Journal of Mathematical Research with Applications
May, 2018, Vol. 38, No. 3, pp. 247–252
DOI:10.3770/j.issn:2095-2651.2018.03.003
Http://jmre.dlut.edu.cn

Notes on McCoy Modules

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Abstract Let R be a ring with an identity and $\mathcal{C}(R)$ be the category of right R-modules. In this paper we introduce the notion of semi-McCoy module. With this notion we show that McCoy modules of $\mathcal{C}(R)$ are closed under kernels of epimorphisms, and they are also closed under extensions and direct sums with certain conditions. We also get some results on the subcategories of McCoy modules of $\mathcal{C}(R[x])$ and $\mathcal{C}(R[x;x^{-1}])$.

Keywords McCoy module; extension; cokernel of monomorphism; kernel of epimorphism; direct sum

MR(2010) Subject Classification 16U80; 16S90

1. Introduction

Let R be a ring with an identity and all modules are right R-modules. $R[x]$ denotes the polynomial ring over R, $M[x]$ denotes the polynomial module over M and $R[x;x^{-1}]$ denotes the Laurent polynomial ring over R. We will use $\mathcal{C}(R)$ to denote the category of all right R-modules. In 1942, McCoy [1] proved that if R is commutative, $f(x)$ is a zero-divisor in $R[x]$, then there exists a nonzero element $r \in R$ such that $f(x)r = 0$. Rege and Chhawchharia [2] and Nielsen [3] introduced the notion of McCoy rings respectively. In 2011, Cui and Chen [4] introduced the notion of McCoy modules over R. McCoy module is the generalization of Armendariz module [2,5]. In 2017, Anderson and Chun [6] showed that if R is commutative, then the modules over R may not be McCoy modules [6, Example 3.1], and the direct sums of McCoy modules over R may not be McCoy modules [6, Example 3.3], either.

In this paper, we introduce the notion of semi-McCoy module. And if R is commutative, then all the modules over R are semi-McCoy. We discuss the short exact sequences of R-modules, $R[x]$-modules and $R[x;x^{-1}]$-modules, respectively. We show that all McCoy modules over R ($R[x], R[x,x^{-1}]$, resp.) are closed under kernels of monomorphisms, and then we give a condition and show that all McCoy modules satisfying certain conditions are closed under extensions and direct sums. But they are not closed under cokernels of epimorphisms.

Received April 15, 2017; Accepted December 16, 2017
Supported by the National Natural Science Foundation of China (Grant No. 11471017), the Natural Science Foundation of Anhui Higher Education Institutions of China (Grant No. KJ2018A0304) and the Doctoral Research Foundation and the Research Culture Foundation of Anhui Normal University (Grant No. 2014xmpy11).
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2. The extensions of McCoy modules

Let $\mathbb{Z}$ be the set of integers and $\mathbb{N}$ be the set of all nonnegative integers. The category of McCoy modules contains the category of Armendariz modules. A McCoy ring R is a McCoy R-module by the definition of McCoy module. That is, for any positive integer s, t, M is called McCoy if whenever $m(x) = \sum_{i=0}^{s} m_i x^i \in M[x]$ and $f(x) = \sum_{j=0}^{t} a_j x^j \in R[x] \setminus \{0\}$ satisfy $m(x)f(x) = 0$, then there exists $r \in R\setminus\{0\}$ such that $m(x)r = 0$. The multiplication is defined as $m(x)f(x) = \sum_{k=0}^{s+t}(\sum_{i+j=k} m_i a_j)x^k$. Now we will consider a kind of R-module.

Definition 2.1 *Let M be a right module over ring R. M is called semi-McCoy if $m(x)g(x) = 0$, where $m(x) = \sum_{i=0}^{m} m_i x^i \in M[x]$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]\setminus\{0\}$, implies that $m(x)rg(x) = 0$ for any $r \in R \setminus \{0\}$.*

From the definition we can get that all the modules over commutative ring R are semi-McCoy modules. Then module M in [6, Example 3.1] is a semi-McCoy R-module, but it is not a McCoy module.

Let $0 \to A \to B \to C \to 0$ be a short exact sequence with $A, B, C \in \mathcal{C}(R)$. We are interested to know for the subcategory of McCoy R-modules of $\mathcal{C}(R)$, whether the short exact sequences are closed under extensions, kernels of epimorphisms and cokernels of monomorphisms.. To answer this question, we first give a relation between short exact sequences of $\mathcal{C}(R)$ and $\mathcal{C}(R[x])$.

Proposition 2.2 *Let R be a ring with an identity and $A, B, C \in \mathcal{C}(R)$. Then*

$$0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$$

is a short exact sequence if and only if

$$0 \to A[x] \xrightarrow{\varphi'} B[x] \xrightarrow{\psi'} C[x] \to 0$$

is a short exact sequence, where $\varphi'(\sum_{i=0}^{m} a_i x^i) = \sum_{i=0}^{m} \varphi(a_i)x^i$, $\psi'(\sum_{j=0}^{n} b_j x^j) = \sum_{j=0}^{n} \psi(b_j)x^j$ with each $a_i \in A, b_j \in B$ and $m, n \in \mathbb{N}$.

Proof First we prove the necessity. We need to show that φ' is a monomorphism, ψ' is an epimorphism and $\mathrm{Ker}\psi' = \mathrm{Im}\varphi'$.

If $\varphi'(\sum_{i=0}^{m} a_i x^i) = \sum_{i=0}^{m} \varphi(a_i)x^i = 0$, then $\varphi(a_i) = 0$, and then $a_i = 0$ for $i = 0, 1, \ldots, m$, it follows that φ' is a monomorphism. For any polynomial $\sum_{k=0}^{t} c_k x^k$ with each $c_k \in C$, there exists $b_k \in B$ such that $\psi(b_k) = c_k$, where $t \in \mathbb{N}$. Then $\psi'(\sum_{k=0}^{t} b_k x^k) = \sum_{k=0}^{t} \psi(b_k)x^k$, that is, ψ' is an epimorphism.

From $\psi'\varphi'(\sum_{i=0}^{m} a_i x^i) = \sum_{i=0}^{m} \psi\varphi(a_i)x^i = 0$, we have $\mathrm{Im}\varphi' \subseteq \mathrm{Ker}\psi'$. If $\psi'(\sum_{j=0}^{n} b_j x^j) = 0$, then $\psi(b_j) = 0$, and then there exists $a_j \in A$ such that $\varphi(a_j) = b_j$ for $j = 0, 1, \ldots, n$, it follows that $\varphi'(\sum_{j=0}^{n} a_j x^j) = \sum_{j=0}^{n} \varphi(a_j)x^j = \sum_{j=0}^{n} b_j x^j$. That is, $\mathrm{Ker}\psi' \subseteq \mathrm{Im}\varphi'$.

Thus we get the conclusion that $0 \to A[x] \xrightarrow{\varphi'} B[x] \xrightarrow{\psi'} C[x] \to 0$ is a short exact sequence.

Now we prove the sufficiency. Take $x = 0$ for the polynomials in the short exact sequence $0 \to A[x] \xrightarrow{\varphi'} B[x] \xrightarrow{\psi'} C[x] \to 0$, then we can get the exactness of $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$. $\square$

If $A, B \in \mathcal{C}(R)$, for any homomorphism $\phi \in \mathrm{Hom}_R(A, B)$, let $f(x) = \sum_{i=0}^{m} a_i x^i \in A[x]$,

$g(x) = \sum_{j=0}^{n} r_j x^j \in R[x]$, where $m, n \in \mathbb{N}$. We have

$$\begin{aligned}\phi'(f(x)g(x)) &= \phi'\Big(\sum_{i+j=0}^{m+n} a_i r_j x^{i+j}\Big) = \sum_{i+j=0}^{m+n} \phi(a_i r_j) x^{i+j} = \sum_{i+j=0}^{m+n} \phi(a_i) r_j x^{i+j} \\ &= \sum_{i+j=0}^{m+n} (\phi(a_i)x^i)(r_j x^j) = \Big(\sum_{i=0}^{m} \phi(a_i)x^i\Big)\Big(\sum_{j=0}^{n} r_j x^j\Big) = \phi'(f(x))g(x). \qquad (*)\end{aligned}$$

In general, the extension of McCoy modules may not be a McCoy module. For example, if $0 \to A \to A \oplus B \to B \to 0$ is a split exact sequence, where A and B are McCoy R-modules. Then $A \oplus B$ may not be a McCoy R-module. Now we will consider McCoy R-modules with the following condition.

Condition (U) Let R be a ring with an identity and n be a positive integer. For any $i = 1, 2, \ldots, n$, A_i are all McCoy R-modules. Let $f_{A_i}(x) \in A_i[x]$ and $g(x) \in R[x] \setminus \{0\}$. If $(f_{A_1}(x), f_{A_2}(x), \ldots, f_{A_n}(x))g(x) = 0$ implies $(f_{A_1}(x)r_1, f_{A_2}(x)r_2, \ldots, f_{A_n}(x)r_n) = 0$ with nonzero elements $r_i \in R$ for $i = 1, 2, \ldots, n$, then $r_1 r_2 \cdots r_n \neq 0$.

Theorem 2.3 *Let R be a ring with an identity and $\mathcal{C}(R)$ be the category of R-modules. If $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ is a short exact sequence with $A, B, C \in \mathcal{C}(R)$, then*

(1) If B and C are McCoy R-modules, then A is a McCoy R-module.

(2) If A and C are McCoy R-modules satisfying condition (U), and B is a semi-McCoy module over R, then B is a McCoy R-module.

Proof (1) By [4, Proposition 2.3(1)], all submodules of B are McCoy R-modules, we can get that A is also a McCoy R-module.

(2) From the short exact sequence $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$, we have the short exact sequence $0 \to A[x] \xrightarrow{\varphi'} B[x] \xrightarrow{\psi'} C[x] \to 0$ by Proposition 2.2. Let $f_B(x) \in B[x]$, $g(x) \in R[x] \setminus \{0\}$, and $f_B(x)g(x) = 0$. We need to find a nonzero element $u \in R$ such that $f_B(x)u = 0$.

If $\psi'(f_B(x)) \neq 0$, since C is a McCoy R-module and $\psi'(f_B(x))g(x) = \psi'(f_B(x)g(x)) = 0$, there exists $r \in R \setminus \{0\}$ such that $\psi'(f_B(x))r = \psi'(f_B(x)r) = 0$. B is a McCoy R-module if $f_B(x)r = 0$. Suppose $f_B(x)r \neq 0$, then there exists $f_A(x) \in A[x]$ such that $\varphi(f_A(x)) = f_B(x)r$, and then $\varphi'(f_A(x)g(x)) = \varphi'(f_A(x))g(x) = f_B(x)rg(x) = 0$ for the condition that B is a semi-McCoy module. It follows that $f_A(x)g(x) = 0$. A is a McCoy module implies that there exists a nonzero element $r' \in R$ such that $f_A(x)r' = 0$. Notice that A and C satisfy condition (U), then there exists $u = rr' \neq 0$ such that $f_B(x)u = 0$.

If $\psi'(f_B(x)) = 0$, then there exists $f_A(x) \in A[x]$ such that $\varphi'(f_A(x)) = f_B(x)$. Note that $\varphi'(f_A(x)g(x)) = \varphi'(f_A(x))g(x) = f_B(x)g(x) = 0$ by $(*)$. Then $f_A(x)g(x) = 0$, and A is a McCoy module implies that there exists a nonzero element $a \in R$ such that $f_A(x)a = 0$, thus there exists $u = \varphi(a) \neq 0$ such that $f_B(x)u = 0$. Thus B is a McCoy R-module. □

For the category of $R[x]$-modules, we can give the similar conclusions in $\mathcal{C}(R[x])$.

Corollary 2.4 *Let R be a ring with an identity. $0 \to A[x] \xrightarrow{\varphi'} B[x] \xrightarrow{\psi'} C[x] \to 0$ is a short* *exact sequence with $A, B, C \in \mathcal{C}(R)$, then*

(1) If B and C are McCoy R-modules, then $0 \to A[x] \xrightarrow{\varphi'} B[x] \xrightarrow{\psi'} C[x] \to 0$ is a short exact sequence of McCoy $R[x]$-modules.

(2) If A and C are McCoy R-modules satisfying condition (U), and B is a semi-McCoy module over R, then $0 \to A[x] \xrightarrow{\varphi'} B[x] \xrightarrow{\psi'} C[x] \to 0$ is a short exact sequence of McCoy $R[x]$-modules.

Proof (1) By Proposition 2.2, $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ is a short exact sequence. If B and C are McCoy modules, then A is a McCoy module by Theorem 2.3. By [4, Theorem 3.2], $A[x]$ is a McCoy $R[x]$-module.

(2) By Proposition 2.2, $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ is a short exact sequence. If A and C are McCoy modules satisfying condition (U), then B is a McCoy module by Theorem 2.3. By [4, Theorem 3.2], $B[x]$ is a McCoy $R[x]$-module. $\square$

However, McCoy modules are not closed under cokernels of monomorphisms, we will give an example to show this fact.

Example 2.5 Let $\mathbb{Z}_2$ be the ring of residue classes modulo 2. Let

$$R = \left\{ \begin{pmatrix} a & a_{12} & a_{13} \\ 0 & a & a_{23} \\ 0 & 0 & a \end{pmatrix}, a, a_{ij} \in \mathbb{Z}_2 \right\}.$$

By [7, Theorem 2], R is McCoy. Let

$$I = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix}, a_{23} \in \mathbb{Z}_2 \right\}$$

be a right ideal of R. Then I is a McCoy R-module by [4, Proposition 2.3].

Take $M = R/I$. We next show that M is not a McCoy R-module. Let

$$m(x) = m_0 + m_1 x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x \in M[x]$$

and

$$f(x) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} x \in R[x].$$

Then $m(x)f(x) = 0$.

Note that for any nonzero A of R, $m_1 A = 0$ implies that $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. However,

$m_0 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \neq 0$. It shows that M is not a McCoy module.

If R is commutative, then McCoy R-modules with condition (U) are closed under extensions, and they are also closed under direct sums.

Corollary 2.6 *Suppose R is a commutative ring, A and B are McCoy R-modules satisfying condition (U), then $A \oplus B$ is a McCoy R-module. Moreover, the direct sum of McCoy R-modules $\bigoplus_{i=1}^n A_i$ is a McCoy module, where $A_1, A_2, \dots, A_n$ are McCoy R-modules satisfying condition (U).*

Proof For the short exact sequence $0 \to A \to A \oplus C \to C \to 0$, where A and C are McCoy R-modules satisfying condition (U), then $A \oplus C$ is a McCoy R-module by Theorem 2.3.

For the finite direct sum of McCoy modules $\bigoplus_{i=1}^n A_i$, where A_i are McCoy modules for $i = 1, 2, \dots, n$, we will prove the conclusion by induction. If each A_i is a McCoy R-module satisfying condition (U), then $A_1 \oplus A_2$ is a McCoy R-module from the short exact sequence $0 \to A_1 \to A_1 \oplus A_2 \to A_2 \to 0$ by Theorem 2.3, and $A_1 \oplus A_2, A_3, \dots, A_n$ also satisfy condition (U). Suppose $\bigoplus_{i=1}^{n-1} A_i$ and A_n are McCoy R-modules satisfying condition (U), then we consider short exact sequence $0 \to \bigoplus_{i=1}^{n-1} A_i \to \bigoplus_{i=1}^{n} A_i \to A_n$. By Theorem 2.3, $\bigoplus_{i=1}^n A_i$ is a McCoy R-module. □

Let S be a multiplicatively closed subset of central regular elements of R and M be a S-torsion free R-module, that is, $ms \neq 0$ for any nonzero element $m \in M$ and $s \in S$. Then $S^{-1}M$ becomes $S^{-1}R$-module. Let $R[x; x^{-1}]$ be the Laurent polynomial ring over R and $M[x; x^{-1}] = \{\sum_{i=m}^n m_i x^i \mid m, n \in \mathbb{Z}, m_i \in M\}$. For $m(x) = \sum_i m_i x^i \in M[x, x^{-1}]$, $f(x) = \sum_j a_j x^j \in R[x; x^{-1}]$, the multiplication is $m(x)f(x) = \sum_k (\sum_{i+j=k} m_i a_j) x^k$, and the addition is defined as usual. Then $M[x; x^{-1}]$ is a $R[x; x^{-1}]$-module.

Proposition 2.7 *Let R be a ring with an identity and A, B, C be S-torsion free R-modules. S is a multiplicatively closed subset of central regular elements of R. Then the following conclusions are equivalent.*

(1) $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ is a short exact sequence.

(2) $0 \to S^{-1}A \xrightarrow{S^{-1}\varphi} S^{-1}B \xrightarrow{S^{-1}\psi} S^{-1}C \to 0$ is a short exact sequence, where $S^{-1}\varphi(\frac{a}{s}) = \frac{\varphi(a)}{s}, a \in A, s \in S$, and $S^{-1}\psi(\frac{b}{t}) = \frac{\psi(b)}{t}, b \in B, t \in S$.

(3) $0 \to S^{-1}A[x] \xrightarrow{S^{-1}\varphi'} S^{-1}B[x] \xrightarrow{S^{-1}\psi'} S^{-1}C[x] \to 0$ is a short exact sequence, where $S^{-1}\varphi'(\sum_{i=0}^m \frac{a_i}{s_i} x^i) = \sum_{i=0}^m \frac{\varphi(a_i)}{s_i} x^i, a_i \in A, s_i \in S$ for $i = 0, 1, \dots, m$, and $S^{-1}\psi'(\sum_{j=0}^n \frac{b_j}{t_j} x^j) = \sum_{j=0}^n \frac{\psi(b_j)}{t_j} x^j, b_j \in B, t_j \in S$ for $j = 0, 1, \dots, n$.

Proof The conclusions from (1) to (2) and (2) to (3) are true since S^{-1} is an exact functor. We just need to prove the conclusion from (3) to (1).

For any element $a \in A, b \in B$ and $c \in C$, take $x = 0$ in $\frac{a}{1} + \sum_{i=1}^m \frac{a_i}{1} x^i \in S^{-1}A[x]$,

$\frac{b}{1}+\sum_{j=1}^{n}\frac{b_j}{1}x^j\in S^{-1}B[x]$ and $\frac{c}{1}+\sum_{k=1}^{t}\frac{c_k}{1}x^k\in S^{-1}C[x]$. The short exact sequence

$$0\to S^{-1}A[x]\xrightarrow{S^{-1}\varphi'}S^{-1}B[x]\xrightarrow{S^{-1}\psi'}S^{-1}C[x]\to 0$$

implies that $0\to A\xrightarrow{\varphi}B\xrightarrow{\psi}C\to 0$ is a short exact sequence. $\square$

Let $S=\{1,x,x^2,\ldots\}$. Then S is a multiplicatively closed subset of central regular elements of $R[x]$. For $M\in\mathcal{C}(R)$, $S^{-1}M[x]=M[x;x^{-1}]$ and $S^{-1}R[x]=R[x;x^{-1}]$, we can give a relation of short exact sequences between modules of $\mathcal{C}(R)$ and $\mathcal{C}(R[x;x^{-1}])$.

Corollary 2.8 *Let R be a ring with an identity and A,B,C be S-torsion free R-modules. Then*

$$0\to A\xrightarrow{\varphi}B\xrightarrow{\psi}C\to 0$$

is exact if and only if

$$0\to A[x;x^{-1}]\xrightarrow{\varphi'}B[x;x^{-1}]\xrightarrow{\psi'}C[x;x^{-1}]\to 0$$

is exact, where $\varphi'(\sum_{i=m_1}^{n_1}a_ix^i)=\sum_{i=m_1}^{n_1}\varphi(a_i)x^i$, $\psi'(\sum_{j=m_2}^{n_2}b_jx^j)=\sum_{j=m_2}^{n_2}\psi(b_j)x^j$ with each $a_i\in A, b_j\in B$ and $m_1,m_2,n_1,n_2\in\mathbb{Z}$.

Finally, we give conclusions of McCoy modules of the categories of R-modules, $R[x]$-modules and $R[x;x^{-1}]$-modules respectively from Theorem 2.3 to Corollary 2.8.

Theorem 2.9 *Let R be a ring with an identity. Then*

(1) The following conclusions are equivalent

(i) The McCoy modules of $\mathcal{C}(R)$ are closed under kernels of epimorphisms.

(ii) The McCoy modules of $\mathcal{C}(R[x])$ are closed under kernels of epimorphisms.

(iii) The McCoy modules of $\mathcal{C}(R[x;x^{-1}])$ are closed under kernels of epimorphisms.

(2) If the extensions of McCoy modules are semi-McCoy, then the following conclusions are equivalent

(i) The McCoy modules satisfy condition (U) of $\mathcal{C}(R)$ are closed under extensions.

(ii) The McCoy modules satisfy condition (U) of $\mathcal{C}(R[x])$ are closed under extensions.

(iii) The McCoy modules satisfy condition (U) of $\mathcal{C}(R[x;x^{-1}])$ are closed under extensions.

Acknowledgements We are grateful to the anonymous referee for carefully reading our manuscript and also for his/her detailed comments.

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