifth largest spectral radii together with the corresponding graphs among all quasi-tree graphs on n vertices.

Keywords quasi-tree graph; spectral radius; extremal graph

MR(2010) Subject Classification 05C50

1. Introduction

Let $G=(V,E)$ be a simple undirected graph with vertex set $V=V(G)=\{v_1,v_2,\ldots,v_n\}$ and edge set $E=E(G)$. As usual, $A(G)$ denotes the adjacency matrix of a graph G and $\rho(G)$ denotes the largest eigenvalue of $A(G)$ which is called spectral radius of G. If G is connected, then $A(G)$ is irreducible and by the Perron-Frobenius theorem, $\rho(G)$ has multiplicity one and there exists a unique positive unit eigenvector corresponding to $\rho(G)$. We will refer to such an eigenvector as the Perron vector of G.

The study of ordering graphs by their spectral radius was stared by Collatz and Sinogowitz [1] in 1957. Cvetković [2] proposed twelve directions for further research in the theory of graph spectra, one of which is "classifying and ordering graphs". From then on, ordering graphs with various properties by their spectra, specially by their largest eigenvalues, becomes an attractive topic. There are many results on ordering graphs by their spectral radii [3–12].

A connected graph $G=(V,E)$ is called a quasi-tree graph, if there exists a vertex $v_0\in V(G)$ such that $G-v_0$ is a tree. Let $\mathcal{Q}_n$ be the set of all quasi-tree graphs on n vertices. Liu and Lu [12] determined the maximal spectral radius together with the corresponding graph among all quasi-tree graphs in the set $\mathcal{Q}_n$. In this paper, we extend their result, and determine the second to the fifth largest spectral radii together with the corresponding graphs among all quasi-tree

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graphs in the set $\mathcal{Q}_n$. The main result of this paper is as follows:

Theorem 1.1 *Let $n \geq 32$, $G \in \mathcal{Q}_n \setminus \{G_n^1, G_n^2, G_n^3, G_n^4, G_n^5\}$, where G_n^i $(i = 1, 2, \ldots, 5)$ is showed in Figure 1. Then*

$$\rho(G_n^1) > \rho(G_n^2) > \rho(G_n^3) > \rho(G_n^4) > \rho(G_n^5) > \rho(G),$$

where $\rho(G_n^i)$ $(i = 1, 2, \ldots, 5)$ is the largest root of the following polynomial $f_i(x)$,

$$f_1(x) := x^3 - (2n-3)x - 2(n-2),$$
$$f_2(x) := x^5 - (2n-3)x^3 - 2(n-2)x^2 + 3(n-4)x + 2(n-4),$$
$$f_3(x) := x^4 - 2(n-2)x^2 - 2(n-3)x + (n-3),$$
$$f_4(x) := x^5 - 2(n-2)x^3 - 2(n-3)x^2 + (2n-7)x + 2(n-4),$$
$$f_5(x) := x^5 - (2n-3)x^3 - 2(n-2)x^2 + 6(n-5)x + 4(n-5).$$

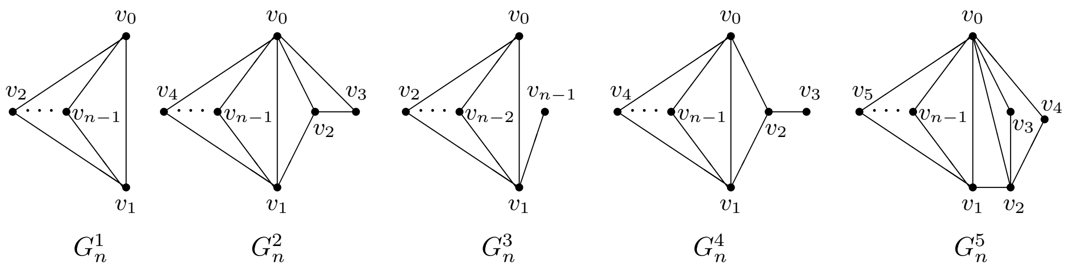


Figure 1 G_n^i $(i = 1, 2, 3, 4, 5)$

The rest of the paper is organized as follows. In Section 2, we recall some basic notions and lemmas used further, and prove three new lemmas. In Section 3, we give a proof of Theorem 1.1.

2. Preliminaries

Denote by $K_{1,n-1}$ and P_n the star and the path on n vertices, respectively. Let $G-u$ denote the graph that arises from G by deleting the vertex $u \in V(G)$ and all the edges incident with u, and $G-uv$ denote the graph that arises from G by deleting the edge $uv \in E(G)$. Similarly, $G+uv$ is the graph that arises from G by adding an edge $uv \notin E(G)$, where $u, v \in V(G)$. For $v \in V(G)$, $N(v)$ denotes the neighborhood of v in G and $d(v) = |N(v)|$ denotes the degree of vertex v. The diameter of a connected graph is the maximum distance between pairs of its vertices. A pendant vertex of G is a vertex of degree 1. A pendant edge of G is an edge incident with a pendant vertex. Let $S_n(s,t)$ $(s \geq t \geq 1, s+t = n-2)$ be the graph on n vertices obtained from a path P_2 by attaching s pendant edges and t pendant edges to each end vertex of P_2, respectively. Denote by $\Phi(G, x)$ the characteristic polynomial of a graph G, where $\Phi(G, x) = \det(xE_n - A(G))$.

Lemma 2.1 ([13]) *Let G be a connected graph of order n and $\rho(G)$ be the spectral radius of $A(G)$. Let u, v be two vertices of G. Suppose $v_1, v_2, \ldots, v_s \in N_G(v) \backslash N_G(u)$ $(1 \leq s \leq d_G(v))$, and $x = (x_1, x_2, \ldots, x_n)^T$ is the Perron vector of $A(G)$, where x_i corresponds to the vertex*

v_i $(1 \le i \le n)$. Let G^* be the graph obtained from G by deleting the edges vv_i and adding the edges uv_i $(1 \le i \le s)$. If $x_u \ge x_v$, then $\rho(G) < \rho(G^*)$.

Lemma 2.2 ([14]) *Let $C(u)$ be the set of all cycles containing u, where $u \in V(G)$. Then*

$$\Phi(G;x) = x\Phi(G-u;x) - \sum_{v\in N(u)} \Phi(G-u-v;x) - 2\sum_{Z\in C(u)} \Phi(G-V(Z);x).$$

Lemma 2.3 ([15]) *Let G_1 and G_2 be two graphs. If $\Phi(G_2,x) > \Phi(G_1,x)$ for $x \ge \rho(G_2)$, then $\rho(G_1) > \rho(G_2)$.*

Lemma 2.4 ([16]) *Let G be a connected graph of order n, and H be a proper subgraph of G. Then $\rho(H) < \rho(G)$.*

Lemma 2.5 ([17]) *For a connected graph G, $V(G) = \{v_1, v_2, \ldots, v_n\}$, then*

$$\rho(G) \le \max\{\sqrt{d_{v_i} m_{v_i}} : 1 \le i \le n\},$$

where $m_{v_i} = \sum_{v_j v_i \in E(G)} d_{v_j}/d_{v_i}$.

Let Q_{n,d_0} be the graph obtained from a star $K_{1,n-2}$ and an isolated vertex v_0 by adding an edge joining v_0 to the center of $K_{1,n-2}$ and $d_0 - 1$ edges joining v_0 to the pendant vertices of $K_{1,n-2}$, respectively. Let Q^*_{n,d_0} $(2 \le d_0 \le n-2)$ be the graph obtained from a graph Q_{n-1,d_0} by attaching a pendant edge to one vertex of degree 2 in Q_{n-1,d_0}. For $1 \le d_0 \le n-1$, let $\mathcal{Q}(n, d_0) := \{G : G \in \mathcal{Q}_n$ with $G - v_0$ being a tree and $d_G(v_0) = d_0\}$.

Lemma 2.6 ([12]) *(i) Let $n \ge 4$, $1 \le d_0 \le n-1$, $G \in \mathcal{Q}(n, d_0)$. Then $\rho(G) \le \rho(Q_{n,d_0})$ and equality holds if and only if $G \cong Q_{n,d_0}$.*

(ii) Let $n \ge 5$, $2 \le d_0 \le n-2$, $G \in \mathcal{Q}(n, d_0)\backslash\{Q_{n,d_0}\}$. Then $\rho(G) \le \rho(Q^*_{n,d_0})$ and equality holds if and only if $G \cong Q^*_{n,d_0}$.

(iii) $\rho(Q_{n,d_0+1}) > \rho(Q_{n,d_0})$ for $1 \le d_0 \le n-2$.

Lemma 2.7 *Let $n \ge 7$, G_n^i $(i = 1, 2, 5)$ be showed in Figure 1, $G \in \mathcal{Q}(n, n-1) \setminus \{G_n^1, G_n^2, G_n^5\}$. Then $\rho(G_n^1) > \rho(G_n^2) > \rho(G_n^5) > \rho(G)$.*

Proof Let $G \in \mathcal{Q}(n, n-1) \setminus \{G_n^1\}$. By Lemma 2.6, we have $\rho(G_n^1) > \rho(G)$.

Let $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $x = \{x_0, x_1, \ldots, x_{n-1}\}^T$ be the Perron vector of $A(G)$, where x_i corresponds to the vertex v_i for $0 \le i \le n-1$. Assume, without loss of generality, that $G - v_0$ is a tree.

Step 1. Choose $G \in \mathcal{Q}(n, n-1) \setminus \{G_n^1\}$ such that $\rho(G)$ is as large as possible, and prove that $G = G_n^2$. Since $G \in \mathcal{Q}(n, n-1) \setminus \{G_n^1\}$, it follows that $G - v_0 \ne K_{1,n-2}$.

Firstly, we show that the diameter d of $G - v_0$ is 3. Otherwise, we suppose $d > 3$. Then there are at least two nonpendant edges in $G - v_0$. Assume that uv is one of them. If $x_u \ge x_v$, let

$$G^* = G - \sum_{w\in N_G(v)\setminus\{u,v_0\}} vw + \sum_{w\in N_G(v)\setminus\{u,v_0\}} uw;$$

if $x_u < x_v$, let

$$G^* = G - \sum_{w \in N_G(u) \setminus \{v, v_0\}} uw + \sum_{w \in N_G(u) \setminus \{v, v_0\}} vw.$$

Then in either case, $G^* \in \mathcal{Q}(n, n-1) \setminus \{G_n^1\}$. By Lemma 2.1, we have $\rho(G^*) > \rho(G)$, a contradiction. Therefore the diameter d of $G - v_0$ is 3, $G - v_0 = S_{n-1}(s, t)$ $(s \geq t \geq 1)$.

Secondly, we show that $G - v_0 = S_{n-1}(n-4, 1)$. Otherwise, we suppose $t > 1$. Let uv be the unique nonpendant edge in $G - v_0$, and ua, vb be two pendant edges in $G - v_0$. If $x_u \geq x_v$, let

$$G^* = G - \sum_{w \in N_G(v) \setminus \{u, v_0, b\}} vw + \sum_{w \in N_G(v) \setminus \{u, v_0, b\}} uw;$$

if $x_u < x_v$, let

$$G^* = G - \sum_{w \in N_G(u) \setminus \{v, v_0, a\}} uw + \sum_{w \in N_G(u) \setminus \{v, v_0, a\}} vw.$$

Then in either case, $G^* \in \mathcal{Q}(n, n-1) \setminus \{G_n^1\}$. By Lemma 2.1, we have $\rho(G^*) > \rho(G)$, a contradiction. Therefore, $G - v_0 = S_{n-1}(n-4, 1)$. Namely, $G = G_n^2$.

Step 2. Choose $G \in \mathcal{Q}(n, n-1) \setminus \{G_n^1, G_n^2\}$ such that $\rho(G)$ is as large as possible, and show that $G = G_n^5$. Since $G \in \mathcal{Q}(n, n-1) \setminus \{G_n^1, G_n^2\}$, it follows that $G - v_0 \neq K_{1,n-2}$, $G - v_0 \neq S_{n-1}(n-4, 1)$.

Similarly, we can show that $G - v_0 = S_{n-1}(s, t)$ $(s \geq t \geq 2)$. Next we show that $G - v_0 = S_{n-1}(n-5, 2)$. Otherwise, we suppose $t \geq 3$. Let uv be the unique nonpendant edge in $G - v_0$, and ua, ub, vc, vd be four pendant edges in $G - v_0$. If $x_u \geq x_v$, let

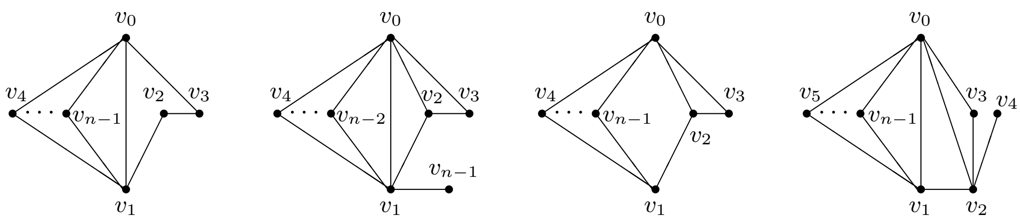
$$G^* = G - \sum_{w \in N_G(v) \setminus \{u, v_0, c, d\}} vw + \sum_{w \in N_G(v) \setminus \{u, v_0, c, d\}} uw;$$

if $x_u < x_v$, let

$$G^* = G - \sum_{w \in N_G(u) \setminus \{v, v_0, a, b\}} uw + \sum_{w \in N_G(u) \setminus \{v, v_0, a, b\}} vw.$$

Then in either case, $G^* \in \mathcal{Q}(n, n-1) \setminus \{G_n^1, G_n^2\}$. By Lemma 2.1, we have $\rho(G^*) > \rho(G)$, a contradiction. Therefore, $G - v_0 = S_{n-1}(n-5, 2)$. Namely, $G = G_n^5$. □

Lemma 2.8 *Let $n \geq 10$, G_n^3, G_n^4 and F_i $(i = 1, 2, \ldots, 7)$ be showed in Figures 1 and 2, respectively, $G \in \mathcal{Q}(n, n-2) \setminus \{G_n^3, G_n^4, F_2\}$. Then $\rho(G_n^3) > \rho(G_n^4) > \rho(F_2) > \rho(G)$.*



F_1 F_2 F_3 F_4

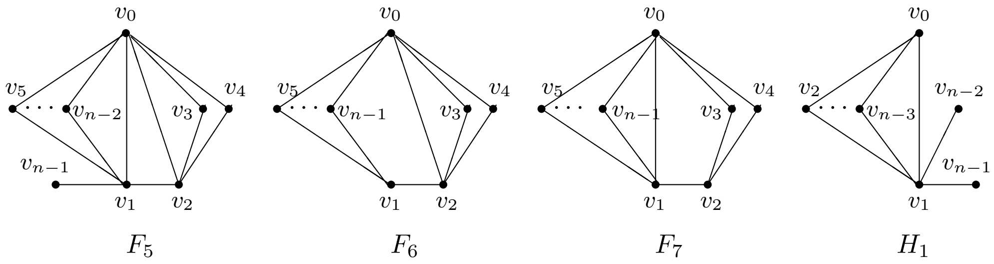


Figure 2 F_i $(i = 1, 2, 3, 4, 5, 6, 7)$, H_1

Proof By Lemma 2.6, we have $\rho(G_n^3) > \rho(G_n^4) > \rho(G)$.

Choose $G \in \mathcal{Q}(n, n-2) \setminus \{G_n^3, G_n^4\}$ such that $\rho(G)$ is as large as possible. We will prove that $G = F_2$. Let $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $x = \{x_0, x_1, \ldots, x_{n-1}\}^T$ be the Perron vector of $A(G)$, where x_i corresponds to the vertex v_i for $0 \le i \le n-1$. Assume, without loss of generality, that $G - v_0$ is a tree.

Firstly, we claim that $G - v_0 = K_{1,\,n-2}$ or $G - v_0 = S_{n-1}(s,t)$ $(s \ge t \ge 1)$. Otherwise, we suppose that the diameter d of $G - v_0$ is more than 3, then there are at least two nonpendant edges in $G - v_0$. It is not difficult to see that there must exist a nonpendant edge uv in $G - v_0$ such that $x_u \ge x_v$ and

$$G^* = G - \sum_{w \in N_G(v)\setminus\{u, v_0\}} vw + \sum_{w \in N_G(v)\setminus\{u, v_0\}} uw \in \mathcal{Q}(n, n-2) \setminus \{G_n^3, G_n^4\}.$$

By Lemma 2.1, we have $\rho(G^*) > \rho(G)$, a contradiction. Therefore, the diameter d of $G - v_0$ is not more than 3, then $G - v_0 = K_{1,\,n-2}$ or $G - v_0 = S_{n-1}(s,t)$ $(s \ge t \ge 1)$.

Secondly, for the case of $G - v_0 = S_{n-1}(s,t)$ $(s \ge t \ge 1)$, we claim that $G - v_0 = S_{n-1}(n-4, 1)$ or $G - v_0 = S_{n-1}(n-5, 2)$. Otherwise, we suppose $t \ge 3$. Let uv be the unique nonpendant edge in $G - v_0$. If $v_0u, v_0v \in E(G)$, then there exists a vertex $v_i \in V(G) \setminus \{v_0, u, v\}$ such that $v_0v_i \notin E(G)$. Without loss of generality, we may assume that $x_u \ge x_v$ and vv_j is a pendant edge in $G - v_0$ such that $v_j \ne v_i$. Let

$$G^* = G - \sum_{w \in N_G(v)\setminus\{u, v_0, v_j\}} vw + \sum_{w \in N_G(v)\setminus\{u, v_0, v_j\}} uw.$$

Then $G^* \in \mathcal{Q}(n, n-2) \setminus \{G_n^3, G_n^4\}$, by Lemma 2.1, we have $\rho(G^*) > \rho(G)$, a contradiction. If $v_0u \notin E(G)$, then let ua, vb be two pendant edges in $G - v_0$. If $x_u \ge x_v$, let

$$G^* = G - \sum_{w \in N_G(v)\setminus\{u, v_0, b\}} vw + \sum_{w \in N_G(v)\setminus\{u, v_0, b\}} uw;$$

if $x_u < x_v$, let

$$G^* = G - \sum_{w \in N_G(u)\setminus\{v, a\}} uw + \sum_{w \in N_G(u)\setminus\{v, a\}} vw.$$

Then in either case, $G^* \in \mathcal{Q}(n, n-2) \setminus \{G_n^3, G_n^4\}$. By Lemma 2.1, we have $\rho(G^*) > \rho(G)$, a contradiction. If $v_0v \notin E(G)$, similarly, we can also get a contradiction. Therefore, $G - v_0 = S_{n-1}(n-4, 1)$ or $G - v_0 = S_{n-1}(n-5, 2)$.

Thirdly, when $G - v_0 = K_{1,n-2}$, it is easy to see that $G = K_{2,n-2}$.

Fourthly, for the case of $G - v_0 = S_{n-1}(n-4, 1)$, we claim that $G = F_2$. In this case, $G \in \{F_1, F_2, F_3\}$. For F_1, if $x_2 \geq x_4$, let $G^* = F_1 - v_4v_0 + v_2v_0$; if $x_2 < x_4$, let $G^* = F_1 - v_2v_3 + v_4v_3$. Then in either case, $G^* \cong F_2 \in \mathcal{Q}(n, n-2) \setminus \{G_n^3, G_n^4\}$. By Lemma 2.1, we have $\rho(F_2) > \rho(F_1)$. This implies that $G \neq F_1$. For F_3, if $x_1 \geq x_4$, let $G^* = F_3 - v_4v_0 + v_1v_0$; if $x_1 < x_4$, let

$$G^* = F_3 - \sum_{u \in N_{F_3}(v_1)\setminus\{v_4\}} v_1u + \sum_{u \in N_{F_3}(v_1)\setminus\{v_4\}} v_4u.$$

Then in either case, $G^* \cong F_2 \in \mathcal{Q}(n, n-2) \setminus \{G_n^3, G_n^4\}$. By Lemma 2.1, we have $\rho(F_2) > \rho(F_3)$. This implies that $G \neq F_3$. From the above arguments, we have $G = F_2$.

Fifthly, for the case of $G - v_0 = S_{n-1}(n-5, 2)$, we claim that $G = F_4$. In this case, $G \in \{F_4, F_5, F_6, F_7\}$. For F_6, if $x_1 \geq x_5$, let $G^* = F_6 - v_5v_0 + v_1v_0$; if $x_1 < x_5$, let

$$G^* = F_6 - \sum_{u \in N_{F_6}(v_1)\setminus\{v_5\}} v_1u + \sum_{u \in N_{F_6}(v_1)\setminus\{v_5\}} v_5u.$$

Then in either case, $G^* \cong F_5 \in \mathcal{Q}(n, n-2) \setminus \{G_n^3, G_n^4\}$. By Lemma 2.1, we have $\rho(F_5) > \rho(F_6)$. This implies that $G \neq F_6$. For F_7, if $x_2 \geq x_3$, let $G^* = F_7 - v_3v_0 + v_2v_0$; if $x_2 < x_3$, let

$$G^* = F_7 - \sum_{u \in N_{F_7}(v_2)\setminus\{v_3\}} v_2u + \sum_{u \in N_{F_7}(v_2)\setminus\{v_3\}} v_3u.$$

Then in either case, $G^* \cong F_4 \in \mathcal{Q}(n, n-2) \setminus \{G_n^3, G_n^4\}$. By Lemma 2.1, we have $\rho(F_4) > \rho(F_7)$. This implies that $G \neq F_7$. For F_5, by Lemma 2.2, we have

$$\begin{aligned}
\Phi(F_4; x) =& x^{n-6}[x^6 - 2(n-2)x^4 - 2(n-3)x^3 + (5n-18)x^2 + 4(n-5)x - (n-5)],\\
\Phi(F_5; x) =& x^{n-6}[x^6 - 2(n-2)x^4 - 2(n-3)x^3 + (7n-37)x^2 + 4(n-5)x - 2(n-6)].
\end{aligned}$$

Noting that $K_{2,n-5}$ is a subgraph of F_5, by Lemma 2.4, we have $\rho(F_5) > \rho(K_{2,n-5}) = \sqrt{2(n-5)}$. It follows that

$$\Phi(F_5; x) - \Phi(F_4; x) = x^{n-6}[(2n-19)x^2 - (n-7)] > 0$$

for $x \geq \rho(F_5)$. By Lemma 2.3, we have $\rho(F_4) > \rho(F_5)$. This implies that $G \neq F_5$. From the above arguments, we have $G = F_4$.

From the above arguments, we have $G \in \{K_{2,n-2}, F_2, F_4\}$. Now we show that $G = F_2$. Noting that $K_{2,n-4}$ is a subgraph of F_4, by Lemma 2.4, we have $\rho(F_4) > \rho(K_{2,n-4}) = \sqrt{2(n-4)}$. By Lemma 2.2, we have

$$\begin{aligned}
\Phi(F_2; x) =& x^{n-6}[x^6 - 2(n-2)x^4 - 2(n-3)x^3 + (4n-17)x^2 + 2(n-4)x - (n-5)],\\
\Phi(F_4; x) =& x^{n-6}[x^6 - 2(n-2)x^4 - 2(n-3)x^3 + (5n-18)x^2 + 4(n-5)x - (n-5)].
\end{aligned}$$

It follows that

$$\Phi(F_4; x) - \Phi(F_2; x) = x^{n-5}[(n-1)x + 2(n-6)] > 0$$

for $x \geq \rho(F_4)$. By Lemma 2.3, we have $\rho(F_2) > \rho(F_4)$. This implies that $G \neq F_4$. Noting that $\rho(K_{2,n-2}) = \sqrt{2(n-2)}$ and $\Phi(F_2; \rho(K_{2,n-2})) < 0$, we have $\rho(F_2) > \rho(K_{2,n-2})$. This implies that $G \neq K_{2,n-2}$. Combining the above arguments, we have $G = F_2$. $\square$

Lemma 2.9 *Let $n \geq 32$, G_n^i $(i = 2, 3, 4, 5)$, F_2 and H_1 be showed in Figures 1 and 2, respectively. Then*

$$\rho(G_n^2) > \rho(G_n^3),\ \rho(G_n^4) > \rho(G_n^5),\ \rho(G_n^5) > \rho(F_2),\ \rho(G_n^5) > \rho(H_1).$$

Proof By Lemma 2.2, we have

$$\begin{aligned}
\Phi(G_n^2; x) =& x^{n-5}[x^5 - (2n-3)x^3 - 2(n-2)x^2 + 3(n-4)x + 2(n-4)],\\
\Phi(G_n^3; x) =& x^{n-4}[x^4 - 2(n-2)x^2 - 2(n-3)x + (n-3)],\\
\Phi(G_n^4; x) =& x^{n-5}[x^5 - 2(n-2)x^3 - 2(n-3)x^2 + (2n-7)x + 2(n-4)],\\
\Phi(G_n^5; x) =& x^{n-5}[x^5 - (2n-3)x^3 - 2(n-2)x^2 + 6(n-5)x + 4(n-5)],\\
\Phi(F_2; x) =& x^{n-6}[x^6 - 2(n-2)x^4 - 2(n-3)x^3 + (4n-17)x^2 + 2(n-4)x - (n-5)],\\
\Phi(H_1; x) =& x^{n-4}[x^4 - (2n-5)x^2 - 2(n-4)x + 2(n-4)].
\end{aligned}$$

Firstly, we show that $\rho(G_n^2) > \rho(G_n^3)$. It is easy to see that $K_{2,\,n-3}$ is a subgraph of G_n^3. By Lemma 2.4, we have $\rho(G_n^3) > \rho(K_{2,\,n-3}) = \sqrt{2(n-3)}$. This implies that

$$\begin{aligned}
\Phi(G_n^3; x) - \Phi(G_n^2; x) &= x^{n-5}[x^3 + 2x^2 - (2n-9)x - 2(n-4)]\\
&= x^{n-5}[x(x^2 - (2n-9)) + 2(x^2 - (n-4))] > 0
\end{aligned}$$

for $x \geq \rho(G_n^3)$. By Lemma 2.3, we have $\rho(G_n^2) > \rho(G_n^3)$.

Secondly, we show that $\rho(G_n^4) > \rho(G_n^5)$. It is easy to see that $K_{2,\,n-4}$ is a subgraph of G_n^5. By Lemma 2.4, we have $\rho(G_n^5) > \rho(K_{2,\,n-4}) = \sqrt{2(n-4)}$. By Lemmas 2.5 and 2.7, we have $\rho(G_n^5) < \rho(G_n^1) \leq \sqrt{3n-5}$. This implies that $\sqrt{2(n-4)} < \rho(G_n^5) < \sqrt{3n-5}$. Note that

$$\Phi(G_n^5; x) - \Phi(G_n^4; x) = x^{n-5}[-x^3 - 2x^2 + (4n-23)x + 2n - 12].$$

Let $f(x) = -x^3 - 2x^2 + (4n-23)x + 2n - 12$. Then

$$f'(x) = -3x^2 - 4x + (4n-23),$$

$$f''(x) = -6x - 4.$$

It follows that $f'(x)$ is decreasing for $x > 0$. Noting that $f'(\sqrt{2(n-4)}\,) < 0$, we have $f(x)$ is decreasing for $x \in [\sqrt{2(n-4)}, \infty)$. Noting that $f(\sqrt{3n-5}\,) > 0$, we have $f(x) > 0$ for $x \in [\sqrt{2(n-4)}, \sqrt{3n-5}\,]$. This implies that

$$\Phi(G_n^5; x) - \Phi(G_n^4; x) > 0 \ \text{ for } \ x \in [\sqrt{2(n-4)}, \sqrt{3n-5}\,].$$

Combining the above arguments, we have $\Phi(G_n^4; \rho(G_n^5)) < 0$. Hence $\rho(G_n^4) > \rho(G_n^5)$.

Thirdly, we show that $\rho(G_n^5) > \rho(F_2)$. It is easy to see that $K_{2,\,n-4}$ is a subgraph of F_2. By Lemma 2.4, we have $\rho(F_2) > \rho(K_{2,\,n-4}) = \sqrt{2(n-4)}$. By Lemmas 2.5 and 2.6, we have $\rho(F_2) < \rho(G_n^1) \leq \sqrt{3n-5}$. This implies that $\sqrt{2(n-4)} < \rho(F_2) < \sqrt{3n-5}$. Note that

$$\Phi(F_2; x) - \Phi(G_n^5; x) = x^{n-6}[x^4 + 2x^3 - (2n-13)x^2 - 2(n-6)x - (n-5)].$$

Let $f(x) = x^4 + 2x^3 - (2n-13)x^2 - 2(n-6)x - (n-5)$. Then

$$f'(x) = 4x^3 + 6x^2 - 2(2n-13)x - 2(n-6),$$

$$f''(x) = 12x^2 + 12x - 2(2n-13),$$

$$f'''(x) = 24x + 12.$$

It is easy to see that $f''(x)$ is increasing for $x > 0$. Since $f''(\sqrt{2(n-4)}\,) > 0$, it follows that $f'(x)$ is increasing for $x \in [\sqrt{2(n-4)}, \infty)$. Noting that $f'(\sqrt{2(n-4)}\,) > 0$, we have $f(x)$ is increasing for $x \in [\sqrt{2(n-4)}, \infty)$. Noting that $f(\sqrt{2(n-4)}\,) > 0$, we have $f(x) > 0$ for $x \in [\sqrt{2(n-4)}, \sqrt{3n-5}\,]$. This implies that $\Phi(F_2; x) - \Phi(G_n^5; x) > 0$ for $x \in [\sqrt{2(n-4)}, \sqrt{3n-5}\,]$. Combining the above arguments, we have $\Phi(G_n^5; \rho(F_2)) < 0$. Hence $\rho(G_n^5) > \rho(F_2)$.

Fourthly, noting that H_1 is a subgraph of G_n^5, by Lemma 2.4, we have $\rho(G_n^5) > \rho(H_1)$. $\Box$

3. A proof of Theorem 1.1

Employing the lemmas above, now we give a proof of Theorem 1.1.

Proof Let $G \in \mathcal{Q}_n \setminus \{G_n^1\}$. By Lemma 2.6, we have $\rho(G_n^1) > \rho(G)$.

Let $G \in \mathcal{Q}_n \setminus \{G_n^1, G_n^2\}$. If $G \in \mathcal{Q}(n, n-1) \setminus \{G_n^1, G_n^2\}$, by Lemma 2.7, we have $\rho(G_n^2) > \rho(G)$. If $G \in \mathcal{Q}(n, d_0) \setminus \{G_n^3\}$ $(1 \le d_0 \le n-2)$, by Lemmas 2.6 and 2.9, we have $\rho(G_n^2) > \rho(G_n^3) > \rho(G)$. It follows that $\rho(G_n^1) > \rho(G_n^2) > \rho(G)$ for the case of $G \in \mathcal{Q}_n \setminus \{G_n^1, G_n^2\}$.

Let $G \in \mathcal{Q}_n \setminus \{G_n^1, G_n^2, G_n^3\}$. If $G \in \mathcal{Q}(n, n-1) \setminus \{G_n^1, G_n^2, G_n^5\}$, by Lemmas 2.6, 2.7 and 2.9, we have $\rho(G_n^3) > \rho(G_n^4) > \rho(G_n^5) > \rho(G)$. If $G \in \mathcal{Q}(n, n-2) \setminus \{G_n^3\}$, by Lemma 2.6, we have $\rho(G_n^3) > \rho(G)$. If $G \in \mathcal{Q}(n, d_0) \setminus \{H_1\}$ $(1 \le d_0 \le n-3)$, by Lemma 2.6, we have $\rho(G_n^3) > \rho(H_1) > \rho(G)$. It follows that $\rho(G_n^1) > \rho(G_n^2) > \rho(G_n^3) > \rho(G)$ for the case of $G \in \mathcal{Q}_n \setminus \{G_n^1, G_n^2, G_n^3\}$.

Let $G \in \mathcal{Q}_n \setminus \{G_n^1, G_n^2, G_n^3, G_n^4\}$. If $G \in \mathcal{Q}(n, n-1) \setminus \{G_n^1, G_n^2, G_n^5\}$, by Lemmas 2.7 and 2.9, we have $\rho(G_n^4) > \rho(G_n^5) > \rho(G)$. If $G \in \mathcal{Q}(n, n-2) \setminus \{G_n^3, G_n^4\}$, by Lemma 2.6, we have $\rho(G_n^4) > \rho(G)$. If $G \in \mathcal{Q}(n, d_0) \setminus \{H_1\}$ $(1 \le d_0 \le n-3)$, by Lemmas 2.6 and 2.9, we have $\rho(G_n^4) > \rho(G_n^5) > \rho(H_1) > \rho(G)$. It follows that $\rho(G_n^1) > \rho(G_n^2) > \rho(G_n^3) > \rho(G_n^4) > \rho(G)$ for the case of $G \in \mathcal{Q}_n \setminus \{G_n^1, G_n^2, G_n^3, G_n^4\}$.

Let $G \in \mathcal{Q}_n \setminus \{G_n^1, G_n^2, G_n^3, G_n^4, G_n^5\}$. If $G \in \mathcal{Q}(n, n-1) \setminus \{G_n^1, G_n^2, G_n^5\}$, by Lemma 2.7, we have $\rho(G_n^5) > \rho(G)$. If $G \in \mathcal{Q}(n, n-2) \setminus \{G_n^3, G_n^4, F_2\}$, by Lemmas 2.8 and 2.9, we have $\rho(G_n^5) > \rho(F_2) > \rho(G)$. If $G \in \mathcal{Q}(n, d_0) \setminus \{H_1\}$ $(1 \le d_0 \le n-3)$, by Lemmas 2.6 and 2.9, we have $\rho(G_n^5) > \rho(H_1) > \rho(G)$. It follows that $\rho(G_n^1) > \rho(G_n^2) > \rho(G_n^3) > \rho(G_n^4) > \rho(G_n^5) > \rho(G)$ for the case of $G \in \mathcal{Q}_n \setminus \{G_n^1, G_n^2, G_n^3, G_n^4, G_n^5\}$. $\Box$

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